A Cournot-Stackelberg Model of Supply Contracts with Financial Hedging

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This revision: 29 February 2016

Abstract

We study the performance of a stylized supply chain where multiple retailers and a single producer compete in a Cournot-Stackelberg game. At time $t = 0$ the retailers order a single product from the producer and upon delivery at time $T > 0$, they sell it in the retail market at a stochastic clearance price. We assume the retailers’ profits depend in part on the realized path of some tradeable stochastic process such as a foreign exchange rate, commodity price or more generally, some tradeable economic index. Because production and delivery do not take place until time $T$, the producer offers a menu of wholesale prices to the retailers, one for each realization of the process up to some time, $\tau$, where $0 \leq \tau \leq T$. The retailers’ ordering quantities therefore depend on the realization of the process until time $\tau$. We also assume, however, that the retailers are budget-constrained and are therefore limited in the number of units they may purchase from the producer. The supply chain might therefore be more profitable if the retailers were able to reallocate their budgets across different states of nature. In order to affect a (partial) reallocation, we assume that the retailers are also able to trade dynamically in the financial market. After solving for the Nash equilibrium we address such questions as: (i) whether or not the players would be better off if the retailers merged and (ii) whether or not the players are better off when the retailers have access to the financial markets. Our model can easily handle variations where, for example, the retailers are located in a different currency area to the producer or where the retailers must pay the producer before their budgets are available. Finally, we consider the case where the producer can choose the optimal timing, $\tau$, of the contract and we formulate this as an optimal stopping problem.


Keywords: Procurement contract, financial constraints, supply chain coordination.
1 Introduction

We study the performance of a stylized supply chain where multiple retailers and a single producer compete in a Cournot-Stackelberg game. At time $t = 0$ the retailers order a single product from the producer and upon delivery at time $T > 0$, they sell it in the retail market at a stochastic clearance price that depends in part on the realized path or terminal value of some observable and tradeable financial process. Because delivery does not take place until time $T$, the producer offers a menu of wholesale prices to the retailer, one for each realization of the process up to time some time, $\tau$, where $0 \leq \tau \leq T$. The retailers’ ordering quantities are therefore contingent upon the realization of the process up to time $\tau$. Production of the good is then completed at time $\tau$.

We also assume, however, that the retailers are budget-constrained and are therefore limited in the number of units they may purchase from the producer. As a result, the supply chain might be more profitable if the retailers were able to reallocate their financial resources, i.e. their budgets, across different states. By allowing the retailers to trade dynamically in the financial markets we enable such a (partial) reallocation of resources. The producer has no need to trade in the financial markets as he is not budget constrained and, like the retailers, is assumed to be risk neutral. After solving for the Cournot-Stackelberg equilibrium we address such questions as whether or not the players would be better off if the retailers merged and whether or not the players are better off when the retailers have access to the financial markets.

We now attempt to position our paper within the vast literature on supply chain management. We refer the reader to the books by de Kok and Graves (2003) and Simchi-Levi et al. (2004) for a general overview of supply chain management issues and to the survey article by Cachon (2003) for a review of supply chain management contracts.

A distinguishing feature of our model with respect to most of the literature in supply chain management is the budget constraint that we impose on the retailers’ procurement decisions. Some recent exceptions include Buzacott and Zhang (2004), Caldentey and Haugh (2009), Dada and Hu (2008), Kouvelis and Zhao (2012), Xu and Birge (2004) and Caldentey and Chen (2012) (see also Part Three in Kouvelis et al., 2012).

Xu and Birge (2004) analyze a single-period newsvendor model which is used to illustrate how a firm’s inventory decisions are affected by the existence of a budget constraint and the firm’s capital structure. In a multi-period setting, Hu and Sobel (2005) examine the interdependence of a firm’s capital structure and its short-term operating decisions concerning inventory, dividends, and liquidity. In a similar setting, Dada and Hu (2008) consider a budget-constrained newsvendor that can borrow from a bank that acts strategically when choosing the interest rate applied to the loan. They characterize the Stackelberg equilibrium and investigate conditions under which channel coordination, i.e., where the ordering quantities of the budget-constrained and non budget-constrained newsvendors coincide, can be achieved.

Buzacott and Zhang (2004) incorporate asset-based financing in a deterministic multi-period production/inventory control system by modeling the available cash in each period as a function of the firm’s assets and liabilities. In their model a retailer finances its operations by borrowing from a commercial bank. The terms of the loans are contingent upon the retailer’s balance sheet and income statement and in particular, upon the inventories and accounts receivable. The authors conclude that asset-based financing allows retailers to enhance their cash return over what it would
otherwise be if they were only able to use their own capital.

The work by Caldentey and Haugh (2009), Kouvelis and Zhao (2012) and Caldentey and Chen (2012) are the most closely related to this paper. They all consider a two-echelon supply chain system in which there is a single budget constrained retailer and investigate different types of procurement contracts between the agents using a Stackelberg equilibrium concept. In Kouvelis and Zhao (2012) the supplier offers different type of contracts designed to provide financial services to the retailer. They analyze a set of alternative financing schemes including supplier early payment discount, open account financing, joint supplier financing with bank, and bank financing schemes. In a similar setting, Caldentey and Chen (2012) discuss two alternative forms of financing for the retailer: (a) internal financing in which the supplier offers a procurement contract that allows the retailer to pay in arrears a fraction of the procurement cost after demand is realized and (b) external financing in which a third party financial institution offers a commercial loan to the retailer. They conclude that in an optimally designed contract it is in the supplier’s best interest to offer financing to the retailer and that the retailer will always prefer internal rather than external financing.

In Caldentey and Haugh (2009) the supplier offers a modified wholesale price contract which is executed at a future time $\tau$. The terms of the contract are such that the actual wholesale price charged at time $\tau$ depends on information publicly available at this time. Delaying the execution of the contract is important because in this model the retailer’s demand depends in part on a financial index that the retailer and supplier can observe through time. As a result, the retailer can dynamically trade in the financial market to adjust his budget to make it contingent upon the evolution of the index. Their model shows how financial markets can be used as (i) a source of public information upon which procurement contracts can be written and (ii) as a means for financial hedging to mitigate the effects of the budget constraint. In this paper, we therefore extend the model in Caldentey and Haugh (2009) by considering a market with multiple retailers in Cournot competition as well as a Stackelberg leader. Our extended model can also easily handle variations where, for example, the retailers are located in a different currency area to the producer or where the retailers must pay the producer before their budgets are available. In addition we consider the case where the producer can choose the optimal timing, $\tau$, of the contract and we formulate this as an optimal stopping problem.

A second related stream of research considers Cournot-Stackelberg equilibria. There is an extensive economics literature on this topic that focuses on issues of existence and uniqueness of the Nash equilibrium. See Okoguchi and Szidarovsky (1999) for a comprehensive review. In the context of supply chain management, there has been some recent research that investigates the design of efficient contracts between the supplier and the retailers. For example, Bernstein and Federgruen (2003) derive a perfect coordination mechanism between the supplier and the retailers. This mechanism takes the form of a nonlinear wholesale pricing scheme. Zhao et al. (2005) investigate inventory sharing mechanisms among competing dealers in a distribution network setting. Li (2002) studies a Cournot-Stackelberg model with asymmetric information in which the retailers are endowed with some private information about market demand. In contrast, the model we present in this paper uses the public information provided by the financial markets to improve the supply chain coordination.

Finally, we mention that there exists a related stream of research that investigates the use of financial markets and instruments to hedge operational risk exposure. See Boyabatli and Toktay (2004) and the survey paper by Zhao and Huchzermeier (2015) for detailed reviews. For example,
Caldentey and Haugh (2006) consider the general problem of dynamically hedging the profits of a risk-averse corporation when these profits are partially correlated with returns in the financial markets. Chod et al. (2009) examine the joint impact of operational flexibility and financial hedging on a firm’s performance and their complementarity/substitutability with the firm’s overall risk management strategy. Ding et al. (2007) and Dong et al. (2014) examine the interaction of operational and financial decisions from an integrated risk management standpoint. Boyabatli and Toktay (2011) analyze the effect of capital market imperfections on a firm’s operational and financial decisions in a capacity investment setting. Babich and Sobel (2004) propose an infinite-horizon discounted Markov decision process in which an IPO event is treated as a stopping time. They characterize an optimal capacity-expansion and financing policy so as to maximize the expected present value of the firm’s IPO. Babich et al. (2012) consider how trade credit financing affects the relationships among firms in the supply chain, supplier selection, and supply chain performance.

The remainder of this paper is organized as follows. Section 2 describes our model, focussing in particular on the supply chain, the financial markets and the contractual agreement between the producer and the retailers. We analyze this model in Section 3 where we obtain explicit expressions for the retailers’ purchasing decisions in the Cournot equilibrium as a function of the producer’s price menu. In this section we also obtain the producer’s optimal price menu, i.e. the Stackelberg equilibrium, in two special cases: (i) when the budget constraints are non-binding (possibly due to the ability to hedge in the financial markets) and (ii) when all retailers have identical budgets. Motivated by the results of these two cases, we propose a class of linear wholesale price contracts and show by way of example that it is straightforward for the producer to optimize numerically over this class. In Section 4 we discuss the value of the financial markets and we consider various extensions to the model in Section 5. These extensions include variations where the retailers are located in a different currency area to the producer, where the retailers must pay the producer before their budgets are available, and where the producer can choose the optimal timing, $\tau$, of the contract. We conclude in Section 6. Most of our proofs together with other details are contained in the Appendices.

2 Model Description

We now describe the model in further detail. We begin with the supply chain description and then discuss the role of the financial markets. At the end of the section we define the contract which specifies the agreement between the producer and the retailers. Throughout this section we will assume for ease of exposition that both the producer and the retailers are located in the same currency area and that interest rates are identically zero. In Section 5 we will relax these assumptions and still maintain the tractability of our model using change of measure arguments.

2.1 The Supply Chain

We model an isolated segment of a competitive supply chain with one producer that produces a single product and $N$ competing retailers that face a stochastic clearance price\(^1\) for this product. This clearance price, and the resulting cash-flow to the retailers, is realized at a fixed future time

\(^1\)Similar models are discussed in detail in Section 2 of Cachon (2003). See also Lariviere and Porteus (2001).
The retailers and producer, however, negotiate the terms of a procurement contract at time 
$t = 0$. This contract specifies three quantities:

(i) A production time, $\tau$, with $0 \leq \tau \leq T$. While $\tau$ will be fixed for most of our analysis, we will also consider the problem of selecting an optimal $\tau$ in Section 5.4.

(ii) A rule that specifies the size of the order, $q_i$, for the $i^{th}$ retailer where $i = 1, \ldots, N$. In general, $q_i$ may depend upon market information available at time $\tau$.

(iii) The payment, $W(q_i)$, that the $i^{th}$ retailer pays to the producer for fulfilling the order. Again, $W(q_i)$ will generally depend upon market information available at time $\tau$. The timing of this payment is not important when we assume that interest rates are identically zero. In Section 5.3, however, we will assume interest rates are stochastic when we consider the case where the retailers must pay the producer before their budgets are available. It will then be necessary to specify exactly when the retailers pay the producer.

We will restrict ourselves to transfer payments that are linear on the ordering quantity. That is, we consider the so-called wholesale price contract where $W(q) = wq$ and where $w$ is the per-unit wholesale price charged by the producer. We assume that the producer offers the same contract to each retailer and while this simplifies the analysis considerably, it is also realistic. For example, it is often illegal for a producer to price-discriminate among its customers. We also assume that during the negotiation of the contract the producer acts as a Stackelberg leader. That is, for a fixed procurement time $\tau$, the producer moves first and at $t = 0$ proposes a wholesale price menu, $w_\tau$, to which the retailers then react by selecting their ordering levels, $q_i$, for $i = 1, \ldots, N$. Note that the $N$ retailers also compete among themselves in a Cournot-style game to determine their optimal ordering quantities and trading strategies.

We assume that the producer has unlimited production capacity and that if production takes place at time $\tau$ then the per-unit production cost is $c_\tau$. We will generally assume that $c_\tau$ is constant but many of our results, however, go through when $c_\tau$ is stochastic. The producer’s payoff as a function of the procurement time, $\tau$, the wholesale price, $w_\tau$, and the ordering quantities, $q_i$, is given by

\[ \Pi_P := \sum_{i=1}^{N} (w_\tau - c_\tau) q_i. \]  

We assume that each retailer is restricted by a budget constraint that limits his ordering decisions. In particular, we assume that each retailer has an initial budget, $B_i$, that may be used to purchase product units from the producer. Without loss of generality, we order the retailers so that $B_1 \geq B_2 \geq \ldots \geq B_N$. We assume each of the retailers can trade in the financial markets during the time interval $[0, \tau]$, thereby transferring cash resources from states where they are not needed to states where they are.

For a given set of order quantities, the $i^{th}$ retailer collects a random revenue at time $T$. We compute this revenue using a linear clearance price model. That is, the market price at which the retailer

\[ 2\text{When we consider the optimal timing of } \tau \text{ in Section 5.4 we will assume that } c_\tau \text{ is deterministic and increasing in } \tau \text{ so that production postponement comes at a cost.} \]

\[ 3\text{In Section 5.2 we will assume that the producer and retailers are located in different currency areas. We will then need to adjust (2) appropriately.} \]
sells these units is a random variable, \( P(Q) := A - (q_i + Q_{i-}) \), where \( A \) is a non-negative random variable, \( Q_{i-} := \sum_{j \neq i} q_j \) and \( Q := \sum_j q_j \). The random variable \( A \) models the market size that we assume is unknown. The realization of \( A \), however, will depend in part on the realization of the financial markets between times 0 and \( T \). The payoff of the \( i \)th retailer, as a function of \( \tau \), \( w_\tau \), and the order quantities, then takes the form

\[
\Pi_{R_i} := (A - (q_i + Q_{i-})) q_i - w_\tau q_i. \tag{2}
\]

A stochastic clearance price is easily justified since in practice unsold units are generally liquidated using secondary markets at discount prices. Therefore, we can view our clearance price as the average selling price across all units and markets. As stated earlier, \( w_\tau \) and the \( q_i \)'s will in general depend upon market information available at time \( \tau \). Since \( W(q) \), \( \Pi_\tau \) and the \( \Pi_{R_i} \)'s are functions of \( w_\tau \) and the \( q_i \)'s, these quantities will also depend upon market information available at time \( \tau \).

The linear clearance price in (2) is commonly assumed in the operations and economics literature for reasons of tractability and estimation. It also helps ensure that the game will have a unique Nash equilibrium. (For further details see Chapter 4 of Vives, 2001.)

### 2.2 The Financial Market

The financial market is modeled as follows. Let \( X_t \) denote\(^4\) the time \( t \) value of a tradeable security and let \( \{F_t\}_{0 \leq t \leq T} \) be the filtration generated by \( X_t \) on a probability space, \((\Omega, F, Q)\). We do not assume that \( F_T = F \) since we want the non-financial random variable, \( A \), to be \( F \)-measurable but not \( F_T \)-measurable. There is also a risk-less cash account available from which cash may be borrowed or in which cash may be deposited. Since we have assumed\(^5\) zero interest rates, the time \( \tau \) gain (or loss), \( G_\tau(\theta) \), that results from following a self-financing\(^6\) \( F_\tau \)-adapted trading strategy, \( \theta_t \), can be represented as a stochastic integral with respect to \( X \). In a continuous-time setting, for example, we have

\[
G_\tau(\theta) := \int_0^\tau \theta_s \, dX_s. \tag{3}
\]

We assume that \( Q \) is an equivalent martingale measure (EMM) so that discounted security prices are \( Q \)-martingales. Since we are currently assuming that interest rates are identically zero, however, it is therefore the case that \( X_t \) is a \( Q \)-martingale. Subject to integrability constraints on the set of feasible trading strategies, we also see that \( G_\tau(\theta) \) is a \( Q \)-martingale for every \( F_\tau \)-adapted self-financing trading strategy, \( \theta_t \).

Our analysis will be simplified considerably by making a complete financial markets assumption. In particular, let \( G_\tau \) be any suitably integrable contingent claim that is \( F_\tau \)-measurable. Then a

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\(^4\) All of our analysis goes through if we assume \( X_t \) is a multi-dimensional price process. For ease of exposition we will assume \( X_t \) is one-dimensional.

\(^5\) As mentioned earlier, we will relax this assumption in Section 5.3.

\(^6\) A trading strategy, \( \theta_t \), is self-financing if cash is neither deposited with nor withdrawn from the portfolio during the trading interval, \([0, T]\). In particular, trading gains or losses are due to changes in the values of the traded securities. Note that \( \theta_t \) represents the number of units of the tradeable security held at time \( s \). The self-financing property then implicitly defines the position at time \( s \) in the cash account. Because we have assumed interest rates are identically zero, there is no term in (3) corresponding to gains or losses from the cash account holdings. See Duffie (2004) for a technical definition of the self-financing property.
complete financial markets assumption amounts to assuming the existence of an \( \mathcal{F}_t \)-adapted self-financing trading strategy, \( \theta_t \), such that \( G_\tau(\theta) = G_\tau \). That is, \( G_\tau \) is attainable. This assumption is very common in the financial literature. Moreover, many incomplete financial models can be made complete by simply expanding the set of tradeable securities. When this is not practical, we can simply assume the existence of a market-maker with a known pricing function or pricing kernel\(^7\) who is willing to sell \( G_\tau \) in the market-place. In this sense, we could then claim that \( G_\tau \) is indeed attainable.

