**IEOR 4703: Homework 9 Solutions**

**Importance sampling**

1. An Erlang distribution with two phases and mean \(1/\alpha\) is denoted by \(E_2(\alpha)\) and a rv \(X\) with such a distribution can be represented via \(X = Y_1 + Y_2\) where the \(Y_i\) are iid with an exponential distribution with rate \(2\alpha\). Such a distribution is thus easy to simulate from: Generate two iid exponentials and sum them up.

Consider a random walk \(R_k = \Delta_1 + \cdots + \Delta_k, R_0 = 0\) in which the \(\Delta\) are iid distributed as \(\Delta = S - T\), where \(S\) and \(T\) are independent, \(S \sim E_2(\mu)\) with mean \(1/\mu\), and \(T \sim E_2(\lambda)\) with mean \(1/\lambda\), and \(\lambda < \mu\). We will use importance sampling to estimate the probability of ruin for an insurance risk business having claims arrive according to a renewal process with iid interarrival times distributed as \(T\) and claim sizes iid distributed as \(S\). (And we assume that money is earned at rate \(c = 1\) in between claims, for simplicity.) We know that if the reserve is initially \(b > 0\), then the ruin probability is given by \(P(M > b)\), where \(M = \max_{k \geq 0} R_k\). Find the Lundberg constant, the unique \(\gamma > 0\) such that \(E(e^{\gamma \Delta}) = 1\).

**SOLUTION:** To find \(\gamma\), we need to solve

\[
E(e^{\gamma \Delta}) = E(e^{\gamma S})E(e^{-\gamma T}) = 1,
\]

where \(T \sim \text{Erlang}(2\lambda)\) and \(S \sim \text{Erlang}(2\mu)\); or

\[
\left(\frac{2\mu}{2\mu - \gamma}\right)^2 \left(\frac{2\lambda}{2\lambda + \gamma}\right)^2 = 1.
\]

(And we must have \(0 < \gamma < 2\mu\).) But note that we can take the \(\sqrt{\cdot}\) of both sides, reducing the problem to

\[
\left(\frac{2\mu}{2\mu - \gamma}\right)\left(\frac{2\lambda}{2\lambda + \gamma}\right) = 1.
\]

The solution is easily seen to be \(\gamma = 2\mu - 2\lambda\).

2. Let \(f(x)\) denote the density of \(\Delta = S - T\). Argue that the twisted density, \(g(x) = e^{\gamma x}f(x)\), is in fact itself identical in distribution to \(T - S\). In other words, the effect of the twist is to simply swap the two rates \(\mu\) and \(\lambda\): Under the new density, claims have a \(E_2(\lambda)\) distribution and interarrival times have a \(E_2(\mu)\) distribution.

In general

\[
K_f(s) \overset{\text{def}}{=} \int_{-\infty}^{\infty} e^{sx} f(x) dx = E(e^{s\Delta}) = E(e^{sS})E(e^{-sT}) = \left(\frac{2\mu}{2\mu - s}\right)^2 \left(\frac{2\lambda}{2\lambda + s}\right)^2.
\]

\[
K_g(s) \overset{\text{def}}{=} \int_{-\infty}^{\infty} e^{sx} e^{\gamma x} f(x) dx = E(e^{(s+\gamma)\Delta}) = K_f(s+\gamma) = \left(\frac{2\mu}{2\mu - (s + \gamma)}\right)^2 \left(\frac{2\lambda}{2\lambda + (s + \gamma)}\right)^2.
\]

Plugging in \(\gamma = 2\mu - 2\lambda\), then yields

\[
K_g(s) = \left(\frac{2\lambda}{2\lambda - s}\right)^2 \left(\frac{2\mu}{2\mu + s}\right)^2 = E(e^{s(T-S)})
\]

so the moment generating function is indeed that of \(T - S\), and hence by uniqueness of moment generating functions we are done.
3. For \( \lambda = 1 \) and \( \mu = 2 \), use the importance sampling method to estimate \( P(M > 5) \): simulate the random walk using (twisted) increments as \( T - S \) until it goes above level 5, let \( B_1 = B_1(5) \) be the overshoot. Let \( X_1 = e^{-\gamma B_1} \). Independently simulate the random walk again to get a sample \( X_2 \) and so on for \( n = 1000 \) such samples and then use the approximation

\[
P(M > 5) \approx e^{-\gamma 5} \frac{1}{n} \sum_{i=1}^{n} X_i.
\]

**SOLUTION:**
the numerical answer is: \( 1.8385 \times 10^{-5} \), and the code:

```python
lambda = 1; mu = 2; b = 5; sum = 0; gamma = 2*mu - 2*lambda; for i=1:1000
    M = 0;
    R = 0;
    true = 1;
    while (true == 1)
        u_1 = rand;
        u_2 = rand;
        T = (-1/(2*lambda))*log(u_1)+(-1/(2*lambda))*log(u_2);
        v_1 = rand;
        v_2 = rand;
        S = (-1/(2*mu))*log(v_1) + (-1/(2*mu))*log(v_2);
        R = R + T - S;
        if (R > b)
            B = R - b;
            true = 0;
        end
    end
    X = exp(-gamma*B);
    sum = sum+X;
end
prob = exp(-gamma*b)*sum/1000
```

**Score function method**
Read Lecture Notes 6, Section 1.2 on the *Score Function Method* for estimating Greeks (Lecture Notes 6 have been revised to include, as Section 1.2, the score function method).

