IEOR 4703: Homework 8

Given a SDE for a diffusion,

\[ dX(t) = a(X(t))dt + b(X(t))dB(t), \quad X(0) = X_0, \]

where \( \{B(t) : t \geq 0\} \) denotes a standard BM, the Euler method for approximating the sample paths of \( X = \{X(t) : 0 \leq t \leq T\} \) is given by choosing a small \( h > 0 \), setting \( N = \lfloor T/h \rfloor \), the integer part of \( T/h \), then generating \( N \) iid \( N(0,1) \) rvs, \( Z_1, \ldots, Z_N \), and using the recursion (with \( X^h_0 = X_0 \))

\[
X^h_1 = X_0 + a(X_0)h + b(X_0)\sqrt{h}Z_1 \\
X^h_2 = X^h_1 + a(X^h_1)h + b(X^h_1)\sqrt{h}Z_2 \\
\vdots \\
X^h_N = X^h_{N-1} + a(X^h_{N-1})h + b(X^h_{N-1})\sqrt{h}Z_N.
\]

This approximation yields \( N \) values, \( \{X^h_i : 1 \leq i \leq N\} \), which approximate \( X(t) \) only at the time points \( t \in \{h, 2h, \ldots, Nh\} \), that is \( X(ih) \approx X^h_i \). To get an approximation over the entire time interval \([0, T]\) (denoted by \( \{X^h(t) : t \geq 0\} \)) we can either linearly interpolate between time points (yielding continuous sample paths), or use the piecewise constant approach via

\[
X^h(t) = X^h_i, \quad ih \leq t < (i+1)h, \quad 0 \leq i \leq N. \tag{1}
\]

In what follows, we shall use the piecewise constant approach for simplicity.

For example, if \( T = 1 \), and we choose \( h = 1/100 \), then \( N = 100 \) (we have partitioned the interval \([0, 100]\) into 100 subintervals of length 1/100) and we have approximated the path \( \{X(t) : 0 \leq t \leq 1\} \) by the discrete set of points \( \{X^h_i : 0 \leq i \leq 100\} \) yielding a stochastic process \( \{X^h(t) : 0 \leq t \leq 1\} \) as defined by (1). As \( h \downarrow 0 \), the approximation gets better and better, but requires more computation. In particular, the point \( X^h_N \) approximates \( X(T) \), and thus (if \( X \) is the price of an asset) could be used to price a derivative of \( X \) involving only the termination date \( T \); \( (X(T) - K)^+ \), for example.

1. Consider GBM under its risk-neutral measure, for the purposes of pricing our derivatives.

Suppose the risk-free interest rate \( r = 0.05 \), and \( \sigma = 0.04 \). Thus the \( \mu \) in the Brownian motion is to be set to \( \mu = \mu^* = r - \sigma^2/2 = 0.05 - 0.0008 = 0.0492 \) yielding \( X(t) = (0.04)B(t) + (0.0492)t \), and \( S(t) = S(0)e^{X(t)} \). The discounted payoff of the Asian call option, that you priced in Homework 3, is given by

\[
Y = e^{-0.20} \left( \frac{1}{4} \sum_{i=1}^{4} S(i) - 40 \right)^+. \tag{2}
\]

The price of this option is thus \( C_0 = E(Y) \) and when \( S_0 = 35 \), the price estimate using Monte Carlo simulation is about 0.64.

Let us re-estimate this price by first approximating the GBM using the Euler method. The purpose of this is only for you to see, first hand, how well the Euler method might work in general for a given diffusion.
The SDE for GBM of the form $S(t) = S_0 e^{rB(t)}$ is
\[
dS(t) = (\mu + \sigma^2/2)S(t)dt + \sigma S(t)dB(t), \quad S(0) = S_0.
\]

With our parameters (recall that $\mu + \sigma^2/2 = r$ here) this becomes
\[
dS(t) = 0.050S(t)dt + 0.04S(t)dB(t), \quad S(0) = 35.
\]

For our Asian option, $T = 4$, and let us use $h = T/N$ for an integer $N$. This yields the Euler approximation
\[
S_h^1 = S(0) + 0.050S(0)h + 0.04S(0)\sqrt{h}Z_1
\]
\[
S_h^2 = S_h^1 + 0.050S_h^1h + 0.04S_h^1\sqrt{h}Z_2
\]
\[\vdots\]
\[
S_h^N = S_h^{N-1} + 0.050S_h^{N-1}h + 0.04S_h^{N-1}\sqrt{h}Z_N.
\]

This simplifies to
\[
S_h^1 = S(0)[1 + 0.05h + 0.04\sqrt{h}Z_1]
\]
\[
S_h^2 = S_h^1[1 + 0.05h + 0.04\sqrt{h}Z_2]
\]
\[\vdots\]
\[
S_h^N = S_h^{N-1}[1 + 0.05h + 0.04\sqrt{h}Z_N].
\]

In effect, we obtain $S_h^i = S(0)V_1V_2 \cdots V_i$ where the $V_i$ are iid distributed as $[1 + 0.05h + 0.04\sqrt{h}Z]$, a normal with mean $1 + 0.05h$ and variance $h$.

Starting with $N = 50$ (so $h = 4/50$), simulate the above so as to obtain an approximate copy of the Asian payoff in (2); denote this by $\hat{Y}_1$. Repeat this (independently) $n = 1000$ times and average to get the approximation for the option price as
\[
C_0 \approx \frac{1}{n} \sum_{j=1}^{n} \hat{Y}_j.
\]

Now double $N$ to $N = 100$ and do this again. Keep doubling until your answer (denote by $A$) satisfies $|A - 0.64| < 0.02$.

2. Consider an Ornstein-Uhlenbeck process
\[
dX(t) = -\alpha X(t)dt + \sigma dB(t), \quad X(0) = X_0.
\]

With $X(0) = 0$, $\sigma = 1$ and $\alpha = 10$, use the Euler method to obtain an approximation for $X(10)$, so as to estimate $Var(X(10))$. Using $N = 100$, generate 1000 iid copies of $X^h(10)$ and take the sample variance. Repeat for $N = 200$. In both cases, graph the resulting approximating process (just the 1rst sample path) over the interval $[0, 10]$. 


3. Consider the Cox-Ingersoll-Ross process (CIR), described by the SDE

\[ dX(t) = \alpha(c - X(t))dt + \beta \sqrt{X(t)}dB(t), \quad X(0) = X_0. \]

Using \( X_0 = 1 = \alpha = c = \beta \) (in particular \( 2\alpha c > \beta^2 \) so the process is positive), use the Euler method to approximate the path of \( X(t) \) over the unit interval \([0, 1]\) (so \( T = 1 \) here). Use \( N = 100 \). Remember to take the positive part in the square root via

\[ X_h = X_{i-1}^{h} + \alpha(c - X_{i-1}^{h})h + \beta \sqrt{X_{i-1}^{h}}+\sqrt{h}Z_N. \]

Graph the resulting approximation over \([0, 1]\). (So in this problem you only need to generate 1 sample path.)