Regardless of how we choose to justify it, assuming complete financial markets means that we will never need to solve for an optimal dynamic trading strategy, \( \theta \). Instead, we will only need to solve for an optimal contingent claim, \( G_\tau \), safe in the knowledge that any such claim is attainable. For this reason we will drop the dependence of \( G_\tau \) on \( \theta \) in the remainder of the paper. The only restriction that we will impose on any such trading gain, \( G_\tau \), is that it satisfies \( \mathbb{E}^{\mathbb{Q}}_0[G_\tau] = G_0 \) where \( G_0 \) is the initial amount of capital that is devoted to trading in the financial market. Without any loss of generality we will typically assume \( G_0 = 0 \). This assumption will be further clarified in Section 2.3.

A key aspect of our model is the dependence between the payoffs of the supply chain and returns in the financial market. Other than assuming the existence of \( \mathbb{E}^{\mathbb{Q}}_\tau[A] \), the expected value of \( A \) conditional on the information available in the financial markets at time \( \tau \), we do not need to make any assumptions regarding the nature of this dependence. We will make the following assumption regarding \( \mathbb{E}^{\mathbb{Q}}_\tau[A] \).

**Assumption 1** For all \( \tau \in [0,T] \), \( \bar{A}_\tau := \mathbb{E}^{\mathbb{Q}}_\tau[A] \geq c_\tau \).

This condition ensures that for every time and state there is a total production level, \( Q \geq 0 \), for which the retailers’ expected market price exceeds the producer’s production cost. In particular, this assumption implies that it is possible to profitably operate the supply chain.

### 2.3 The Flexible Procurement Contract with Financial Hedging

The final component of our model is the contractual agreement between the producer and the retailers. We consider a variation of the traditional wholesale price contract in which the terms of the contract are specified contingent upon the public history, \( \mathcal{F}_\tau \), that is available at time \( \tau \). Specifically, at time \( t = 0 \) the producer offers an \( \mathcal{F}_\tau \)-measurable wholesale price, \( w_\tau \), to the retailers. In response to this offer, the \( i^{th} \) retailer decides on an \( \mathcal{F}_\tau \)-measurable ordering quantity, \( q_i = q_i(w_\tau) \), for \( i = 1, \ldots, N \). Note that the contract itself is negotiated at time \( t = 0 \) whereas the actual order quantities are only realized at time \( \tau \geq 0 \).

The retailers’ order quantities at time \( \tau \) are constrained by their available budgets at this time. Besides the initial budget, \( B_i \), the \( i^{th} \) retailer has access to the financial markets where he can hedge his budget constraint by purchasing at date \( t = 0 \) a contingent claim, \( G_\tau^{(i)} \), that is realized

\(^7\)See Duffie (2004). More generally, Duffie may be consulted for further technical assumptions (that we have omitted to specify) regarding the filtration, \( \{\mathcal{F}_t\}_{0 \leq t \leq T} \), feasible trading-strategies, etc.

\(^8\)There is a slight abuse of notation here and throughout the paper when we write \( q_i = q_i(w_\tau) \). This expression should not be interpreted as implying that \( q_i \) is a function of \( w_\tau \). We only require that \( q_i \) be \( \mathcal{F}_\tau \)-measurable and so a more appropriate interpretation is to say that \( q_i = q_i(w_\tau) \) is the retailer’s response to \( w_\tau \).
at date $\tau$ and that satisfies $E_Q^0[G_{\tau}^{(i)}] = 0$. Given an $\mathcal{F}_\tau$-measurable wholesale price, $w_\tau$, the retailer purchases an $\mathcal{F}_\tau$-measurable contingent claim, $G_{\tau}^{(i)}$, and selects an $\mathcal{F}_\tau$-measurable ordering quantity, $q_i = q_i(w_\tau)$, in order to maximize the economic value of his profits. Because of his access to the financial markets, the retailer can therefore mitigate his budget constraint so that it becomes

$$w_\tau q_i \leq B_i + G_{\tau}^{(i)} \quad \text{for all } \omega \in \Omega \text{ and } i = 1, \ldots, N.$$ 

Since the no-trading strategy with $G_{\tau}^{(i)} \equiv 0$ is always an option, it is clear that for a given wholesale price, $w_\tau$, the retailers are always better-off having access to the financial market. Whether or not the retailers will remain better off in equilibrium will be discussed in Section 3.

Before proceeding to analyze this contract a number of further clarifying remarks are in order.

1. The model assumes a common knowledge framework in which all parameters of the model are known to all agents. Because of the Stackelberg nature of the game, this assumption implies that the producer knows the retailers’ budgets and the distribution of the market demand. We also make the implicit assumption that the only information available regarding the random variable, $A$, is what we can learn from the evolution of $X_t$ in the time interval $[0, \tau]$. If this were not the case, then the trading strategy in the financial market could depend on some non-financial information and so it would not be necessary to restrict the trading gains to be $\mathcal{F}_\tau$-measurable. More generally, if $Y_t$ represented some non-financial noise that was observable at time $t$, then the trading strategy, $\theta_t$, would only need to be adapted with respect to the filtration generated by $X$ and $Y$. In this case the complete financial market assumption is of no benefit and it would be necessary for the retailers to solve the much harder problem of finding the optimal $\theta$ in order to find the optimal $G_{\tau}^{(i)}$’s.

2. In this model the producer does not trade in the financial markets because, being risk-neutral and not restricted by a budget constraint, he has no incentive to do so.

3. A potentially valid criticism of this model is that, in practice, a retailer is often a small entity and may not have the ability to trade in the financial markets. There are a number of responses to this. First, we use the word ‘retailer’ in a loose sense so that it might in fact represent a large entity. For example, an airline purchasing aircraft is a ‘retailer’ that certainly does have access to the financial markets. Second, it is becoming ever cheaper and easier for even the smallest ‘player’ to trade in the financial markets. Finally, even if the retailer does not have access to the financial market, then the producer, assuming he is a big ‘player’, can offer to trade with the retailer or act as his financial broker.

4. We claimed earlier that, without loss of generality, we could assume $G_{0}^{(i)} = 0$. This is clear for the following reason. If $G_{0}^{(i)} = 0$ then then the $i^{th}$ retailer has a terminal budget of $B_{\tau}^{(i)} := B_i + G_{\tau}^{(i)}$ with which he can purchase product units at time $\tau$ and where $E_Q^0[G_{\tau}^{(i)}] = 0$. If he allocated $a > 0$ to the trading strategy, however, then he would have a terminal budget of $B_{\tau}^{(i)} = B_i - a + G_{\tau}^{(i)}$ at time $\tau$ but now with $E_Q^0[G_{\tau}^{(i)}] = a$. That the retailer is indifferent between the two approaches follows from the fact any terminal budget, $B_{\tau}^{(i)}$, that is feasible under one modeling approach is also feasible under the other and vice-versa.

These clarifications were also made in Caldentey and Haugh (2009) who study the single-retailer case.
5. Another potentially valid criticism of this framework is that the class of contracts is too complex. In particular, by only insisting that \( w_r \) is \( \mathcal{F}_\tau \)-measurable we are permitting wholesale price contracts that might be too complicated to implement in practice. If this is the case then we can easily simplify the set of feasible contracts. By using appropriate conditioning arguments, for example, it would be straightforward to impose the tighter restriction that \( w_r \) be \( \sigma(X_\tau) \)-measurable instead where \( \sigma(X_\tau) \) is the \( \sigma \)-algebra generated by \( X_\tau \). In section 3.2, for example, we will consider wholesale price contracts that are linear in \( c_\tau \) and \( \bar{A}_\tau \).

We complete this section with a summary of the notation and conventions that will be used throughout the remainder of the paper. The subscripts \( R, P, \) and \( C \) are used to index quantities related to the retailers, producer and central planner, respectively. The subscript \( \tau \) is used to denote the value of a quantity conditional on time \( \tau \) information. For example, \( \Pi_{P|\tau} \) is the producer’s expected payoff conditional on time \( \tau \) information. The expected value, \( E_0[\Pi_{R|\tau}] \), is simply denoted by \( \Pi_r \) and similar expressions hold for the retailers and central planner. Any other notation will be introduced as necessary.

3 The Cournot-Stackelberg Game

As is customary in games of this form, we begin by solving the Cournot game played by the retailers and then use the solution of this game to characterize the producer’s Stackelberg optimal best response. Taking \( Q_{i-} \) and the producer’s price menu, \( w_\tau \), as fixed, the \( i^{th} \) retailer’s problem is formulated\(^\text{10}\) as

\[
\Pi_{R_i}(w_\tau) = \max_{q_i \geq 0, G_\tau} E_0^\sigma \left[ (\bar{A}_\tau - (q_i + Q_{i-}) - w_\tau) q_i \right]
\]  \hspace{1cm} (4)

subject to \( w_\tau q_i \leq B_i + G_\tau, \) for all \( \omega \in \Omega \)
\[
E_0^\sigma \left[ G_\tau \right] = 0.
\]  \hspace{1cm} (5)

Each of the \( N \) retailers must solve this problem and our goal is to characterize the resulting Cournot equilibrium. Without loss of generality, we recall that the retailers have been ordered so that \( B_1 \geq B_2 \geq \ldots \geq B_N \).

A first key step in solving (4)-(6) is to note that we can replace the set of pathwise budget constraints in (5) by a single average budget constraint, namely, \( E_0^\sigma[w_\tau q_i] \leq B_i \). To see this, consider the relaxed retailer’s problem:

\[
\bar{\Pi}_{R_i}(w_\tau) = \max_{q_i \geq 0} E_0^\sigma \left[ (\bar{A}_\tau - (q_i + Q_{i-}) - w_\tau) q_i \right]
\]  \hspace{1cm} (7)

subject to \( E_0^\sigma[w_\tau q_i] \leq B_i, \) for all \( \omega \in \Omega \).
\[
\]  \hspace{1cm} (8)

It should be clear that the feasible region of (7)-(8) contains the feasible region of (4)-(6) and so \( \bar{\Pi}_{R_i}(w_\tau) \geq \Pi_{R_i}(w_\tau) \). On the other hand, for any feasible solution \( q_i \) of (7)-(8), we can set a trading strategy such that \( G_\tau = w_\tau q_i - E_0^\sigma[w_\tau q_i] \). But the pair \( (q_i, G_\tau) \) is feasible for (4)-(6) and generates the same expected payoff. It follows that \( \bar{\Pi}_{R_i}(w_\tau) = \Pi_{R_i}(w_\tau) \) and we can safely focus on solving

\(^{10}\)To be precise, the \( i^{th} \) retailer’s objective function is \( E_0^\sigma[(A - (q_i + Q_{i-}) - w_\tau) q_i] \). Since \( w_\tau \) and the \( q_i \)’s must be \( \mathcal{F}_\tau \)-measurable, however, we can write this objective as \( E_0^\sigma[E_0^\sigma[(A - (q_i + Q_{i-}) - w_\tau) q_i]] \) and obtain (4).
the simpler optimization problem (7)-(8) to determine the Cournot equilibrium in the retailer’s market.

Taking $Q_i$ and the producer’s price menu, $w_r$, as fixed, it is straightforward to obtain

$$ q_i = \frac{(\bar{A}_r - w_r (1 + \lambda_i) - Q_i)^+}{2} \tag{9} $$

where $\lambda_i \geq 0$ is the deterministic Lagrange multiplier corresponding to the $i^{th}$ retailer’s budget constraint in (8). In particular, $\lambda_i \geq 0$ is the smallest real such that $\mathbb{E}_0^Q[w, q_i] \leq B_i$. Given the ordering of the budgets, $B_i$, it follows that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$ when they are chosen optimally. Equation (9) and the ordering of the Lagrange multipliers then implies that for each outcome $\omega \in \Omega$, there is a function $n_r(\omega) \in \{0, 1, \ldots, N\}$ such that $q_j(\omega) = 0$ for all $j > n_r$. In other words, $n_r(\omega)$ is the number of active retailers in state $\omega$.

Continuing to drop the dependence of random variables on $\omega$ (e.g., writing $n_r$ for $n_r(\omega)$), we therefore obtain the following system of equations

$$ q_i = \bar{A}_r - w_r (1 + \lambda_i) - Q, \quad \text{for } i = 1, \ldots, n_r \tag{10} $$

where $Q = \sum_{i=1}^{n_r} q_i$. For each $\omega \in \Omega$, this is a system with $n_r$ linear equations in $n_r$ unknowns which we can easily solve. Summing the $q_i$’s we obtain

$$ Q = \frac{1}{n_r + 1} \left[ n_r \bar{A}_r - w_r \sum_{i=1}^{n_r} (1 + \lambda_i) \right]. \tag{11} $$

Substituting this value of $Q$ in (10), and using the fact that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$, we see the optimal ordering quantities, $q_i$ for $i = 1, \ldots, N$, satisfy

$$ q_i = \frac{\left[ \bar{A}_r - w_r \left( (n_r + 1) (1 + \lambda_i) - \sum_{j=1}^{n_r} (1 + \lambda_j) \right) \right]^+}{(n_r + 1)}, \quad i = 1, 2, \ldots, N. \tag{12} $$

To complete the characterization of the Cournot equilibrium in the retailers’ market, we must compute the values of the Lagrange multipliers $\{\lambda_i, i = 1, \ldots, N\}$ as well as the random variable $n_r$. For reasons that will soon become apparent, it will be convenient to replace the Lagrange multipliers by an equivalent set of unknowns $\{\alpha_i, i = 1, \ldots, N\}$ that we define below.

Suppose $q_i(\omega) = 0$ in some outcome, $\omega$. Then (9) implies $\bar{A}_r - w_r (1 + \lambda_i) - Q \leq 0$ which, after substituting for $Q$ using (11), implies that

$$ (1 + \lambda_i) (1 + n_r) \geq \alpha_r + \sum_{j=1}^{n_r} (1 + \lambda_j), \tag{13} $$

where $\alpha_r := \bar{A}_r/w_r$. Since $\bar{A}_r$ is the expected maximum clearing price (corresponding to $Q = 0$) and $w_r$ is the procurement cost, we may interpret $\alpha_r - 1$ as the expected maximum per unit margin of the retail market. It follows that in equilibrium the producer chooses $w_r$ so that $\alpha_r \geq 1$. We also note that equation (13) implies that $n_r$ depends on $\omega$ only through the value of $\alpha_r$, that is, $n_r = n_r(\alpha_r)$. 
Let $\alpha_i$ denote that value of $\alpha_r$ where the $i^{th}$ retailer moves from ordering zero to ordering a positive quantity. Abusing notation slightly, we see\footnote{We are assuming that the $N$ budgets are distinct so that $B_k > B_{k-1}$. This then implies $q_i(\alpha_k) > 0$ for all $i \leq k - 1$. The case where some budgets coincide is straightforward to handle.} that $n(\alpha_i) = i - 1$ and so (13) implies
\begin{equation}
\alpha_i = i(1 + \lambda_i) - \sum_{j=1}^{i-1} (1 + \lambda_j) \quad \text{for} \quad i = 1, \ldots, N. \tag{14}
\end{equation}

Using (14) recursively, one can show that
\begin{equation}
1 + \lambda_i = \frac{\alpha_i}{i} + \sum_{j=1}^{i-1} \frac{\alpha_j}{j(j+1)}. \tag{15}
\end{equation}

Substituting this expression in (12), it follows that for all $i = 1, \ldots, N$
\begin{equation}
q_i = w_r \left[ \frac{\alpha_r}{1 + n_r} - (1 + \lambda_i) + \sum_{j=1}^{n_r} \frac{1 + \lambda_j}{1 + n_r} \right]^{+} = w_r \left[ \frac{\alpha_r}{1 + n_r} - \frac{\alpha_i}{i + 1} + \sum_{j=1}^{n_r} \frac{1 + \lambda_j}{1 + n_r} - \sum_{j=1}^{i-1} \frac{\alpha_j}{j(j+1)} \right]^{+} = w_r \left[ \frac{\alpha_r}{n_r + 1} - \frac{\alpha_i}{i + 1} + \sum_{j=i+1}^{n_r} \frac{\alpha_j}{j(j+1)} \right]^{+}
\end{equation}

where the last equality follows from the identity:
\begin{equation}
\sum_{j=1}^{n_r} \frac{1 + \lambda_j}{1 + n_r} = \frac{1}{1 + n_r} \sum_{j=1}^{n_r} \left( \frac{\alpha_j}{j} + \sum_{k=1}^{j-1} \frac{\alpha_k}{k(k+1)} \right) = \sum_{j=1}^{n_r} \frac{\alpha_j}{j(j+1)},
\end{equation}

which in turns follows from (15).

It should be clear from the discussion above that
\begin{equation}
n_r = \max \{ i \in \{0, 1, \ldots, N \} \text{ such that } \alpha_i \leq \alpha_r \}
\end{equation}

and we therefore only need to derive the values of the $\alpha_i$’s. We have relegated this derivation to Appendix A and we summarize the main results in Proposition 1 below whose statement makes use of the following definition:

**Definition 3.1** Let $w_r$ be an $F_r$-measurable wholesale price contract. For any $B \geq 0$, we define
\begin{equation}
H(B) := \inf \{ x \geq 1 \text{ such that } \mathbb{E}[w_r^2 (\alpha_r - x)^+] \leq B \}.
\end{equation}

Note that $H(B)$ is a non-increasing function in $B > 0$.