1. Go back to HMWK 7, and re-do Exercise 1 by using the score function method (use the same \( n = 5000 \)), to estimate the \( \Delta \) for the Asian call option given.

**SOLUTION:**
the numerical answer is: 0.4564
T = 4; r = 0.05; mu = 0.0492; sigma = 0.04; n = 5000; S_0 = 35; K = 40; X=zeros(1,n);

for i=1:n
    Z=randn(1,4);
    S_1 = S_0*exp(sigma*Z(1)+(r-(sigma^2)/2));
    S_2 = S_1*exp(sigma*Z(2)+(r-(sigma^2)/2));
    S_3 = S_2*exp(sigma*Z(3)+(r-(sigma^2)/2));
    S_4 = S_3*exp(sigma*Z(4)+(r-(sigma^2)/2));
    avg = (1/4)*(S_1+S_2+S_3+S_4);
    X(i) = max(avg-K,0)*Z(1)/(S_0*sigma);
end

mean(X)*exp(-r*T)

2. Using the same parameters (and n = 5000) as in (1) (S_0 = 35, etc.), estimate the ∆ for a digital call option with payoff \(I\{S(4) > 40\}\) using the score function method. Compare with the exact answer (using the price of the option, \(e^{-rT}P(S(4) > 40)\), and directly differentiating it with respect to \(S_0\)).

the numerical answer is: 0.0871.
Moreover, the price of a digital call is given by:

\[e^{-rT}P(S(T) > K) = e^{-rT}\Theta(c),\]

where

\[c = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}},\]

and \(\Theta(x)\) is the unit normal cdf. Thus it can explicitly be differentiated with respect to \(S_0\) yielding

\[\Delta = \frac{e^{-rT}\Theta'(c)}{\sigma S_0\sqrt{T}},\]

where

\[\Theta'(x) = \theta(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}, x \in \mathbb{R},\]

is the unit normal density.
sum = sum+Z/(S_0*sqrt(T)*sigma);
end
end
exp(-r*T)*sum/n

3. Consider any option with payoff of the form $h(S(T))$ at terminal time $T$ (under GBM). And let’s consider computing the Vega of such an option. Let $\dot{f}(x) = \frac{df(x)}{d\sigma}$, where $f(x)$ is the density of $S(T)$.

Show (show all the algebra involved) that the score function can be given by

$$\frac{\dot{f}(S(T))}{f(S(T))} = \frac{Z^2 - 1}{\sigma} - Z\sqrt{T}. $$

(Here, $Z$ is the unit normal used to construct $S(T)$.)

**SOLUTION:**

$$\frac{df(x)}{d\sigma} = \frac{-1}{x\sigma^2 T} \theta(l(x)) + \frac{1}{x\sigma\sqrt{T}} \frac{d\theta(l(x))}{d\sigma}$$

$$\frac{d\theta(l(x))}{d\sigma} = -l(x)\theta(l(x)) \frac{\sigma T}{\sigma T} \theta(l(S(T))) + \frac{1}{S(T)\sigma \sqrt{T}} \dot{\theta}(l(S(T)))$$

$$\dot{f}(S(T)) = \frac{-1}{S(T)\sigma^2 \sqrt{T}} \theta(l(S(T))) + \frac{1}{S(T)\sigma \sqrt{T}} [-Z\theta(Z) \frac{\sigma^2 T \sqrt{T} - \sigma^2 Z}{\sigma^2 T}]$$

where we use $l(S(T)) = Z$

$$f(S(T)) = \frac{1}{S(T)\sigma \sqrt{T}} \theta(l(S(T))) = \frac{1}{S(T)\sigma \sqrt{T} \theta(Z)}$$

$$\Rightarrow \frac{\dot{f}(S(T))}{f(S(T))} = \frac{-1}{\sigma} \left( -Z \frac{\sigma^2 T \sqrt{T} - \sigma^2 Z}{\sigma^2 T} \right)$$

$$\Rightarrow \frac{\dot{f}(S(T))}{f(S(T))} = \frac{Z^2 - 1}{\sigma} - Z\sqrt{T}$$

4. Use (3) to estimate the Vega of the digital option in (2).

**SOLUTION:**

The numerical answer is: -5.0891

And: In this case, as for the $\Delta$ (in (2) above), a closed form can be derived. (Computing a Vega for a digital option is somewhat meaningless, since it can be negative, or even 0, as well as positive.)

```plaintext
T = 4; r = 0.05; mu = 0.0492; sigma = 0.04; n = 5000; S_0 = 35; K = 40; X=zeros(1,n); sum = 0;

for i=1:n
    Z=randn;
    S_4 = S_0*exp(sigma*sqrt(T)*Z+(r-(sigma^2)/2)*T);
    if (S_4 > 40)
```