**Proposition 1** (Cournot Equilibrium in the Retailers’ Market)

For a given $F_r$-measurable wholesale price menu, $w_r$, the optimal ordering quantities, $q_i$, satisfy
\begin{equation}
q_i = w_r \left[ \frac{\alpha_r}{n_r + 1} - \frac{\alpha_i}{i + 1} + \sum_{j=i+1}^{n_r} \frac{\alpha_j}{j(j+1)} \right]^{+} \quad \text{for all} \quad i = 1, 2, \ldots, N
\end{equation}

with the budget constraint $\mathbb{E}[w_r q_i] \leq B_i$ binding if $\alpha_i > 1$ where
\begin{align}
\alpha_i &:= H((i + 1)B_i + B_{i+1} + \cdots + B_N) \quad \text{for all} \quad i = 1, 2, \ldots, N, \tag{16}
n_r &:= \max \{ i \in \{0, 1, \ldots, N \} \text{ such that } \alpha_i \leq \alpha_r \}. \tag{17}
\end{align}
The ordering $B_1 \geq B_2 \geq \cdots \geq B_N$ implies that $q_i > 0$ if and only if $i \leq n_r$. The parameter $\alpha_i$ is therefore the cutoff\footnote{This construction has similarities to the equilibrium constructions found in Golany and Rothblum (2008) and Ledvina and Sircar (2012) who also obtain cutoff points below which firms produce and above which firms are costed out.} point such that the $i^{th}$ retailer orders a positive quantity only if $\alpha_r \geq \alpha_i$. It is interesting to note that equation (16) implies that $\alpha_i$ does not depend on the $i - 1$ highest budgets, $B_j$, for $j = 1, \ldots, i - 1$. In fact $\alpha_i$ only depends on $B_i$, the sum of the $N - i$ smallest budgets and the number of retailers, $i - 1$, that have a budget larger than $B_i$. As a result, $q_i$ only depends on $B_i$, $(B_{i+1} + \cdots + B_N)$ and $i$. In other words, the procurement decisions of small retailers are unaffected by the size (but not the number) of larger retailers for a given wholesale price $w_r$. In equilibrium, however, we expect the wholesale price $w_r$ to depend on the entire vector of budgets. Proposition 1 also implies that

$$q_i - q_{i+1} = w_r \left( \frac{\alpha_r - \alpha_i}{i+1} \right), \quad i = 1, 2, \ldots, N$$

and this confirms our intuition that larger retailers order more than smaller ones so that $q_i$ is non-increasing in $i$. This follows from the fact that $H(B)$ is non-increasing in $B$ which implies that the $\alpha_i$'s are non-decreasing in $i$. Having characterized the Cournot equilibrium of the $N$ retailers, we can now determine the producer’s expected profits, $\Pi_\tau = \mathbb{E}_0^\tau[ (w_r - c_r) Q(w_r) ]$, for a fixed price menu, $w_r$. We have the following proposition.

**Proposition 2** (Producer’s Expected Profits)

An $\mathcal{F}_\tau$-measurable wholesale price menu $w_r$ is a Stackelberg equilibrium in the producer’s market if it maximizes the producer’s expected payoff, $\Pi_\tau$, given by

$$\Pi_\tau = \left( \frac{m - 1}{m} \right) \mathbb{E}_0^\tau[ (w_r - c_r) (\bar{A}_\tau - w_r)^+ ] + \sum_{j=m}^N \left( \frac{B_j}{j+1} \right) \mathbb{E}_0^\tau[ (\bar{A}_\tau - \alpha_j w_r)^+]$$

(18)

where

$$m = m(\{w_r\}) := \max \{ i \geq 1 \text{ such that } \alpha_{i-1} = 1 \}$$

(19)

and $\alpha_{i} = H((i+1)B_i + B_{i+1} + \cdots + B_N)$ (as defined in Proposition 1) with $\alpha_{0} := 1$.

**Proof:** See Appendix A.

The producer’s problem is then to maximize $\Pi_\tau$ in (18) over price menus $w_r$. A first important observation regarding this problem is that it cannot be solved path-wise since the $\alpha_i$'s are deterministic and depend implicitly in a non-trivial way on $w_r$ through the function $H(\cdot)$. Note also that $m$ is the index of the first retailer whose budget constraint is binding\footnote{We say a player is binding if his budget constraint is binding in the Cournot equilibrium. Otherwise a player is non-binding.} with the understanding that if $m = N + 1$ then all $N$ retailers are non-binding. We can characterize those values of $m \in \{1, \ldots, N + 1\}$ that are possible. In particular, if the producer sets $w_r = \bar{A}_\tau$ then all of the retailers are non-binding and so $m = N + 1$. We can also find the smallest possible value of $m$, $m_{\min}$, say, by setting $w_r = c_r$, solving for the resulting $\alpha_i$'s using (16) and then taking $m_{\min}$
according to (19). Assuming the $B_i$’s are distinct, the achievable values of $m$ are given by the set $M_{\text{feas}} := \{m_{\min}, \ldots, N + 1\}$. This can be seen by taking $w_\tau = \gamma c_\tau + (1 - \gamma) \bar{A}_\tau$ with $\gamma = 0$ initially and then increasing it to 1. In the process each of the values in $M_{\text{feas}}$ will be obtained.

We could use this observation to solve numerically for the producer’s optimal menu, $w^*_\tau$, by solving a series of sub-problems. In particular we could solve for the optimal price menu subject to the constraint that $m = m^*$ for each possible value of $m^* \in M_{\text{feas}}$. Each of these $N - m^* + 2$ sub-problems could be solved numerically after discretizing the probability space. The overall optimal price menu, $w^*_\tau$, is then simply the optimal price menu in the sub-problem whose objective function is maximal. In an effort to be more concrete in identifying the structure of an optimal wholesale price menu, we consider two special cases in the next subsections. In each special case the producer’s optimization problem can be solved explicitly and we use these results to motivate a class of linear price contracts that should yield a good solution for the producer’s problem in general.

3.1 Special Cases

I) Non-Binding Budget Constraints:

We first consider the case in which the retailers’ budgets $(B_1, \ldots, B_N)$ are sufficiently large so that their budget constraints are never binding in expectation in equilibrium. Under this condition, it follows that $\alpha_i = 1$ for all $i = 1, \ldots, N$ and $m = N + 1$ in (19). As a result the producer’s expected payoff in (18) becomes

$$\Pi_P = \frac{N}{N + 1} \mathbb{E}_0[(w_\tau - c_\tau)(\bar{A}_\tau - w_\tau)^+]$$.

Suppose initially that the producer optimizes this payoff by maximizing the argument inside the expectation pathwise. Under the condition $\bar{A}_\tau \geq c_\tau$ of Assumption 1, the resulting optimal wholesale price menu is equal to

$$w^*_\tau = \frac{\bar{A}_\tau + c_\tau}{2}$$ \hfill (20)

To check the non-binding budget condition, we use the result in Proposition 1 for $w^*_\tau$ and $\alpha_i = 1$ (for all $i$) to obtain each retailer’s ordering quantity and corresponding expected budget requirement. After some straightforward manipulations, we get

$$q^*_i = \frac{\bar{A}_\tau - c_\tau}{2(N + 1)}$$ and $$\mathbb{E}_0[w^*_\tau q^*_i] = \frac{\mathbb{E}_0[(\bar{A}_\tau^2 - c_\tau^2)]}{4(N + 1)}$$.

It follows that if each $B_i$ is greater than $\mathbb{E}_0[w^*_\tau q^*_i]$ then all budget constraints will be non-binding in equilibrium. We summarize the previous steps in the following result.

**Proposition 3 (Non-Binding Budget Constraints)**

Let $\bar{B}_N := \mathbb{E}_0[(\bar{A}_\tau^2 - c_\tau^2)]/(4(N + 1))$ and suppose $B_i \geq \bar{B}_N$ for all $i = 1, \ldots, N$. Then, the Cournot-Stackelberg equilibrium of the game is given by

$$w^*_\tau = \frac{\bar{A}_\tau + c_\tau}{2}$$ and $$q^*_i = \frac{\bar{A}_\tau - c_\tau}{2(N + 1)}$$, \hspace{1em} i = 1, \ldots, N.

Furthermore, the agents payoffs are given by

$$\Pi_P = \frac{N}{4(N + 1)} \mathbb{E}_0[(\bar{A}_\tau - c_\tau)^2]$$ and $$\Pi_{R_i} = \frac{1}{4(N + 1)^2} \mathbb{E}_0[(\bar{A}_\tau - c_\tau)^2]$$.
A few comments regarding this result are in order. First, it is worth highlighting the fact that the minimum budget \( \tilde{B}_N \) only guarantees that the retailers’ budget constraints are not binding in expectation. In other words, is certainly possible that \( w^*_i(\omega)q^*_i(\omega) > \tilde{B}_N \) for a set \( E \) of states \( \omega \in \Omega \) with \( Q(E) > 0 \). However, because of the retailers ability to trade in the financial market, the budget constraint is met in expectation, i.e. \( \mathbb{E}^0[w^*_i q^*_i] \leq \tilde{B}_N \). It is interesting to note that an optimal wholesale price menu simply averages the per-unit manufacturing cost \( c_r \) and maximum conditional expected retail price \( \bar{A}_r \). This simplicity is certainly appealing from a practical/implemention standpoint. Also, and as one would expect, the producer is better of as the number of retailers \( N \) increases. Actually, in the limit as \( N \) goes to infinity the producer achieves the same expected payoff as a central planner who controls both production and retail sales. Since the threshold budget \( \tilde{B}_N \) decreases with \( N \), it follows that the manufacturer has incentives to promote competition in the retailer’s market even if this competition is driven by encouraging small retailers to enter the market. Note that by increasing competition we do not mean increasing the number of retailers so that their cumulative budget increases, as this would trivially benefit the producer. Actually, from the value of \( \tilde{B}_N \) in Proposition 3, we see that \( \lim_{N \to \infty} N \tilde{B}_N = \mathbb{E}_0^0[\bar{A}_r^2 - c_r^2]/4 =: \bar{B} \), and the producer can asymptotically (in \( N \)) achieve the central planner payoff as as long as (i) the cumulative budget of all retailers exceeds \( \bar{B} \) and (ii) is evenly distributed among them.

II) Symmetric Retailers:

Let us consider now our second special case where all retailers have identical budgets, i.e., there exists a \( B > 0 \) such that \( B_i = B \) for all \( i = 1, \ldots, N \). While not a realistic assumption in practice, restricting the model to the case of symmetric agents is a standard “artifice” in game theory as it typically provides more analytical tractability. In our case, we will be able to solve for the producer’s optimal price menu in Proposition 2 and therefore solve for the overall Cournot-Stackelberg equilibrium. In terms of solution techniques, some of the single-retailer results of Caldentey and Haugh (2009) will prove useful in this multi-retailer symmetric case.

Consider then the case where each of the retailers has the same budget so that \( B_i = B \) for all \( i = 1, \ldots, N \). For a given price menu, \( w_r \), the \( i \)th retailer’s problem is

\[
\Pi_R(w_r) = \max_{q_i \geq 0} \mathbb{E}_0^0 \left[ (\bar{A}_r - (q_i + Q_i) - w_r) q_i \right] \tag{21}
\]

subject to \( \mathbb{E}_0^0[w_r q_i] \leq B \). \hspace{1cm} \tag{22}

While the retailer’s problem in this symmetric-budget setting is a special case of the results in Proposition 1, it is instructive to see an alternative solution to the retailers problem:

**Proposition 4** (Optimal Strategy for the 4 N Retailers in the Symmetric Case)

Let \( w_r \) be an \( \mathcal{F}_r \)-measurable wholesale price offered by the producer and let \( Q_r, \mathcal{X} \) and \( \mathcal{X}^c \) be defined as follows. \( Q_r := \frac{(\bar{A}_r - w_r)^+}{(N+1)} \), \( \mathcal{X} := \{ \omega \in \Omega : B \geq Q_r w_r \} \) and \( \mathcal{X}^c := \Omega - \mathcal{X} \). The following two cases arise in the computation of the optimal ordering quantities and the financial claims:

\[14\] Indeed, note that a central planner (with no budget constraint) would determine the optimal production level by maximizing (pathwise) the expected payoff

\[ \Pi_C = \max_{Q} \mathbb{E}_0^0[(\bar{A}_r - Q - c_r)Q]. \]

Under Assumption 1, it follows that the optimal centralized production quantity and expected payoff are given by \( Q^*_C = (\bar{A}_r - c_r)/2 \) and \( \Pi_C = \mathbb{E}_0^0[(\bar{A}_r - c_r)^2]/4. \)
Case 1: Suppose that $E^0_\tau [Q, w_\tau] \leq B$. Then $q_i(w_\tau) = Q_\tau$ and there are infinitely many choices of the optimal claim, $G_\tau = G^{(i)}_\tau$, for $i = 1, \ldots, N$. One natural choice is to take

$$G_\tau = [Q_\tau, w_\tau - B] : \begin{cases} \delta & \text{if } \omega \in X \\ 1 & \text{if } \omega \in X^c \end{cases}$$

where $\delta := \int_{X^c} [Q_\tau, w_\tau - B] dQ / \int_X [B - Q_\tau, w_\tau] dQ$.

In this case (possibly due to the ability to trade in the financial market), the budget constraint is not binding for any of the $N$ retailers.

Case 2: Suppose $E^0_\tau [Q, w_\tau] > B$. Then

$$q_i(w_\tau) = q(w_\tau) = \frac{(\bar{A}_\tau - w_\tau (1 + \lambda))^+}{(N + 1)}$$

and $G_\tau := q(w_\tau)w_\tau - B$ (23)

is optimal for each $i$ where $\lambda \geq 0$ solves $E^0_\tau [q(w_\tau)w_\tau] = B$.

**Proof:** See Appendix A.

The manufacturer’s problem is straightforward to solve. Given the best response of the $N$ retailers, his problem may be formulated as

$$\Pi_p = \max_{w_\tau, \lambda \geq 0} N E^0_\tau \left( (w_\tau - c_\tau) \frac{(\bar{A}_\tau - w_\tau (1 + \lambda))^+}{(N + 1)} \right)$$

subject to $E^0_\tau \left[ w_\tau \frac{(\bar{A}_\tau - w_\tau (1 + \lambda))^+}{(N + 1)} \right] \leq B$. (24)

Note that the factor $N$ outside the expectation in (24) is due to the fact that there are $N$ retailers and that the producer earns the same profit from each of them. Note also that there should be $N$ constraints in this problem, one corresponding to each of the $N$ retailers. However, these $N$ constraints are identical since each retailer solves the same problem. The producer’s problem then only requires the one constraint given in (25). We can easily re-write this problem as

$$\Pi_p = \max_{w_\tau, \lambda \geq 0} \frac{2N}{N + 1} E^0_\tau \left( (w_\tau - c_\tau) \frac{(\bar{A}_\tau - w_\tau (1 + \lambda))^+}{2} \right)$$

subject to $E^0_\tau \left[ w_\tau \frac{(\bar{A}_\tau - w_\tau (1 + \lambda))^+}{2} \right] \leq \frac{(N + 1)}{2} B$ (27)

and now it is clearly identical\(^\text{15}\) to the producer’s problem where the budget constraint has been replaced by $(N + 1)B/2$ and there is just one retailer. We have the following result.

**Proposition 5** (Producer’s Optimal Strategy and the Cournot-Stackelberg Solution)

Let $\phi_p$ be the minimum $\phi \geq 1$ that solves $E^0_\tau \left[ \frac{(\bar{A}_\tau - (\phi c_\tau))^2}{8} \right] \leq \frac{(N + 1)}{2} B$. Then the optimal wholesale price and ordering level for each retailer satisfy

$$w_\tau^* = \frac{\bar{A}_\tau + \phi_p c_\tau}{2} \quad \text{and} \quad q_\tau^* = \frac{\bar{A}_\tau - \phi_p c_\tau}{2(N + 1)}.$$ 

\(^\text{15}\)The factor $2N/(N + 1)$ in the objective function has no bearing on the optimal $\lambda$ and $w_\tau$. 

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The players’ expected payoffs conditional on time $\tau$ information satisfy

$$\Pi_{p|\tau} = \frac{N}{4(N+1)} (\bar{A}_\tau - (2 - \phi_P) c_\tau) (\bar{A}_\tau - \phi_P c_\tau)^+ \quad \text{and} \quad \Pi_{r|\tau} = \frac{(\bar{A}_\tau - \phi_P c_\tau)^+)^2}{4(N+1)^2}. \quad (29)$$

Let us discuss some of the properties of the Cournot-Stackelberg equilibrium in this symmetric case. First of all, it is worth noticing that the structure of the solution is very similar to the one in Proposition 3 for the non-binding budget case. Specifically, the optimal price menu is a linear combination of the manufacturing cost $c_\tau$ and the maximum retail price $\bar{A}_\tau$. Furthermore, if we think of $\delta_\tau := \phi_P c_\tau$ as some type of (budget modified) per-unit manufacturing cost then the optimal wholesale price menu $w^*_\tau$ is again the average between $\bar{A}_\tau$ and this modified manufacturing cost $\delta_\tau$. Similarly, the retailers’ optimal ordering quantity $q^*_\tau$ is proportional to the difference between $\bar{A}_\tau$ and $\delta_\tau$ and decays in proportion to $N + 1$.

Another interesting feature of this solution is the fact that the supply chain could completely shut down in some states $\omega \in \Omega$. Indeed, if $\bar{A}_\tau \leq \delta_\tau$ then $q^*_\tau = 0$ and both retailers and producers make zero profit in those states. Intuitively, by shutting down the market in some low-demand states, the producer can reallocate the retailer’s budget to high-demand states in a profitable way. This budget reallocation is only possible because of the retailer’s ability to trade on the financial market. Furthermore, it is not hard to see that if the retailers had no access to the financial markets then the producer would never close the market. So although the producer obviously benefits from the retailers’ access to the financial markets it is not clear that this benefit also extends to the retailers in equilibrium. We will return to this point in Section 4 where we discuss in detail the economic value of financial markets from the perspective of each agent and the supply chain as a whole.

### 3.2 Constant and Linear Wholesale Price Contracts

As stated earlier, solving for the wholesale price contract that optimizes the producer’s expected profits in Proposition 2 for an arbitrary vector of budgets $(B_1, \ldots, B_N)$ is a complex optimization problem. In the previous subsection, however, we identified two notable exceptions for which the optimal contract is explicitly available. A common feature of these two cases is the fact that in equilibrium the producer chooses a wholesale price menu $w_\tau$ that is a linear combination of $\bar{A}_\tau$ and $c_\tau$. Motivated by this observation, we restrict ourselves in this section to the family of price contracts $w_\tau$ that are linear in $\bar{A}_\tau$ and $c_\tau$. Through a numerical example we demonstrate that it is straightforward to optimize numerically for the optimal contract within this family for an arbitrary vector of budgets. We also note that since $c_\tau$ is deterministic this family includes the important case of a constant wholesale price contract. Our example will also consider the benefit (to the producer) of offering an optimal linear contract versus an optimal constant wholesale price contract. We first discuss the constant wholesale price contract as a special case as some additional explicit results are available in this case.

**A Constant Wholesale Price**

The problem of numerically optimizing the expected payoff in (18) is considerably more tractable if the producer offers a constant wholesale price $\bar{w}$ instead of a random menu $w_\tau$. From a practical standpoint, this is an important special case since a constant wholesale price is also a much simpler contract to implement. In this case, Proposition 2 can be specialized as follows.
Corollary 1 Under a constant wholesale price, \( \bar{w} \), the producer’s expected payoff is given by

\[
\Pi_P = \frac{\bar{w} - c}{m} \left( (m - 1) \mathbb{E}_0^\bar{w}[(\bar{A}_r - \bar{w})^+] + \sum_{j=m}^N \frac{B_j}{\bar{w}} \right).
\]

Let \( \bar{w}^*(B_1, \ldots, B_N) \) be the constant wholesale price that maximizes \( \Pi_P \) as a function of the budgets \( B_1 \geq B_2 \geq \cdots \geq B_N \). Then it follows that \( \bar{w}^*(B_1, \ldots, B_N) \geq \bar{w}^*(\infty, \ldots, \infty) \) for all such budgets.

**Proof:** See Appendix A.

While \( m \) is a function of \( \bar{w} \) it is nonetheless straightforward to check that \( \Pi_P \) in (30) is a continuous function of \( \bar{w} \). Note also that if some budget is transferred from one non-binding player to another non-binding player and both players remain non-binding after the transfer then \( \Pi_P \) is unchanged. Similarly if some budget is transferred from one binding player to another binding player and both players remain binding after the transfer then \( \Pi_P \) is again unchanged. Both of these statements follow from (30) and because it is easy to confirm that in each case the value of \( m \) is unchanged.

If some budget is transferred from a binding player to a non-binding player, however, then the ordering of the \( B_i \)'s and the definition of the \( \alpha_i \)'s imply that both players remain binding and non-binding, respectively, after the transfer. Therefore \( m \) remains unchanged and \( \Pi_P \) decreases according to (30).

Conversely, we can increase \( \Pi_P \) by transferring budget from a non-binding player to a binding player in such a way that both players remain non-binding and binding, respectively, after the transfer. It is also possible to increase \( \Pi_P \) if budget is transferred from one non-binding player to another non-binding player so that the first player becomes binding after the transfer.

Note that the statements above are consistent with the idea that the producer would like to see the budgets spread evenly among the various retailers. See Proposition 8 below for a similar result.

The final part of Corollary 1 asserts that it is in the producer’s best interest to increase the wholesale price when selling to budget-constraint retailers. By doing so the producer is inducing the retailers to reallocate their limited budgets into those states in which demand is high and for which the retailers have the incentives to invest more of their budgets in procuring units from the producer. As a result, the producer is able to extract a larger fraction of the retailers initial budgets.

**Optimizing Numerically Over the Family of Linear Wholesale Price Contracts**

We return now to linear price contracts of the form \( w_\tau = w_c c_\tau + w_A \bar{A}_\tau \) for constants \( w_c \) and \( w_A \). The constant wholesale price contract is of course a special case corresponding to \( w_A \equiv 0 \) since \( c_\tau \) is assumed constant. Motivated by the results of the previous section, it seems reasonable to conjecture that this class should contain contracts that are optimal or close to optimal for the producer. We now demonstrate through a numerical example that it is straightforward for the producer to optimize over \( (w_c, w_A) \) and therefore compute the Cournot-Stackelberg equilibrium solution for this class of contracts.
We consider a simple additive model of the form \( A = F(X_T) + \varepsilon \) with \( \mathbb{E}^Q[\varepsilon] = 0 \). The random perturbation \( \varepsilon \) captures the non-financial component of the market price uncertainty and is assumed to be independent of \( X_t \). Note that if \( F(x) = \bar{A} \), we recover a model for which demand is independent of the financial market. We assume \( X_t \), follows a geometric Brownian motion with dynamics

\[
dX_t = \sigma X_t dW_t
\]

where \( W_t \) is a \( Q \)-Brownian motion. Note that since interest rates are zero, \( X_t \) must be a \( Q \)-martingale and therefore does not have a drift. To model the dependence between the market clearance price and the process, \( X_t \), we assume a linear model for \( F(\cdot) \) so that \( F(X) = A_0 + A_1 X \) where \( A_0 \) and \( A_1 \) are positive constants. It follows that \( \bar{A}_t = A_0 + A_1 X_t \) and \( \bar{A} = \mathbb{E}^Q_0[A] = A_0 + A_1 X_0 \).

We consider a problem with \( N = 5 \) retailers with budgets given by \( B = [1400, 1000, 500, 300, 100] \). Other problem parameters are \( X_0 = \sigma = 1 \), \( \tau = .5 \), \( A_1 = 100 \), \( A_0 = 10 \), \( c\tau = 7 \) and \( B = [1400, 1000, 500, 300, 100] \). In Figure 1 below we have plotted the producer’s expected profits \( \Pi_P \) as a function of \( w_c \) and \( w_A \) on the grid \([0, 5] \times [0, 1.2]\). Note that \( \Pi_P \) was computed numerically using the results of Proposition 2. We first observe that \( \Pi_P \) is very sensitive to \( w_A \) while it is less sensitive to \( w_c \). This of course is due in part to our selected parameter values. Moreover, while not shown in the figure, it is indeed the case that \( \Pi_P \) decreases to zero for sufficiently large values of \( w_c \). The behavior of \( \Pi_P \) as a function of \( w_A \) (with \( w_c \) fixed) is perhaps more interesting: we see that \( \Pi_P \) initially increases in \( w_A \) but eventually decreases to zero at which point the producer earns zero profits and the supply chain has been shut down. We also see that the producer incurs losses for values of \( (w_c, w_A) \) close to \( (0, 0) \) which of course is not surprising since the producer is undercharging at these values.

For this example, we found an optimal equilibrium value of \( \Pi_P = 3,016 \) with corresponding optimal parameters \( w_c^* = .975 \) and \( w_A^* = .48 \). At this equilibrium solution we found that the budget constraints were binding for all but the largest retailer. It is also interesting to note that the optimal value of \( w_A \) was very close to .5 which we know to be the optimal coefficient in the special cases of the previous...
section. This lends further support to our consideration of linear contracts for the general problem with non-symmetric and possibly binding budgets. Incidentally, we also solved the symmetric problem with \( N = 5 \) retailers where each \( B_i = 660 \). In this case the total retailer budget in both problems are identical and equal to 3,300. In this symmetric case we found optimal producer profits of \( \Pi_P = 3,131 \) corresponding to \( w^*_c = 4.15 \) and \( w^*_A = 487 \). In this case the (identical) budget constraints for all five retailers were binding. We were also not surprised to see the optimal value of \( \Pi_P \) increase in the symmetric case since this intuitively corresponds to a situation of increased competition among the retailers.

It is also interesting to compare the optimal producer profits, \( \Pi^*_P \), assuming linear wholesale contracts are available, with his optimal profits, \( \Pi^C_P \), when only constant wholesale price contracts are available. We perform this comparison as a function of \( \sigma \), the volatility parameter driving the dynamics of the financial market in Figure 2 below.

![Figure 2: Expected equilibrium profits as a function of \( \sigma \) for the producer and retailers: optimal linear wholesale price contract with \( w_r = w_{c}\tau + w_A \bar{A}_\tau \) versus the optimal constant contract (with \( w_A \equiv 0 \)). The other parameters are \( N = 5 \) retailers, \( X_0 = 1 \), \( \tau = .5 \), \( A_1 = 100 \), \( A_0 = 10 \), \( c_\tau = 7 \) and \( B = [1400 \ 1000 \ 500 \ 300 \ 100] \).](image)

Our first observation is that \( \Pi^*_P = \Pi^C_P \) when \( \sigma = 0 \). This is as expected, however, since if \( \sigma = 0 \) then there is no informational content in the market regarding the value of \( A \) and the ability to offer price menus that depend on \( \bar{A}_\tau \) therefore has no value. It is also interesting to note that \( \Pi^*_P \) also equals \( \Pi^C_P \) for large values of \( \sigma \). This may at first appear surprising but we must first recall that even when the producer offers a constant wholesale price, the retailers’ ordering quantities can and will depend on \( \bar{A}_\tau = A_0 + A_1 X_0 e^{-\sigma^2 \tau/2 + \sigma W_\tau} \). Since \( \text{Var} (A_\tau) = (A_1 X_0)^2 \left( e^{\sigma^2 \tau} - 1 \right) \) increases in \( \sigma \) it is clear that larger values of \( \sigma \) imply that larger values of \( A \) become increasingly probable. The producer can take advantage of this by charging a very large constant wholesale price and forcing the retailers to purchase at that large price only in states where \( \bar{A}_\tau \) is sufficiently large. For large values of \( \sigma \) the producer therefore becomes indifferent between offering a simple constant wholesale price menu and offering a linear pricing menu. This can be seen clearly in Figure 3 below where we again plot the producer’s expected profits as a function of \( w_c \) and \( w_A \) but now on the grid \([300, 600] \times [0, 1] \). The parameters are identical to those of Figure 1 with the exception that we now take \( \sigma = 2 \), corresponding to the largest
value of $\sigma$ in Figure 2. We see there is a large range of $(w_c, w_A)$ values which optimize the producer’s profits in Figure 3. Most importantly, and consistent with our observations above, this range contains a strip of values for which $w_A = 0$ which of course corresponds to constant wholesale price contracts.

A similar line of reasoning also explains why $\Pi^c_0$ and $\Pi^c_1$ are both increasing in $\sigma$ in Figure 2. This is despite the fact that $\bar{A} = E_0^c[A] = E_0^c[\bar{A}_\tau] = A_0 + A_1 X_0$ does not depend on $\sigma$. We note that for intermediate values of $\sigma$ the linear price contract outperforms (from the producer’s perspective) the constant wholesale price contract by up to approximately 50%, demonstrating the potential value of such contracts. We note in passing that the volatility $\sigma$ of financial indices might typically range from 20% to 50% depending on market conditions and the indices in question. We also note that the linear contract performs very close to optimal for intermediate and large values of $\sigma$ because in this range his expected profits are close to 3,300 which is the total budget of the retailers. (For very small values of $\sigma$ the constant – and therefore also the linear – contract is very close to optimal as there is nothing to be gained by contracting on the financial market in that case.)

In Figure 2 we have also plotted the total equilibrium profits of the retailers as a function of $\sigma$. The first main observation is that the retailers always do better with the constant contract than with the linear contract. This is because with a constant contract, the producer cannot force the retailers to only order in certain highly profitable states (for the producer) and shut the supply chain down in other less profitable states. We also note a kink in both of the retailers’ curves after which their expected equilibrium profits start to decrease with $\sigma$. Some intuition for this (in the symmetric retailer case) is provided by the expression in (29) for the retailers’ expected payoffs conditional on time $\tau$ information. Since $\text{Var}(\bar{A}_\tau)$ is increasing in $\sigma$ we can argue (because of the positive part) that the expectation of $\bar{A}_\tau$ component in (29) will increase in $\sigma$. In contrast, the value of $\phi_P$ decreases in $\sigma$. The expectation of the retailers’ equilibrium conditional profits in (29) therefore consists of two competing terms. The first term dominates initially which is why we see the retailers’ curves in Figure 2 increasing for smaller values of $\sigma$. Beyond a certain point, however, the second term starts to dominate and so the retailers’

![Figure 3: Producer’s expected profits as a function of $w_a$ and $w_c$ with $w_x = w_x c_r + w_A \bar{A}_x$. The parameters are identical to those of Figure 1 with the exception that $\sigma = 2$ and we have changed the values of $w_a$ and $w_c$ over which the producer’s expected profits are plotted.](image-url)
expected profits start to decrease in $\sigma$ at this point.

We also mention that the results of Figure 2 were computed on a relatively coarse grid beginning at $\sigma = 0$ and using a step-size of 0.2. It would have been easy of course to use a much finer grid but for values of $\sigma$ greater than 1, we encountered some numerical instabilities in computing the retailers’ total equilibrium profits for both types of contracts. The main reason for this is the need to compute the $\alpha_i$’s numerically and the relatively large range of $(w_c, w_A)$ values which are (more or less) optimal\(^{16}\) for the producer. While the producer is almost indifferent to which contract he chooses within this optimal range, the retailers are not, so that the reported equilibrium profits of the retailers (for large values of $\sigma$) are very sensitive to the particular solution chosen by the producer, i.e. found by the numerical optimizer. Had we therefore chosen a finer grid for $\sigma$, we would have obtained curves for the expected total retailer profits that were less smooth\(^{17}\) than those of Figure 2 for values of $\sigma$ greater than 1.

On a closely related note, given that the producer appears to be indifferent to the choice of constant or linear wholesale contracts for values of $\sigma$ close to 2, one might expect the total expected retailers’ profits for the two contracts to also coincide for values of $\sigma$ close to 2. We do not see this, however, because while the producer is almost indifferent between the two contract types, he still marginally prefers the linear contract (although this is not apparent from Figure 2) and this results (for the reasons given earlier) in the retailers being considerably worse off than if only a constant contract was available.

4 The Value of Financial Markets

In this section we discuss the value that the financial markets add to the competitive supply chain. There are two means by which the financial markets add value: (i) as a mechanism for mitigating the retailers’ budget constraints via dynamic trading and (ii) as a source of public information upon which the ordering quantities and prices are contingent. We begin with (i) and towards this end we need to discuss the so-called\(^{18}\) F-contract. The F-contract is in fact identical to our earlier contract but we now assume that the retailers can no longer trade in the financial markets.

4.1 The F-Contract

Drawing on the results of Caldentey and Haugh (2009) in the single-retailer case, we can compare the performance of the supply chain across the two contracts in the symmetric-budget case as well as determining the players’ preferences over each contract. We begin with (i) and towards this end we need to discuss the so-called\(^{18}\) F-contract. The F-contract is in fact identical to our earlier contract but we now assume that the retailers can no longer trade in the financial markets.

For a fixed price menu, $w_\tau$, it is straightforward to solve for the retailers’ Cournot equilibrium. In particular, the $i^{th}$ player solves

\[
\Pi_R^F(w_\tau) = \max_{q_i \geq 0} \mathbb{E}_0^S \left[ (\bar{A}_\tau - (q_i + Q_{i-}) - w_\tau) q_i \right] \\
\text{subject to } w_\tau q_i \leq B_i \quad \text{for all } \omega \in \Omega.
\]

\[^{16}\]We saw precisely this phenomenon in Figure 3 for the case $\sigma = 2$.

\[^{17}\]But they would still have had the same general shape.

\[^{18}\]The term “F-contract” was introduced by Caldentey and Haugh (2009) and so we will use the same term here.
A direct comparison of this optimization problem and the one in (7)-(8) reveals that for a given contract \( w_\tau \), retailers are always better off (in a weak sense) by having access to the financial market. However, whether this conclusion holds in equilibrium is not that clear.

Problem (31)-(32) decouples and is solved separately for each outcome, \( \omega \). The first order conditions imply

\[
q_i = \min \left( \frac{B_i}{w_r}, \frac{(\bar{A}_r - Q_i - w_\tau)^+}{2} \right).
\]

We see that there is a function \( m(\omega) \in \{0, 1, \ldots, N\} \) so that the budget constraints are not binding in state \( \omega \) for the first \( m \) retailers only. The solution then takes the form

\[
q_{i}^F = \begin{cases} q^F := \left( \frac{\bar{A}_r - \sum_{j=m+1}^{N} \frac{B_j}{w_r} - w_\tau}{m+1} \right)^+, & i = 1, \ldots, m \\ B_i/w_r, & i = m+1, \ldots, N. \end{cases} \tag{33}
\]

Note that \( m \) was a constant in Section 3 (Proposition 2) whereas here \( m \) is random. In order to determine the value of \( m = m(\omega) \), we must determine that value of \( m \) where the \( m \)th retailer can afford to order \( q^F \) units but where the \( (m+1) \)th retailer cannot afford \( q^F \) units. Mathematically, this translates to determining the value of \( m \) such that \( B_{m+1} < q^F w_\tau \leq B_m \) with the understanding that \( B_{N+1} := 0 \). If no such \( m \geq 1 \) exists then we take \( m = 0 \) and the budget constraints bind for all \( N \) retailers. It is also necessary to check that there is not more than one value of \( m \) for which the above conditions hold. While this may seem intuitively clear, it is not immediately obvious and so we state it as a Remark which we prove in Appendix A.