4
\[
\text{sum} = \text{sum} + ((Z^2 - 1)/\text{sigma} - Z*\text{sqrt}(T)) ;
\]
\end
\end
\]
\[
\exp(-r*T)*(\text{sum}/n)
\]
5. We know that the joint density \( f(x) \) of \((S(t_1), \ldots, S(t_k))\), can be written as the product of conditional densities
\[
f(x) = f_1(x_1|S_0)f_2(x_2|x_1) \times \cdots \times f_k(x_k|x_{k-1});
\]
where
\[
f_1(x_1|S_0) \text{ is the density of } S(t_1). \ f_2(x_2|x_1) \text{ is the conditional density of } S(t_2) \text{ given that } S(t_1) = x_1, \text{ so it is the density of }
\]
\[
x_1e^{o\sqrt{t_2-t_1}Z_2+(r-o^2/2)(t_2-t_1)},
\]
and so on with \( f_k(x_k|x_{k-1}) \) the conditional density of \( S(t_k) \) given that \( S(t_{k-1}) = x_{k-1} \); it has the density of
\[
x_{k-1}e^{o\sqrt{t_k-t_{k-1}}Z_k+(r-o^2/2)(t_k-t_{k-1})}.
\]
Let \( \dot{f}(x) = \frac{df(x)}{d\sigma} \), and let \( \dot{f}(x_i) = \frac{df(x_i)}{d\sigma} \). Show that
\[
\frac{\dot{f}(x)}{f(x)} = \sum_{i=1}^{k} \frac{\dot{f}_i(x_i)}{f_i(x_i)}.
\]
Thus conclude from the score function derived in (3), that the score function for estimating the Vega of an Asian call is given by
\[
\frac{\dot{f}((S(t_1), \ldots, S(t_k)))}{f((S(t_1), \ldots, S(t_k)))} = \sum_{i=1}^{k} \frac{Z_i^2 - 1}{\sigma} - Z_i\sqrt{t_i - t_{i-1}},
\]
where we are using the iid normals, \( Z_1, \ldots, Z_k \) to construct \((S(t_1), \ldots, S(t_k))\). \((t_0 = 0.) \)
(Payoff = \((\mathcal{S} - K)^+ \text{ where } \mathcal{S} = \frac{1}{k} \sum_{i=1}^{k} S(t_i).\))

**SOLUTION:**
\[
f(x) = \prod_{i=1}^{k} x_i e^{o \sum_{i=1}^{k} \sqrt{t_i-t_{i-1}}Z_i+(r-o^2/2)(t_i-t_{i-1})} \]
\[
\dot{f}(x) = f(x)[\sum_{i=1}^{k} \sqrt{t_i-t_{i-1}}Z_i - \sigma(t_i-t_{i-1})]
\]
\[
\Rightarrow \frac{\dot{f}(x)}{f(x)} = \sum_{i=1}^{k} \sqrt{t_i-t_{i-1}}Z_i - \sigma(t_i-t_{i-1})
\]
But,
\[
\dot{f}_i(x_i) = \sqrt{t_i-t_{i-1}}Z_i - \sigma(t_i-t_{i-1})f_i(x_i)
\]
which leads to,
$\frac{f(x)}{f(x)} = \sum_{i=1}^{k} \frac{f_i(x_i)}{f_i(x_i)}$

To get the Vega for the Asian call, simply plug in the formula in (3) which leads to:

$$\frac{f_i(x_i)}{f_i(x_i)} = Z_i^2 - Z_i \sqrt{t_i - t_{i-1}}$$

and,

$$\frac{f((S(t_1), \ldots, S(t_k)))}{f((S(t_1), \ldots, S(t_k)))} = \sum_{i=1}^{k} \frac{Z_i^2 - 1}{\sigma} - Z_i \sqrt{t_i - t_{i-1}},$$

6. Using (5), re-do HMK 7, Exercise 3 (Vega for an Asian call) using the same parameters, but now using the score function method.

the numerical answer is: 18.2745

T = 4; r = 0.05; mu = 0.0492; sigma = 0.04; n = 5000; S_0 = 35; K = 40; X=zeros(1,n);

for i=1:n
    sum = 0;
    Z=randn(1,4);
    S_1 = S_0*exp(sigma*Z(1)+(r-(sigma^2)/2));
    S_2 = S_1*exp(sigma*Z(2)+(r-(sigma^2)/2));
    S_3 = S_2*exp(sigma*Z(3)+(r-(sigma^2)/2));
    S_4 = S_3*exp(sigma*Z(4)+(r-(sigma^2)/2));
    avg = (1/4)*(S_1+S_2+S_3+S_4);
    for j=1:4
        sum=sum+((Z(j)^2-1)/sigma-Z(j));
    end
    X(i) = max(avg-K,0)*sum;
end

mean(X)*exp(-r*T)