**Remark 1** There is at most one value of \( m \in \{1, \ldots, N\} \) satisfying \( B_{m+1} < q(m) w_\tau \leq B_m \).

The retailers’ problem is then solved separately for each \( \omega \in \Omega \) by determining the number of non-binding retailers, \( m(\omega) \). The producer’s problem also decouples and he simply chooses \( w_r(\omega) \) to optimize his expected profits given the retailers reaction function. We could characterize the values of \( w_r \) for which exactly \( i \) retailers are non-binding for \( i = 0, \ldots, N \) and then determine an expression for the producer’s expected profits. Since our focus in this paper is not on the F-contract, *per se*, we will restrict ourselves to the symmetric budget case where it is possible to make statements concerning the players preferences over the two contracts. Hence, for the rest of this section we assume that \( B_i = B \) for all \( i = 1, \ldots, N \).

When the \( N \) retailers all have the same budget, \( B \), then (33) is easily seen to reduce to

\[
q_i^F = q_r^F := \min \left( \frac{(\bar{A}_r - w_\tau)^+}{N+1}, \frac{B}{w_r} \right) \quad \text{for all } i = 1, \ldots, N. \tag{34}
\]

The producer’s optimal objective function then becomes

\[
\Pi^F = N \mathbb{E}^0 \left[ \max_{w_r \geq c_r} \left\{ (w_r - c_r) \min \left( \frac{(\bar{A}_r - w_\tau)^+}{N+1}, \frac{B}{w_r} \right) \right\} \right] \tag{35}
\]

\[
= \mathbb{E}^0 \left[ \max_{w_r \geq c_r} \left\{ (w_r - c_r) \min \left( \frac{N(\bar{A}_r - w_\tau)^+}{N+1}, \frac{NB}{w_r} \right) \right\} \right] \tag{36}
\]

\[
= \mathbb{E}^0 \left[ \max_{w_r \geq c_r} \left\{ (w_r - c_r) \min \left( \frac{N(\bar{A}_r - w_\tau)^+}{N+1}, \frac{2NB}{w_r} \right) \right\} \right].
\]
But (36) is the producer’s problem when the \( N \) retailers merge and have a combined budget of \( NB \). We have therefore shown that the producer also prefers the retailers to remain in competition when the flexible contract is under consideration. Explicit solutions for the maximization problems in (35) and (36) are easily computed. We also obtain the following result.

**Proposition 6** In the symmetric budget case the producer is always better off if the \( N \) retailers have access to the financial markets.

**Proof:** When the retailers have access to the financial markets the producer’s objective function is given by (26). But this is equivalent to \( 2N/(N+1) \) times the objective function of the producer when there is just a single retailer with a budget of \((N+1)B/2\). Similarly, the producer’s objective function in (35) is equivalent to \( 2N/(N+1) \) times the producer’s objective function in the flexible setting with just a single retailer having a budget of \((N+1)B/2\). But then the result follows immediately from Proposition 8 in Caldentey and Haugh (2009) who show in the single retailer setting that the producer always prefers the retailer to have access to the financial markets. \( \square \)

The situation is more complicated for the retailers. In particular, the retailers may or may not prefer having access to the financial markets in equilibrium. The relationship between \( c_\tau \) and \( \delta_\nu \) (as defined in Proposition 5) is key: if \( c_\tau = \delta_\nu \) the retailers also prefer having access to the financial markets. If \( c_\tau < \delta_\nu \), however, then their preferences can go either way.

### 4.2 The Value of Information in the Financial Markets

The financial markets also add value to the supply chain by allowing the retailers to mitigate their budget constraints via dynamic trading. The next proposition emphasizes the value of information in a competitive supply chain. Under the assumption of zero marginal production costs, it states that for an \( \mathcal{F}_\tau \)-measurable price menu, \( w_\tau \), the producer is always better off when the retailers’ orders are allowed to be contingent upon time \( \tau_2 \) information where \( \tau_2 > \tau_1 \). Later in Section 5.4 we will discuss the optimal timing, \( \tau \), of the contract. Clearly the optimal \( \tau \) achieves the optimal tradeoff between the value of additional information and the cost associated with delaying production.

**Proposition 7** Consider two times \( \tau_1 < \tau_2 \) and let \( w_\tau \) be an \( \mathcal{F}_\tau \)-measurable price menu. Consider the following two scenarios: (1) the producer offers price menu \( w_\tau \) and the retailers choose their Cournot-optimal \( \mathcal{F}_\tau \)-measurable ordering quantities which is then produced at time \( \tau_1 \) and (2) the producer again offers price menu \( w_\tau \) but the retailers now choose their Cournot-optimal \( \mathcal{F}_\tau \)-measurable ordering quantities which is then produced at time \( \tau_2 \). If \( c_{\tau_1} = c_{\tau_2} = 0 \) then the producer always prefers scenario (2).

**Proof:** See Appendix A.

The conclusion of Proposition 7 might appear to be obvious as it is clearly true that the retailers would prefer scenario (2). After all, scenario (2) gives them (at no extra cost) additional information upon which to base their ordering decisions and additional time to run their financial hedging strategy. However, it is not immediately clear that the producer should also benefit from this delay. Proposition 7 states that the producer does indeed benefit from this delay, at least when marginal production costs are zero.
5 Extensions

5.1 Should the Retailers Merge or Remain in Competition?

A question of particular interest is whether or not the retailers should merge or remain in competition. We now give a partial\textsuperscript{19} answer to that question from the producer’s perspective.

Constant Wholesale Price or Zero Marginal Production Cost

The following proposition, which we prove in Appendix A, describes conditions under which the producer always prefers the retailers to remain in competition for the general case of non-identical budgets.

Proposition 8

(a) For any $w_\tau$, the producer prefers the $N$ retailers to remain in competition rather than merging and combining their budgets when the marginal production cost, $c_\tau$, is zero. In particular, this is true in the Cournot-Stackelberg equilibrium where the producer optimizes over $w_\tau$.

(b) If $w_\tau$ is restricted to a constant, then the producer prefers the $N$ retailers to remain in competition rather than merging and combining their budgets. In particular, this is true in the Cournot-Stackelberg equilibrium where the producer optimizes over the constant, $w_\tau$.

\textbf{Proof:} See Appendix A for the proof of (a). The proof of (b) follows from the discussion immediately following Corollary 1.

The Symmetric Case

In the symmetric case we can answer the question as to whether or not the producer and retailers would be better off if the retailers were to merge into a single entity with a combined budget of $N \times B$. Here we will use the superscript $M$ to denote quantities associated with the merged retailers, respectively. The constraint in (27) implies that from the perspective of the producer’s optimization problem, the merged entity’s budget would increase by only a factor of $2N/(N + 1)$. Similarly it is clear from (26) that the producer’s objective function would be reduced by this same factor of $2N/(N + 1)$. As before, the subscripts $P$ and $R$ refer to the producer and retailer, respectively. We will use the subscript $AR$ to denote a quantity that is summed across all retailers. This will only apply in the competitive retailer case so, for example, $\Pi_{AR|R}$ refers to the total profits of the $N$ retailers when they remain in competition. Our first result is that the producer always prefers the retailers to remain in competition when they have identical budgets.

Proposition 9 (Producer Prefers Competitive Retailers) \textit{The expected profits of the producer when there are $N$ retailers, each with a budget of $B$, is greater than or equal to his expected profits when there is just one retailer with a budget of $N \times B$.}

\textsuperscript{19}While we expect this result to be true in general, we have been unable to prove it except for the specific cases discussed in this subsection.
PROOF: See Appendix A.

It is worth emphasizing that the producer is only better off in expectation when there are multiple competing retailers. On a path-by-path basis, the producer will not necessarily be better off. In particular, there will be some outcomes where the ordering quantity is zero under the competing retailers model and strictly positive under the merged retailer model. The producer will earn zero profits on such paths under the competing retailer model, but will earn strictly positive profits under the merged retailer model.

**Proposition 10** (Retailers Are Always Better Off Merging) The profits of the merged retailer are greater than the total profits of the N competing retailers on a path-by-path basis.

**Proof:** The profits of the merged retailer is given by $\Pi_{M|\tau}^R = \frac{((\bar{A}_\tau - \delta_M)^+)^2}{16}$ where $\delta_M$ is the value of $\delta_H$ in Proposition 7 of Caldentey and Haugh (2009) but with $B$ replaced by $N \times B$. The total profits of the retailers in the Cournot version of the game, however, is given by $\Pi_{AR|\tau}^C = \frac{N((\bar{A}_\tau - \delta_P)^+)^2}{4(N+1)^2}$ where $\delta_P$ is given by Proposition 5. It is clear that $\delta_P \geq \delta_M$ and so the result follows immediately. □

### 5.2 Retailers Based in a Foreign Currency Area

We now assume that the retailers and producer are located in different currency areas and use change-of-numeraire arguments to show that our analysis still goes through. Without any loss of generality, we will assume that the retailers and producer are located in the “foreign” and “domestic” currency areas, respectively. The exchange rate, $Z_t$ say, denotes the time $t$ domestic value of one unit of the foreign currency. When the producer proposes a contract, $w_\tau$, we assume that he does so in units of the foreign currency. Therefore the $i^{th}$ retailer pays $q_i w_\tau$ units of foreign currency to the producer. The retailers’ problem is therefore unchanged from the problem we considered at the beginning of Section 3.1 if we take $Q$ to be an EMM of a foreign investor who takes the foreign cash account as his numeraire security. As explained in Appendix B.1, this same $Q$ can also be used by the producer as a domestic EMM where he takes the domestic value of the foreign cash account as the numeraire security.

We could take our financial process, $X_t$, to be equivalent to $Z_t$ so that the retailers hedge their foreign exchange risk in order to mitigate the effects of their budget constraints. This would only make sense if $\bar{A}_\tau$ and the exchange rate, $Z_t$, were dependent. More generally, we could allow $X_t$ to be multi-dimensional so that it includes $Z_t$ as well as other tradeable financial processes that influence $\bar{A}_\tau$.

The producer must convert the retailers’ payments into units of the domestic currency and he therefore earns a per-unit profit of either (i) $w_\tau Z_\tau - c_\tau$ if production costs are in units of the domestic currency or (ii) $Z_\tau (w_\tau - c_\tau)$ if production costs are in units of the foreign currency. Case (i) would apply if production takes place domestically whereas case (ii) would apply if production takes place in the foreign currency area. We will assume that interest rates in both the domestic and foreign currency areas are identically zero.

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20 See Ding et al. (2007) for a comprehensive review of the literature discussing exchange rate uncertainty in a production/inventory context.

21 Which is the domestic currency from the retailers’ perspective.

22 We assume zero interest rates only so that we can focus on the issues related to foreign exchange.
Analogously to (26) and (27) we find in the equibudget case that the producer’s problem in case (i) is given by

\[ \Pi_p = \max_{w, \lambda \geq 0} Z_0 \frac{2N}{N+1} E_0^Q \left[ \frac{(w Z_r - c_r) (\bar{A}_r - w_r (1 + \lambda))^+}{2} \right] \]  

subject to \[ E_0^Q \left[ w_r \left( \frac{\bar{A}_r - w_r (1 + \lambda))^+}{2} \right] \right] \leq \frac{(N + 1)}{2} B. \]  

Note that \( Z_r \) appears in the denominator inside the expectation in (37) because, as explained above, the domestic value of the foreign cash account is the appropriate numeraire corresponding to the EMM, \( Q \). Since we have assumed interest rates are identically zero, the foreign value of the foreign cash-account is identically one and so its domestic value is \( Z_t \) at time \( t \). For the same reason, \( Z_0 \) appears outside the expectation in (37). Solving the producer’s problem in (37) and (38) is equivalent to solving the problem he faced earlier in this section but now with a stochastic cost, \( \hat{c}_r := c_r / Z_r \). However, it can easily be seen that the proof of Proposition 5, or more to the point, Proposition 7 in Caldentey and Haugh (2009), goes through unchanged when \( c_r \) is stochastic. We therefore obtain the same result as Proposition 5 with \( c_r \) replaced by \( \hat{c}_r \) and \( Q \) interpreted as a foreign EMM with the domestic value of the foreign cash account as the numeraire security.

**Remark:** If instead case (ii) prevailed so that the producer’s per-unit profit was \( Z_r (w_r - c_r) \) then the \( Z_r \) term in both the numerator and denominator of (37) would cancel, leaving the producer with an identical problem to that of Section 3.1 (Part II) albeit with different EMMs. So while the analysis for case (ii) is identical to that of Section 3.1 (Part II), the probability measures under which the solutions are calculated are different.

### 5.3 Stochastic Interest Rates and Paying the Producer in Advance

We now consider the problem where the retailers’ budgets are only available at time \( T \) but that the producer must be paid at time \( \tau < T \). We will assume that interest rates are stochastic and no longer identically zero so that the retailers’ effective time \( \tau \) budgets are also stochastic. In particular, we will assume that the \( Q \)-dynamics of the short rate are given by the Vasicek\(^23\) model so that

\[ dr_t = \alpha (\mu - r_t) \, dt + \sigma dW_t \]  

where \( \alpha, \mu \) and \( \sigma \) are all positive constants and \( W_t \) is a \( Q \)-Brownian motion. The short-rate, \( r_t \), is the instantaneous continuously compounded risk-free interest rate that is earned at time \( t \) by the ‘cash account’, i.e., cash placed in a deposit account. In particular, if $1 is placed in the cash account at time \( t \) then it will be worth \( \exp \left( \int_t^T r_s \, ds \right) \) at time \( T > t \). It may be shown that the time \( \tau \) value of a zero-coupon-bond with face value $1 that matures at time \( T > \tau \) satisfies

\[ Z_r^T := e^{\alpha(T-\tau)+b(T-\tau)r_t} \]  

\(^23\)See, Duffie (2004) for a description of the Vasicek model and other related results that we use in this subsection. Note that it is not necessary to restrict ourselves to the Vasicek model. We have done so in order to simplify the exposition but our analysis holds for more general models such as the multi-factor Gaussian and CIR processes that are commonly employed in practice.
where \( a(\cdot) \) and \( b(\cdot) \) are known deterministic functions of the time-to-maturity, \( T - \tau \). In particular, \( Z_\tau^T \) is the appropriate discount factor for discounting a known deterministic cash flow from time \( T \) to time \( \tau < T \).

Returning to our competitive supply chain, we assume as before that the \( N \) retailers’ profits are realized at time \( T \geq \tau \). Since the producer now demands payment from the retailers at time \( \tau \) when production takes place this implies that the retailers will be forced to borrow against the capital \( B \) that is not available until time \( T \). As a result, the \( i^{th} \) retailer’s effective budget at time \( \tau \) is given by

\[
B_i(\tau) := B_i Z_\tau^T = B_i e^{a(T-\tau) + b(T-\tau)r_\tau}.
\]

As before, we assume that the stochastic clearance price, \( A - Q \), depends on the financial market through the co-dependence of the random variable \( A \), and the financial process, \( X_t \). To simplify the exposition, we could assume that \( X_t \equiv r_\tau \) but this is not necessary. If \( X_t \) is a financial process other than \( r_\tau \), we simply need to redefine our definition of \( \{F_t\}_{0\leq t\leq T} \) so that it represents the filtration generated by \( X_t \) and \( r_\tau \). Before formulating the optimization problems of the retailers and the producer we must adapt our definition of feasible \( F_\tau \)-measurable financial gains, \( G_\tau \). Until this point we have insisted that any such \( G_\tau \) must satisfy \( \mathbb{E}_0^\sigma [G_\tau] = 0 \), assuming as before that zero initial capital is devoted to the financial hedging strategy. This was correct when interest rates were identically zero but now we must replace that condition with the new condition\(^{24}\)

\[
\mathbb{E}_0^\sigma [D_\tau G_\tau] = 0 \quad (41)
\]

where \( D_\tau := \exp \left( -\int_0^\tau r_s \, ds \right) \) is the stochastic discount factor. The \( i^{th} \) retailer’s problem for a given \( F_\tau \)-measurable wholesale price, \( w_\tau \), is therefore given\(^{25}\) by

\[
\Pi_i(w_\tau) = \max_{q_i \geq 0, G_\tau} \mathbb{E}_0^\sigma [D_\tau (A_T - (q_i + Q_{i\cdot})) q_i - D_\tau w_\tau q_i] \quad (42)
\]

subject to

\[
w_\tau q_i \leq B_i(\tau) + G_\tau, \quad \text{for all } \omega \in \Omega \quad (43)
\]

\[
\mathbb{E}_0^\sigma [D_\tau G_\tau] = 0 \quad (44)
\]

and \( F_\tau \)-measurability of \( q_i \). \quad (45)

Note that both \( D_T \) and \( D_\tau \) appear in the objective function (42) and reflect the times at which the retailer makes and receives payments. We also explicitly imposed the constraint that \( q_i \) be \( F_\tau \)-measurable. This was necessary\(^{26}\) because of the appearance of \( D_T \) in the objective function. We can easily impose the \( F_\tau \)-measurability of \( q_i \) by conditioning with respect to \( F_\tau \) inside the expectation appearing in (42). We then obtain

\[
\mathbb{E}_0^\sigma [D_\tau (\bar{A}_T^\omega - (q_i + Q_{i\cdot}) - w_\tau) q_i] \quad (46)
\]

as our new objective function where \( \bar{A}_T^\omega := \mathbb{E}_0^\sigma [D_T A_T]/D_\tau \). With this new objective function it is no longer necessary to explicitly impose the \( F_\tau \)-measurability of \( q_i \).

It is still straightforward to solve for the retailers’ Cournot equilibrium. One could either solve the problem directly as before or alternatively, we could use the change-of-numeraire method of

\(^{24}\) See the first paragraph of Appendix B.1 for why this is the case.

\(^{25}\) We write \( A_T \) for \( A \) to emphasize the timing of the cash-flow.

\(^{26}\) To be precise, terms of the form \( D_T (A_T - q_i) \) should also have appeared in the problem formulations of earlier sections in this paper. In those sections, however, \( D_t \equiv 1 \) for all \( t \) and so the conditioning argument we use above allows us to replace \( A_T \) with \( A_\tau \) in those sections.
Section 5.2 that is described in Appendix B.1. In particular, we could switch to the so-called forward measure where the EMM, $Q_r$, now corresponds to taking the zero-coupon bond maturing at time $\tau$ as the numeraire. In that case the $i$th retailer’s objective function in (46) can be written\(^{27}\) as

\[
Z_0^0 \mathbb{E}_0^{Q_r} \left[ \left( \bar{A}_r^p - (q_i + Q_{i,-}) - w_r \right) q_i \right].
\]

We can therefore solve for the retailers’ Cournot equilibrium using our earlier analysis but with $\bar{A}_r$ and $Q$ replaced by $A_r^p$ and $Q_r$, respectively. Note that the constant factor, $Z_0^T$, in (47) is the same for each retailer and therefore makes no difference to the analysis. Following the first approach we obtain the following solution to the retailer’s problem. We omit the proof as it is very similar to the proof of Proposition 4.

**Proposition 11** (Retailers’ Optimal Strategy)

Let $w_r$ be an $\mathcal{F}_\tau$-measurable wholesale price offered by the producer and define $Q_r := \left( \bar{A}_r^p - w_r \right)^+ / (N+1)Z_r^T$. This is the optimal ordering quantity for each retailer in the absence of any budget constraints. The following two cases arise:

**Case 1:** Suppose $\mathbb{E}_0^Q [D_\tau Q_r w_r] \leq \mathbb{E}_0^Q [D_\tau B(r_\tau)] = Z_0^T B$. Then $q_i(w_r) := q_r := Q_r$ for all $i$ and (possibly due to the ability to trade in the financial market) the budget constraints are not binding.

**Case 2:** Suppose $\mathbb{E}_0^Q [D_\tau Q_r w_r] > Z_0^T B$. Then

\[
q_i(w_r) := q_r := \frac{(\bar{A}_r^p - w_r(1 + \lambda))^+}{(N+1)Z_r^T}
\]

for all $i = 1, \ldots, N$ (48)

where $\lambda \geq 0$ solves

\[
\mathbb{E}_0^Q [D_\tau w_r q_r] = \mathbb{E}_0^Q [B(r_\tau)D_\tau] = Z_0^T B.
\]

Given the retailers’ best response, the producer’s problem may now be formulated\(^{28}\) as

\[
\Pi_p = \max_{w_r, \lambda \geq 0} \mathbb{E}_0^Q \left[ D_\tau (w_r - c_r) \frac{(\bar{A}_r^p - w_r (1 + \lambda))^+}{(N+1)Z_r^T} \right]
\]

subject to

\[
\mathbb{E}_0^Q \left[ D_\tau w_r \frac{(\bar{A}_r^p - w_r(1 + \lambda))^+}{(N+1)Z_r^T} \right] \leq Z_0^T B.
\]

The Cournot-Stackelberg equilibrium and solution of the producer’s problem in the equibudget case is given by the following proposition. We again omit the proof of this proposition as it it very similar to the proof of Proposition 5.

**Proposition 12** (The Equilibrium Solution)

Let $\phi_p$ be the minimum $\phi \geq 1$ that satisfies

\[
\mathbb{E}_0^Q \left[ \frac{D_\tau}{8Z_r^T} \left( (\bar{A}_r^p)^2 - (\phi c_r)^2 \right) \right] \leq \frac{(N+1)}{2} Z_0^T B
\]

and let $\delta_p := \phi c_r$. Then the optimal wholesale price and ordering level satisfy

\[
w_r = \frac{\delta_p + \bar{A}_r^p}{2} \quad \text{and} \quad q_r = \frac{(\bar{A}_r^p - \delta_p)^+}{2(N+1)Z_r^T}.
\]

\(^{27}\)The condition (44) can also be written in terms of $Q_r$ as $Z_0^0 \mathbb{E}_0^{Q_r} [G_r] = 0$, i.e., $\mathbb{E}_0^Q [G_r] = 0$.

\(^{28}\)We assume here and in the foreign retailer setting that the production costs, $c_r$, are paid at time $\tau$. 

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5.4 Optimal Timing of Production Under Non-Binding Budget Constraints

We now extend the contract so that $\tau$, the time at which the contract is executed and physical production quantities are decided, is an endogenous decision variable that is determined as part of the equilibrium. Our interest in the optimal timing of the contract is motivated by our desire to understand the tradeoff between delaying production when the production cost, $c_\tau$, is increasing, and allowing the ordering quantities and price menu to be contingent upon a larger information set. Allowing the producer to choose the optimal timing of the contract therefore allows him to take optimal advantage of the information made available by the financial markets. For simplicity, our discussion will focus on the case where retailers’ budget constraints are not binding in equilibrium, in the sense of Section 3.1, Part I. Although restrictive, this assumption is in part justified by the retailers ability to trade in the financial markets. Furthermore, the analysis of an optimal production postponement strategy for an arbitrary vector of budgets $(B_1, \ldots, B_N)$ is beyond the scope of this paper. Finally, from the results in Proposition 3, it follows that when budgets are not binding, all retailers act similarly, and so with a slight abuse of notation (and without loss of generality) we can restrict ourselves to the single retailer case, i.e., $N = 1$.

We consider two alternatives formulations. In the first alternative, $\tau$ is restricted to be a deterministic time in $[0, T]$ that is selected at time $t = 0$. Motivated by the terminology of dynamic programming, we refer to this alternative as the optimal open-loop production postponement model. In the second alternative, we permit $\tau$ to be an $F_\tau$-stopping time that is bounded above by $T$. We call this alternative the optimal closed-loop production postponement model. In both cases, the procurement contract offered by the producer takes the form of a pair, $(\tau, w_\tau)$, where the wholesale price menu, $w_\tau$, is required to be $F_\tau$-measurable. We note that the producer always prefers the closed-loop model though from a practical standpoint the open-loop model may be easier to implement in practice.

Independently of whether $\tau$ is a deterministic time or a stopping time, the optimal ordering level for the retailer, given a contract $(\tau, w_\tau)$, is an $F_\tau$-measurable menu, $q_\tau$, that satisfies\(^{29}\) the conditions in Proposition 3 with $N = 1$ (since the budget constraint is not binding).

As a result, the producer’s problem of selecting the optimal time $\tau$ is given by

$$\max_{\tau} \mathbb{E}_0^0 \left[ (\bar{A}_\tau - c_\tau)^2 \right] = \max_{\tau} \left\{ \text{Var}(\bar{A}_\tau) + (\bar{A} - c_\tau)^2 \right\}.$$

To solve this optimization problem we would first need to specify the functional forms of $\bar{A}_\tau$ and $c_\tau$ and depending on these specifications, the solution may or may not be easy to find. For the remainder of this section, however, we will show how this problem may be solved when additional assumptions are made. In particular, we make the following two additional assumptions:

1. $X_t$ is a diffusion process with dynamics satisfying

$$dX_t = \sigma(X_t) dW_t,$$

where $W_t$ is a $Q$-Brownian motion. Note that we have not included a drift term in the dynamics of $X_t$ since it must be the case that $X_t$ is a $Q$-martingale. This is not a significant assumption and we could easily consider alternative processes for $X_t$.

\(^{29}\)It is easy to check that the proof of Proposition 4 remains unchanged if $\tau$ is allowed to be a stopping time.
2. We adopt a specific functional form to model the dependence between the market clearance price and the financial market. In particular, we assume that there behaves a well-behaved\textsuperscript{30} function, $F(x)$, and a random variable, $\varepsilon$, such that one of the following two models holds.

\textbf{Additive Model:} \quad A = F(X_T) + \varepsilon, \quad \text{with } \mathbb{E}^q[\varepsilon] = 0, \quad \text{or} \quad (53)

\textbf{Multiplicative Model:} \quad A = \varepsilon F(X_T), \quad \text{with } \varepsilon \geq 0 \text{ and } \mathbb{E}^q[\varepsilon] = 1. \quad (54)

The random perturbation $\varepsilon$ captures the non-financial component of the market price uncertainty and is assumed to be independent of $X_t$. Note that if $F(x) = \bar{A}$, we recover a model for which demand is independent of the financial market.

\textbf{Optimal Open-Loop Production Postponement}

We now restrict $\tau$ to be a deterministic time in $[0, T]$. The producer’s optimization problem reduces to

$$\max_{\tau \in [0, T]} \left\{ \text{Var}(\bar{A}_\tau) + (\bar{A} - c\tau)^2 \right\}. \quad (55)$$

We note that in this optimization problem there is a trade-off between demand learning as represented by the variance term, $\text{Var}(\bar{A}_\tau)$, and production costs as represented by $(\bar{A} - c\tau)^2$. The first term is increasing in $\tau$ while the second term is decreasing in $\tau$ so that, in general, the optimization problem in (55) does not admit a trivial solution and depends on the particular form of the functions $\text{Var}(\mathbb{E}^q[A | X_{\tau}])$ and $c\tau$.

The Itô Representation Theorem\textsuperscript{31} implies the existence of an $\mathcal{F}_t$-adapted process, $\{\theta_t : t \in [0, T]\}$, such that

$$A = \bar{A} + \int_0^T \theta_t \, dX_t + \varepsilon \quad \text{or} \quad A = \varepsilon \left( \bar{A} + \int_0^T \theta_t \, dX_t \right)$$

for the additive or multiplicative model, respectively. In both cases the $\mathbb{Q}$-martingale property of $X_t$ implies

$$\bar{A}_\tau = \bar{A} + \int_0^\tau \theta_t \, dX_t. \quad (56)$$

In order to compute the variance of $\bar{A}_\tau$ we use the $\mathbb{Q}$-martingale property of the stochastic integral and invoke Itô’s isometry to obtain

$$\text{Var}(\bar{A}_\tau) = \mathbb{E}_0^q \left[ \left( \int_0^\tau \theta_t \, dX_t \right)^2 \right] = \mathbb{E}_0^q \left[ \int_0^\tau \theta_t^2 \, d[X]_t \right],$$

where the process $[X]_t$ is the quadratic variation of $X_t$ with dynamics $d[X]_t = \sigma^2(X_t) \, dt$. It follows that

$$\text{Var}(\bar{A}_\tau) = \int_0^\tau \mathbb{E}_0^q[(\theta_t \, \sigma(X_t))^2] \, dt.$$

\textsuperscript{30}It is necessary, for example, that $F(\cdot)$ satisfy certain integrability conditions so that the stochastic integral in (56) be a $\mathbb{Q}$-martingale. In order to apply Itô’s Lemma it is also necessary to assume that $F(\cdot)$ is twice differentiable. Because this section is intended to be brief, we omit the various technical conditions that are required to make our arguments completely rigorous.

\textsuperscript{31}See Øksendal (1998) for a formal statement. Øksendal (1998) may also be consulted for a statement of Itô’s isometry.
The open-loop optimal problem therefore reduces to solving
\[
\max_{\tau \in [0,T]} \left\{ \int_0^\tau \mathbb{E}_0^\mathbb{Q}[\theta_t \sigma(X_t)^2] \, dt + (\bar{A} - c_\tau)^2 \right\}. \tag{57}
\]
If there is an interior solution to this problem (i.e., \(\tau^* \in (0, T)\)), then it must satisfy the first-order optimality condition
\[
\mathbb{E}_0^\mathbb{Q}[\theta_{\tau^*} \sigma(X_{\tau^*})^2] - 2(\bar{A} - c_{\tau^*}) \dot{c}_{\tau^*} = 0, \quad \text{where} \quad \dot{c}_{\tau^*} := \frac{dc_{\tau^*}}{d\tau}.
\]

**Example 2** Consider the case in which the security price, \(X_t\), follows a geometric Brownian motion with dynamics
\[
dX_t = \sigma X_t \, dW_t
\]
where \(\sigma \neq 0\) and \(W_t\) is a \(Q\)-Brownian motion. The quadratic variation process then satisfies \(d[X]_t = \sigma^2 X_t^2 \, dt\). To model the dependence between the market clearance price and the process, \(X_t\), we assume a linear model for \(F(\cdot)\) so that \(F(X) = A_0 + A_1 X\) where \(A_0\) and \(A_1\) are positive constants. Therefore, depending on whether we consider the additive or multiplicative model, we have
\[
A = A_0 + A_1 X_T + \varepsilon \quad \text{or} \quad A = \varepsilon (A_0 + A_1 X_T),
\]
where \(\varepsilon\) is a zero-mean or unit-mean random perturbation, respectively, that is independent of the process, \(X_t\). It follows that \(\bar{A}_\tau = A_0 + A_1 X_\tau\) and \(\bar{A} = \mathbb{E}_0^\mathbb{Q}[A] = A_0 + A_1 X_0\). In addition, it is clear that \(\theta_t\) is identically equal to \(A_1\) for all \(t \in [0, T]\). We assume that the per unit production cost increases with time and is given by
\[
c_{\tau} = c_0 + \alpha \tau^\kappa, \quad \text{for all} \quad \tau \in [0, T],
\]
where \(\alpha\) and \(\kappa\) are positive constants.

To impose the additional constraint that \(\bar{A}_\tau \geq c_\tau\) for all \(\tau\) (Assumption 1), we restrict our choice of the parameters \(A_0, T, c_0, \kappa,\) and \(\alpha\) so that \(A_0 \geq c_0 + \alpha T^\kappa\). Since \(\mathbb{E}_0^\mathbb{Q}[X_T^2] = X_0^2 \exp(\sigma^2 T)\) the optimization problem in (57) reduces to
\[
\max_{\tau \in [0,T]} \left\{ (A_1 X_0)^2 \left( \exp(\sigma^2 \tau) - 1 \right) + (\bar{A} - c_0 - \alpha \tau^\kappa)^2 \right\}.
\]
In general, a closed form solution is not available unless \(\kappa = 0\). This is a trivial case in which \(c_\tau\) is constant and the optimal strategy is to postpone production until time \(T\) so that \(\tau^* = T\). Figure 4 shows the value of the objective function as a function of \(\tau\) for four different values of \(\kappa\). The cost functions are such that it becomes cheaper to produce as \(\kappa\) increases. Note that for \(\kappa \in \{4, 8\}\), it is convenient to postpone production. For the more expensive production cost functions that occur when \(\kappa \in \{0.25, 1\}\), production postponement is not profitable and it is optimal to produce immediately, that is, \(\tau^* = 0\). \(\square\)

**Optimal Closed-Loop Production Postponement**

Instead of selecting a fixed transaction time, \(\tau\), at \(t = 0\), the producer now optimizes over the set of stopping times bounded above by \(T\). In this case, the producer’s optimization problem reduces to solving for
\[
\max_{\tau \in T} \mathbb{E}_0^\mathbb{Q} \left[ (\bar{A}_\tau - c_\tau)^2 \right], \tag{58}
\]
Figure 4: Optimal open-loop production postponement for four different production cost functions parameterized by $\kappa$. The other parameters are $X_0 = \sigma = T = 1$, $A_1 = 8$, $A_0 = 16$, $c_0 = 2.4$ and $\alpha = 5.6$.

where $\mathcal{T}$ is the set of $\mathcal{F}_t$-adapted stopping times bounded above by $T$. According to the modeling of $A$ in (53) or (54), it follows that $v(\tau, X_\tau) :=  \bar{A}_\tau = \mathbb{E}_\mathbb{P}^\mathbb{Q}[F(X_T)]$ is a $\mathbb{Q}$-martingale that satisfies

$$\frac{v(t, x)}{\partial t} + \frac{1}{2} \sigma^2(x) \frac{\partial^2 v(t, x)}{\partial x^2} = 0, \quad v(T, x) = F(x).$$

We define $U$ to be the set $\{(t, x) : Gg(t, x) > 0\}$ where $g(t, x) := (v(t, x) - c_t)^2$ is the payoff function and $G$ is the generator

$$G := \frac{\partial}{\partial t} + \frac{1}{2} \sigma^2(x) \frac{\partial^2}{\partial x^2}.$$ 

We then obtain

$$U = \{(t, x) : (\sigma(x) v_x(t, x))^2 > 2(v(t, x) - c_t) \dot{c}_t \},$$

where $v_x$ is the first partial derivative of $v$ with respect to $x$. In general, the set $U$ is a proper subset of the optimal continuation region for the stopping problem in (58). Computing the optimal stopping time analytically is a difficult task and is usually done numerically. However, if $U$ turns out to equal the entire state space then it is clear that it is always optimal to continue so that $\tau = T$.

Example 2: (continued)
Consider the setting of Example 2 but where now \( \tau \) is a stopping time instead of a deterministic time. For the linear function \( F(X) = A_0 + A_1 X \), the auxiliary function \( v \) satisfies \( v(t, x) = A_0 + A_1 x \), and the region \( U \) is given by
\[
U = \left\{ (t, x) : (\sigma x A_1)^2 > 2(A_0 + A_1 x - c_t) \dot{c}_t \right\}.
\]

Straightforward calculations allow us to rewrite \( U \) as
\[
U = \left\{ (t, x) : x > \dot{c}_t + \frac{\sqrt{\dot{c}_t^2 + 2\sigma^2 (A_0 - c_t) \dot{c}_t}}{\sigma^2 A_1} \right\}.
\]

Let us define the auxiliary function
\[
\rho(t) := \dot{c}_t + \frac{\sqrt{\dot{c}_t^2 + 2\sigma^2 (A_0 - c_t) \dot{c}_t}}{\sigma^2 A_1}.
\]

Since \( U \) is a subset of the optimal continuation region, we know that it is never optimal to stop if \( X_t > \rho(t) \). Of course, it is possible that \( X_t < \rho(t) \) and yet still be optimal to continue.

We solved for the optimal continuation region numerically by using a binomial model to approximate the dynamics of \( X_t \). In so doing, we can assess the quality of the (suboptimal) strategy that uses \( \rho(t) \) to define the continuation region. Figure 5 shows the optimal continuation region and the threshold \( \rho(t) \) for four different cost functions. These cost functions are given by \( c_\tau = c_0 + \alpha \tau^\kappa \) with \( \kappa = 0.25, 1, 4, \) and 8. When \( X(\tau) \) is above the optimal threshold it is optimal to continue. The vertical dashed line corresponds to the optimal open-loop deterministic time computed in Figure 4. For \( \kappa = 0.25 \) or \( \kappa = 1 \) this optimal deterministic time equals 0 since \( X_0 \) lies below the optimal threshold. For \( \kappa = 4 \) it equals 0.476, and for \( \kappa = 8 \) it equals 0.678.

Interestingly, for high values of \( \kappa \) the auxiliary threshold \( \rho(t) \) is a good approximation for the optimal solution. However, as \( \kappa \) decreases the quality of the approximation deteriorates rapidly. Except for the case where \( \kappa = 0.25 \), the optimal threshold increases with time. This reflects the fact that the producer becomes more likely to stop and exercise the procurement contract as the end of the horizon approaches.

We conclude this example by computing the optimal expected payoff for the producer under both the optimal open-loop policy and the optimal closed-loop policy.

<table>
<thead>
<tr>
<th>( \kappa )</th>
<th>Open-Loop Payoff</th>
<th>Closed-Loop Payoff</th>
<th>% Increase</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>7.29</td>
<td>7.29</td>
<td>0.0%</td>
</tr>
<tr>
<td>1</td>
<td>7.29</td>
<td>7.305</td>
<td>0.2%</td>
</tr>
<tr>
<td>4</td>
<td>7.71</td>
<td>7.99</td>
<td>3.7%</td>
</tr>
<tr>
<td>8</td>
<td>8.09</td>
<td>8.33</td>
<td>3.8%</td>
</tr>
</tbody>
</table>

Producer’s expected payoff for four different production cost functions parameterized by \( \kappa \).

The other parameters are \( X_0 = \sigma = T = 1, A_1 = 8, A_0 = 16, c_0 = 2.4 \) and \( \alpha = 5.6 \).

Naturally, the optimal stopping time (closed-loop) policy produces a higher expected payoff than the optimal deterministic time (open-loop) policy. The improvement, however, is only a few percentage points which might suggest that a simpler contract based on a deterministic time captures most of the benefits of allowing \( \tau \) to be a decision variable. In practice, of course, it would be necessary to model the operations and financial markets more accurately and to calibrate the resulting model correctly before such conclusions could be drawn. □
Figure 5: Optimal continuation region for four different manufacturing cost functions parameterized by $\kappa$. The other parameters are $X_0 = \sigma = T = 1$, $A_1 = 8$, $A_0 = 16$, $c_0 = 2.4$ and $\alpha = 5.6$.

6 Conclusions and Further Research

We have studied the performance of a stylized supply chain where multiple retailers and a single producer compete in a Cournot-Stackelberg game. At time $t = 0$ the retailers order a single product from the producer and upon delivery at time $T > 0$, they sell it in the retail market at a stochastic clearance price that depends in part on the realized path or terminal value of some tradeable financial process. Because production and delivery do not take place until time $T$, the producer offers a menu of wholesale prices to the retailer, one for each realization of the process up to time some time, $\tau$, where $0 \leq \tau \leq T$. The retailers’ ordering quantities can therefore depend on the realization of the process until time $\tau$. We also assumed, however, that the retailers were budget-constrained and were therefore limited in the number of units they could purchase from the producer. Because the supply chain is potentially more profitable if the retailers can allocate their budgets across different states we allow them to trade dynamically in the financial market.

Propositions 1 and 2 are the main results of our paper in which we characterize the retailers’ Cournot equilibrium and the producer’s Stackelberg solution, respectively. An explicit solution to the Cournot-Stackelberg equilibrium is not available in general and we must rely on computational methods to solve for it. However, we identify two important examples for which an analytical solution is available, namely, the case in which retailers’ budget constraints are not binding and the case in which retailers are symmetric. In both cases, the optimal wholesale price menu is a linear
combination of the producer’s per unit manufacturing cost ($c_\tau$) and the maximum expected retail price ($\bar{A}_\tau$). Motivated by this pattern, we investigate numerically the performance of an optimal linear (in $c_\tau$ and $\bar{A}_\tau$) wholesale price. Our results suggest that this simple family of contract performs well especially when market volatility is high. (When market volatility is low there is much less to be gained from contracting on the financial index and so constant contract would be lose to optimal in that case).

After solving for the Nash equilibrium we discussed a number of extensions including: (i) whether or not the players would be better off if the retailers merged and (ii) whether or not the players are better off when the retailers have access to the financial markets. We also considered variations of the model where, for example, the retailers were located in a different currency area to the producer. Finally, we consider the situation where the producer could choose the optimal timing, $\tau$, of the contract and we formulated this as an optimal stopping problem.

There are several possible directions for future research. First, it would be interesting to model and solve the game where each retailer’s budget constitutes private information that is known only to him. This problem formulation would therefore require us to solve for a Bayesian Nash equilibrium. It would also be of interest to identify and calibrate settings where supply chain payoffs are strongly dependent on markets. We would then like to estimate just how much value is provided by the financial markets in its role as (i) a source of public information upon which contracts may be written and (ii) as a means of mitigating the retailers’ budget constraints. A further direction is to consider alternative contracts such as 2-part tariffs for coordinating the supply chain. Of course we would still like to have these contracts be contingent upon the outcome of the the financial markets. Finally, we would like to characterize the producer’s optimal price menu in the general non-equibudget case and where the marginal cost, $c_\tau$, is not zero.

Acknowledgment
The research of the first author was supported in part by FONDECYT grant N° 1070342.

References


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**A Appendix: Proofs**

**Proof of Proposition 1:**

The proof of the proposition makes use of the following preliminary result.

**Lemma 1** Let $V_\tau$ be an $\mathcal{F}_\tau$-measurable random variable then

$$(i+1)E^*_0[V_\tau q_i] + \sum_{j=i+1}^N E^*_0[V_\tau q_j] = \int_{\alpha_i<\alpha_\tau} V_\tau (\bar{A}_\tau - \alpha_i w_\tau) \, dQ. \quad (A-1)$$

**Proof of the Lemma:** From equation (14) and the fact that the $\lambda_i$'s are non-decreasing in $i$, it is easy to see that the $\alpha_i$'s are also non-decreasing in $i$ (as we would expect). We also see that $n(\alpha_\tau) = k - 1$ for all $\alpha_\tau$ satisfying $\alpha_{k-1} \leq \alpha_\tau < \alpha_k$. Setting $\alpha_{N+1} := \infty$, we can combine these results and (12) to write

$$q_i = \sum_{k=i}^N \frac{1}{k+1} \left[ \bar{A}_\tau - w_\tau \left( (k+1)(1+\lambda_i) - \sum_{j=1}^k (1+\lambda_j) \right) \right] \mathbb{I} \left( \alpha_\tau \in [\alpha_k, \alpha_{k+1}) \right). \quad (A-2)$$

Letting $\Omega_k := \{\omega : \alpha_k \leq \alpha_\tau < \alpha_{k+1}\}$, we see that (A-2) implies

$$\sum_{j=i+1}^N E^*_0[V_\tau q_j] = \sum_{j=i+1}^N \sum_{k=j}^N \int_{\Omega_k} \frac{V_\tau}{k+1} \left[ \bar{A}_\tau - w_\tau \left( (k+1)(1+\lambda_j) - \sum_{s=1}^k (1+\lambda_s) \right) \right] dQ = \sum_{k=i+1}^N \sum_{j=i+1}^k \int_{\Omega_k} \frac{V_\tau}{k+1} \left[ \bar{A}_\tau - w_\tau \left( (k+1)(1+\lambda_j) - \sum_{s=1}^k (1+\lambda_s) \right) \right] dQ = \sum_{k=i+1}^N \int_{\Omega_k} \frac{V_\tau}{k+1} \left[ (k-i) \bar{A}_\tau - w_\tau \left( (k+1) \sum_{j=i+1}^k (1+\lambda_j) - (k-i) \sum_{s=1}^k (1+\lambda_s) \right) \right] dQ.$$
Combining this last identity and the fact that

\[ E_Q[V_\tau q_i] = \sum_{k=i}^{N} \int_{\Omega_k} \frac{V_\tau}{k + 1} \left[ \bar{A}_r - w_\tau \left( (k + 1) (1 + \lambda_i) - \sum_{j=1}^{k} (1 + \lambda_j) \right) \right] dQ \]

we obtain

\[ E_Q[V_\tau q_i] + \frac{1}{i + 1} \sum_{j=i+1}^{N} E_Q[V_\tau q_j] = \sum_{k=i}^{N} \int_{\Omega_k} \frac{V_\tau}{k + 1} Z_{ik} dQ \tag{A-3} \]

where

\[ Z_{ik} := \left[ \bar{A}_r - w_\tau \left( (k + 1) (1 + \lambda_i) - \sum_{j=1}^{k} (1 + \lambda_j) \right) \right. \]

\[ + \frac{(k - i) \bar{A}_r - w_\tau \left( (k + 1) \sum_{j=i+1}^{k} (1 + \lambda_j) - (k - i) \sum_{s=1}^{k} (1 + \lambda_s) \right)}{i + 1} \left. \right] . \]

and where we have used the convention \( \sum_{j=i+1}^{i} (1 + \lambda_j) = 0 \). After some straightforward manipulations, one can show that

\[ Z_{ik} = \frac{k + 1}{i + 1} (\bar{A}_r - \alpha_i w_\tau) \]

and so by the definition of \( \Omega_k \) we can substitute for \( Z_{ik} \) in (A-3) and obtain (A-1). \( \square \)

Let us now return to the proof of Proposition 1. Let us recall that the Lagrange multipliers for the retailer’s budget constraints \( \{\lambda_i\} \) and the cutoff values \( \{\alpha_i\} \) satisfy the relationship

\[ 1 + \lambda_i = \frac{\alpha_i}{i} + \sum_{j=1}^{i-1} \frac{\alpha_j}{j (j + 1)} . \]

Recall also that we have ordered the retailers’ budget in decreasing order \( B_1 \geq B_2 \geq \cdots \geq B_N \). As a result, if retailer’s \( i \) budget constraint is not binding then retailer’s \( j \) budget constraint is also not binding, for all \( j = 1, \ldots, i \). It follows that if \( E_Q[w, q_i] < B_i \) then \( \lambda_j = 0 \) for all \( j = 1, \ldots, i \) and from the previous equation we conclude that \( \alpha_j = 1 \) for all \( j = 1, \ldots, i \). Hence, if \( \alpha_i > 1 \) the budget constraint is binding and \( E_Q[w, q_i] = B_i \) as required.

To complete the proof, we need to show that \( \alpha_i \) is given by (16). With \( V_\tau \) set to \( w_\tau \), Lemma 1 above implies

\[ E_Q[w_\tau q_i] + \frac{1}{i + 1} \sum_{j=i+1}^{N} E_Q[w_\tau q_j] = \int_{\alpha_i < \alpha_r} \frac{w_\tau}{i + 1} (\bar{A}_r - \alpha_i w_\tau) dQ \tag{A-4} \]

for \( i = 1, \ldots, N \). But the budget constraints for the \( N \) retailers also imply

\[ E_Q[w_\tau q_i] + \frac{1}{i + 1} \sum_{j=i+1}^{N} E_Q[w_\tau q_j] \leq B_i + \frac{1}{i + 1} \sum_{j=i+1}^{N} B_j \tag{A-5} \]

which, when combined with (A-4), leads to

\[ \int_{\alpha_i < \alpha_r} w_\tau (\bar{A}_r - \alpha_i w_\tau) dQ \leq (i + 1) B_i + \sum_{j=i+1}^{N} B_j \tag{A-6} \]
for \( i = 1, \ldots, N \). We can use (A-6) sequentially to determine the \( \alpha_i 's \). Beginning at \( i = N \), we see that the \( N^{th} \) retailer’s budget constraint is equivalent to

\[
\int_{\alpha_N < \alpha_r} w_r (\bar{A}_r - \alpha_N w_r) dQ \leq (N + 1) B_N. \tag{A-7}
\]

The optimality condition on \( \lambda_i \) implies that it is the smallest non-negative real that satisfies the \( i^{th} \) budget constraint. Since the optimal \( \lambda_i 's \) are non-decreasing in \( i \), we see from (A-5) that \( \alpha_i \) is therefore the smallest real greater than or equal to 1 satisfying the \( i^{th} \) budget constraint. Therefore, beginning with \( i = N \) we can check if \( \alpha_N = 1 \) satisfies (A-7) and if it does, then we know the \( N^{th} \) budget constraint is not binding. If \( \alpha_N = 1 \) does not satisfy (A-7) then we set \( \alpha_N \) equal to that value (greater than one) that makes (A-7) an equality. In particular, we obtain that the optimal value of \( \alpha_N \) is \( H((N + 1)B_N) \), as desired.

Note that if \( \alpha_N = 1 \) then none of the budget constraints are binding. In particular, this implies \( \alpha_i = 1 \) and \( \lambda_i = 0 \) for all \( i = 1, \ldots, N \). Moreover, (16) must be satisfied for all \( i \) since it is true for \( i = N \) and since the \( B_i 's \) are decreasing. Suppose now that the budget constraint is binding for retailers \( i + 1, \ldots, N \) and consider the \( i^{th} \) retailer. Then the \( i^{th} \) retailer’s budget constraint is equivalent\(^{32}\) to (A-6) and we can again use precisely the same argument as before to argue that (16) holds. □

**Proof of Proposition 2**: To compute the value of \( \Pi_r \) we will compute the expected revenue \( E_0^r[w_r Q] \) and expected cost, \( c_r E_0^r[Q] \), separately. The first step is to determine those retailers that will be using their entire budgets in the Cournot equilibrium. We know from Proposition 1 and the definition of \( m \) in (19) that only the budget constraints of the first \( m - 1 \) retailers will not be binding and so \( E_0^r[w_r q_i] = B_i \) for \( i = m, m + 1, \ldots, N \). It also follows that \( \lambda_1 = \lambda_2 = \cdots = \lambda_{m-1} = 1 \) and so (10) implies that

\[
q_i = (\bar{A}_r - Q - w_r)^+, \quad i = 1, \ldots, m - 1. \tag{A-8}
\]

Using this identity we obtain

\[
E_0^r[w_r Q] = \sum_{j=1}^{m-1} E_0^r[w_r q_j] + \sum_{j=m}^{N} E_0^r[w_r q_j]
\]

\[
= (m - 1) E_0^r[w_r (\bar{A}_r - Q - w_r)^+] + \sum_{j=m}^{N} B_j
\]

\[
= (m - 1) E_0^r[w_r (\bar{A}_r - w_r)^+] - w_r Q + \sum_{j=m}^{N} B_j \tag{A-9}
\]

where we have used the observation that \( (\bar{A}_r - Q - w_r)^+ = (\bar{A}_r - w_r)^+ - Q \). This observation follows because (i) if \( A_r \leq w_r \) then by (9) \( Q = 0 \) and (ii) if \( Q > 0 \) then \( A_r \geq w_r \) and we can argue using (10), say, that \( (\bar{A}_r - Q - w_r)^+ = \bar{A}_r - Q - w_r \). We can now re-arrange (A-9) to obtain

\[
E_0^r[w_r Q] = \frac{1}{m} \left( \sum_{j=m}^{N} B_j + (m - 1) E_0^r[w_r (\bar{A}_r - w_r)^+] \right). \tag{A-10}
\]

\(^{32}\)Equivalence follows because the second terms on either side of the inequality sign in (A-5) are equal by assumption.
In order to calculate the expected cost, we can use Lemma 1 with \( V_\tau \equiv 1 \) to see that for any \( i \) we have

\[
(i + 1) E_0^0[q_i] + \sum_{j=i+1}^{N} E_0^0[q_j] = E_0^0[(\bar{\alpha}_r - \alpha_i w_r)^+]. \tag{A-11}
\]

(A-11) then defines a system of \( N \) linear equations in the \( N \) unknowns, \( E_0^0[q_i], \quad i = 1, \ldots, N \). If we let \( M = [M_{ij}] \) be the \( N \times N \) upper-triangular matrix defined as

\[
M_{ij} := \begin{cases} 
0 & \text{if } i > j \\
i + 1 & \text{if } i = j \\
1 & \text{if } i < j 
\end{cases}
\]

then it is easy to check that \( [M_{ij}^{-1}] = 1_{\{j=i\}}/(j + 1) - 1_{\{j>i\}}/(j(j + i)) \). The system (A-11) then implies

\[
\sum_{i=1}^{N} E_0^0[q_i] = \sum_{i=1}^{N} \sum_{j=1}^{N} [M_{ij}^{-1}] E_0^0[(\bar{\alpha}_r - \alpha_j w_r)^+] = \sum_{j=1}^{N} \frac{1}{j(j + 1)} E_0^0[(\bar{\alpha}_r - \alpha_j w_r)^+] \tag{A-12}
\]

and so we obtain

\[
E_0^0[c_r Q] = \frac{(m - 1)c_r}{m} E_0^0[(\bar{\alpha}_r - w_r)^+] + \sum_{j=m}^{N} \frac{c_r}{j(j + 1)} E_0^0[(\bar{\alpha}_r - \alpha_j w_r)^+]. \tag{A-13}
\]

where we have used the fact that \( \alpha_1 = \ldots = \alpha_{m-1} = 1 \). We can now combine (A-10) and (A-13) to obtain (18) as desired. \( \square \)

**Proof of Proposition 4:** It is straightforward to see that \( Q_\tau \) is each of the \( N \) retailer’s optimal ordering level given the wholesale price menu, \( w_\tau \), in the absence of a budget constraint. This follows from a standard Cournot-style analysis in which all of the retailers, owing to their identical budgets, order the same quantity. In order to implement this solution, each retailer would need a budget \( Q_\tau, w_\tau \) for all \( \omega \in \Omega \). Therefore, if each retailer can generate a financial gain, \( G_\tau \), such that \( Q_\tau, w_\tau \) are large enough to cover the optimal purchasing cost for all \( \omega \in \mathcal{X} \). However, for \( \omega \in \mathcal{X}^c \), the initial budget is not sufficient. The financial gain, \( G_\tau \), then allows the retailer to transfer resources from \( \mathcal{X} \) to \( \mathcal{X}^c \).

Suppose the condition in Case 1 holds so that \( E_0^0[Q_\tau, w_\tau] \leq B \). Note that according to the definition of \( G_\tau \) in this case, we see that \( B + G_\tau = Q_\tau, w_\tau \) for all \( \omega \in \mathcal{X}^c \). For \( \omega \in \mathcal{X} \), however, \( B + G_\tau = (1 - \delta) B + \delta Q_\tau, w_\tau \geq Q_\tau, w_\tau \). The inequality follows since \( \delta \leq 1 \). \( G_\tau \) therefore allows the retailer to implement the unconstrained optimal solution. The only point that remains to check is that \( G_\tau \) satisfies \( E_0^0[G_\tau] = 0 \). This follows directly from the definition of \( \delta \).

Suppose now that the condition specified in Case 2 holds. We solve the \( i^{th} \) retailer’s optimization problem in (21) by relaxing the gain constraint with a Lagrange multiplier, \( \lambda_i \). We also relax the...
budget constraint for each realization of $X$ up to time $\tau$. The corresponding multiplier for each such realization is denoted by $\beta^{(i)}(t) dQ$ where $\beta^{(i)}(t)$ plays the role of a Radon-Nikodym derivative of a positive measure that is absolutely continuous with respect to $Q$. The first-order optimality conditions for the relaxed version of the retailer’s problem are then given by

$$q_i = \frac{(\bar{A}_r - w_r (1 + \beta^{(i)}(t)) - Q_i)^+}{2}$$

$$\beta^{(i)}(t) = \lambda_i,$$

$$\beta^{(i)}(w_r q_i - B + G_r) = 0, \quad \beta^{(i)}(t) \geq 0, \quad \text{and } E_Q^0[G_r] = 0.$$  

We look for a symmetric equilibrium of the above system of equations where $\lambda_i = \lambda$ and $q_i = q$ for all $i = 1, \ldots, N$.

It is straightforward to show that the solution given in Case 2 of the proposition satisfies these optimality conditions; only the non-negativity of $\beta^{(i)}(t)$ needs to be checked separately. To prove this, note that $\beta^{(i)}(t) = \lambda_i = \lambda$, therefore it suffices to show that $\lambda \geq 0$. This follows from three observations

(a) Since $0 \leq w_r$ the function $E_Q^0 \left[ w_r \left( \frac{\bar{A}_r - w_r (1 + \lambda)}{(N+1)} \right)^+ \right]$ is decreasing in $\lambda_i$.

(b) In Case 2, by hypothesis, we have

$$E_Q^0 \left[ w_r \frac{(\bar{A}_r - w_r)^+}{(N+1)} \right] = E_Q^0 [Q_r w_r] > B$$

(c) Finally, we know that $\lambda$ solves

$$E_Q^0 \left[ w_r \frac{(\bar{A}_r - w_r (1 + \lambda))}{(N+1)}^+ \right] = B.$$  

(a) and (b) therefore imply that we must have $\lambda \geq 0$. □

PROOF OF PROPOSITION 5: The statements regarding the producer follow immediately from Proposition 7 in Caldentey and Haugh (2009) with the budget replaced by $(N + 1)B/2$ and the objective function multiplied by $2N/(N + 1)$. The statements regarding the retailers are due to the fact that the optimal value of $\lambda$ in (26) is 0. This value of $\lambda$ and the optimal value of $w_r$ can then be substituted into the expression for the optimal ordering quantity in either\textsuperscript{33} Case 1 or Case 2 of Proposition 4. The expressions for $q_r$ and $\Pi_{R|r}$ then follow immediately. □

PROOF OF COROLLARY 1:

Using the definition of $\alpha_j$ for $j \geq m$ and assuming a constant $\bar{w}$, we see that the expectation $E_Q^0[(\bar{A}_r - \alpha_j w_r)^+]$ in equation (18) can be replaced by $((j + 1)B_j + B_{j+1} + \cdots + B_N)/\bar{w}$. The rest of the derivation of (30) follows directly after some simple calculations.

\textsuperscript{33}Both cases lead to the same value of $q_r$ as the producer chooses the price menu in such a way that the budget is at the cutoff point between being binding and non-binding with $\lambda = 0$.  

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Let us now prove the second part. For notational convenience, we will write \( \bar{w}^* := \bar{w}^*(B_1, \ldots, B_N) \) and \( \bar{w}^\infty := \bar{w}^*(\infty, \ldots, \infty) \). Note that the producer payoff can be rewritten as follows

\[
\Pi_p(\bar{w}) = \frac{m-1}{m} \Pi_p^\infty(\bar{w}) + \left(1 - \frac{c_r}{\bar{w}}\right) \sum_{j=m}^N B_j \frac{1}{m}, \tag{A-14}
\]

where \( \Pi_p^\infty(\bar{w}) := (\bar{w} - c_r) \mathbb{E}[(\bar{A}_r - \bar{w})^+] \) is the producer payoff when each retailers has an infinity budget (modulo the constant \((N-1)/N\)). It follows that \( \bar{w}^\infty \) is the unique maximizer of \( \Pi_p^\infty(\bar{w}) \). The uniqueness of \( \bar{w}^\infty \) follows from the fact that \( \Pi_p^\infty(\bar{w}) \) is unimodal. Hence, both \( \Pi_p^\infty(\bar{w}) \) and \( (1 - c_r/\bar{w}) \) are increasing functions of \( \bar{w} \) in \([c_r, \bar{w}^\infty)\). This observation together with the facts that \( m \) is a piece-wise constant function of \( \bar{w} \) and \( \Pi_p(\bar{w}) \) is a continuous function of \( \bar{w} \) imply that \( \bar{w}^* \geq \bar{w}^\infty \). The continuity of \( \Pi_p^\infty(\bar{w}) \) is not immediate since \( m \) is a discontinuous function of \( \bar{w} \). However, checking this property is straightforward and is left to reader. □

**Proof of Remark 1:** The proof is by contradiction so suppose\(^{34} \) \( \bar{w}_r > 0, \ m < n \), and that both \( n \) and \( m \) satisfy

\[
B_{n+1} < q^{(n)} w_r \leq B_n, \tag{A-15}
\]

\[
B_{m+1} < q^{(m)} w_r \leq B_m. \tag{A-16}
\]

If we use (33) to substitute for \( q^{(n)} \) and \( q^{(m)} \) in (A-15) and (A-16), and then rearrange terms we obtain

\[
(n+1) \frac{B_{n+1}}{w_r} + \sum_{j=n+1}^N \frac{B_j}{w_r} < \bar{A}_r - w_r \leq (n+1) \frac{B_n}{w_r} + \sum_{j=n+1}^N \frac{B_j}{w_r} \tag{A-17}
\]

\[
(m+1) \frac{B_{m+1}}{w_r} + \sum_{j=m+1}^N \frac{B_j}{w_r} < \bar{A}_r - w_r \leq (m+1) \frac{B_m}{w_r} + \sum_{j=m+1}^N \frac{B_j}{w_r} \tag{A-18}
\]

Since \( m < n \), however, we see that the ordering of the \( B_i \)'s implies the expression on the right of the second inequality in (A-17) is less than or equal to the expression on the left of the first inequality in (A-18). A contradiction follows immediately since we then obtain \( \bar{A}_r - w_r < \bar{A}_r - w_r \). □

**Proof of Proposition 7:** Let \( m_j, \alpha_i^{(j)} \) for \( i = 1, \ldots, N \) and \( j = 1, 2 \) denote the usual Cournot optimal quantities for scenario \( j \) and retailer \( i \). Since \( \mathbb{E}_0^0 [\bar{A}_{r_2}/x] = \bar{A}_{r_1}/x \) for any \( x > 0 \) Jensen’s Inequality implies

\[
\mathbb{E}_0^0 [w_r (\bar{A}_{r_2}/x - w_r)^+] \geq \mathbb{E}_0^0 [w_r (\bar{A}_{r_1}/x - w_r)^+]. \tag{A-19}
\]

After multiplying across (A-19) by \( x \) it then follows from the definition of the \( \alpha_i \)'s in (16) that

\[
\alpha^{(2)}_i \geq \alpha^{(1)}_i \quad \text{for } i = 1, \ldots, N.
\]

This in turn implies that \( m_2 \leq m_1 \). Let \( \Pi_p^{(1)} \) and \( \Pi_p^{(2)} \) denote the producer’s expected revenue in

\(^{34}\)When \( \bar{w}_r = 0 \), none of the budget constraints are binding and we obtain the standard Cournot equilibrium with \( n = N \) in (33).
scenarios (1) and (2) respectively. Then (A-10) implies

\[
\Pi_p^{(2)} - \Pi_p^{(1)} = \frac{(m_1 - m_2)}{m_1 m_2} \sum_{j=m_1}^N B_j + \frac{1}{m_2} \sum_{j=m_2}^{m_1-1} B_j + \frac{(m_2 - 1)}{m_2} \mathbb{E}_0^0[w_r (\hat{A}_{r_2} - w_r)] \\
- \frac{(m_1 - 1)}{m_1} \mathbb{E}_0^0[w_r (\hat{A}_{r_1} - w_r)]
\]

\[
\geq \frac{(m_1 - m_2)}{m_1 m_2} \sum_{j=m_1}^N B_j + \frac{1}{m_2} \sum_{j=m_2}^{m_1-1} B_j + \frac{(m_2 - m_1)}{m_1 m_2} \mathbb{E}_0^0[w_r (\hat{A}_{r_1} - w_r)]
\]

\[
= \frac{(m_1 - m_2)}{m_1 m_2} \left[ \sum_{j=m_1}^N B_j - \mathbb{E}_0^0[w_r (\hat{A}_{r_1} - w_r)] \right] + \frac{1}{m_2} \sum_{j=m_2}^{m_1-1} B_j
\]

\[
\geq \frac{(m_1 - m_2)}{m_1 m_2} \left[ \sum_{j=m_1}^N B_j - (m_1 + 1)B_{m_1} - \cdots - B_N \right] + \frac{1}{m_2} \sum_{j=m_2}^{m_1-1} B_j \quad \text{(by def. of } m_1) \]

\[
= \frac{(m_2 - m_1)}{m_2} B_{m_1} + \frac{1}{m_2} \sum_{j=m_2}^{m_1-1} B_j. \quad \text{(A-20)}
\]

Note that the right-hand-side of (A-20) equals zero if \( m_1 = m_2 \). Otherwise \( m_2 < m_1 \) and the right-hand-side of (A-20) is greater than or equal to \( (m_1 - m_2)(B_{m_2} - B_{m_1})/m_2 \) which in turn is non-negative. The result therefore follows. □

**Proof of Proposition 8:**

(a) We will prove a more general result where we compare the \( N \)-retailer case to the \( (N-1) \)-retailer case that is created by merging the largest two retailers. We will use the notation in Proposition 2 to describe quantities associated with the \( N \) retailer case and use the ‘hat’ notation to describe quantities associated with the \( (N-1) \)-retailer case. We must therefore show that \( \Pi_p - \hat{\Pi_p} \geq 0 \). Proposition 2 then implies

\[
\Pi_p - \hat{\Pi_p} = \frac{(m - \hat{m})}{m \hat{m}} \mathbb{E}_0^0[w_r (\hat{A}_r - w_r)] + \sum_{j=m}^N \frac{B_j}{m} - \sum_{j=m}^{N-1} \frac{\hat{B}_j}{\hat{m}}. \quad \text{(A-21)}
\]

Note that \( \hat{B}_1 = B_1 + B_2 \) and that \( \hat{B}_i = B_{i+1} \) for \( i = 2, \ldots, N - 1 \). We also define \( C_i \) and \( \hat{C}_i \) as

\[
C_i := (i + 1)B_i + \cdots + B_N \quad \text{for } i = 1, \ldots, N
\]

\[
\hat{C}_i := \begin{cases} 
2(B_1 + B_2) + \cdots + B_N, & \text{for } i = 1 \\
(i + 1)B_{i+1} + \cdots + B_N, & \text{for } i = 2, \ldots, N - 1.
\end{cases}
\]

Note that \( C_i \) and \( \hat{C}_i \) are the arguments of the function, \( H \), that defines the corresponding \( \alpha_i \)'s and \( \hat{\alpha}_i \)'s in (16). By the ordering assumption on the budgets we see that \( C_1 \leq \hat{C}_1 \) which implies \( \alpha_1 \geq \hat{\alpha}_1 \). Similarly for \( i = 2, \ldots, N - 1 \) we have \( C_i \geq \hat{C}_i \) which implies \( \alpha_i \leq \hat{\alpha}_i \). In fact it is also clear that \( C_{i+1} \geq \hat{C}_i \) for \( i \geq 2 \) so that \( \alpha_{i+1} \leq \hat{\alpha}_i \). We now consider the various possible values of \( m \) and \( \hat{m} \). The following cases follow from our previous observations:
1. \( m = 1 \): in this case all of the budget constraints in the original system are binding. The only possible values of \( \hat{m} \) are 1 and 2. In particular \( \hat{m} - m \in \{0,1\} \).

2. \( m = 2 \): in this case only the first budget constraint in the original system is non-binding, \( \hat{m} \) must also equal 2 and so \( \hat{m} - m = 0 \).

3. \( 3 \leq m \leq N + 1 \): in this case at least three budget constraints in the original system are non-binding, \( \hat{m} \) can take on any value in \( \{2,\ldots,m-1\} \) and \( \hat{m} - m \in \{2-m,\ldots,-1\} \).

We now prove the result:

**Case (i):** Suppose \( m = 1 \) and \( \hat{m} = 2 \). Then (A-21) reduces to

\[
\Pi_p - \hat{\Pi}_p = -\frac{1}{2} \mathbb{E}_0^q[w_r (\hat{A}_r - w_r)^+] + \sum_{j=1}^{N} B_j - \frac{N-1}{2} \hat{B}_j \geq 0 \quad (A-22)
\]

since \( \hat{m} = 2 \), implying the first constraint is the new system is non-binding.

**Case (ii):** Suppose \( m = \hat{m} \). Then (A-21) clearly implies \( \Pi_p - \hat{\Pi}_p \geq 0 \). Together with Case (i), we have now covered the first two possibilities above.

**Case (iii):** Suppose \( m \geq 3 \) so that \( \hat{m} - m < 0 \). Then (A-21) implies

\[
\Pi_p - \hat{\Pi}_p = \frac{(m - \hat{m})}{m \hat{m}} \mathbb{E}_0^q[w_r (\hat{A}_r - w_r)^+] + \left( \sum_{j=m}^{N} B_j \right) \left( \frac{1}{m} - \frac{1}{\hat{m}} \right) - \frac{1}{m} \sum_{j=m+1}^{m-1} B_j \\
= \frac{(m - \hat{m})}{m \hat{m}} \left[ \mathbb{E}_0^q[w_r (\hat{A}_r - w_r)^+] - \sum_{j=m}^{N} B_j \right] - \frac{1}{m} \sum_{j=m+1}^{m-1} B_j \\
\geq \frac{(m - \hat{m})}{m \hat{m}} \left[ (\hat{m} + 1)B_{\hat{m}+1} + \cdots + B_N - \sum_{j=m}^{N} B_j \right] - \frac{1}{m} \sum_{j=m+1}^{m-1} B_j \quad (A-23)
\]

where (A-23) follows since \( \hat{\alpha}_{\hat{m}} > 1 \) and so \( \mathbb{E}_0^q[w_r (\hat{A}_r - w_r)^+] \geq \hat{C}_m \). Note that the right-hand-side of (A-23) equals \( B_{\hat{m}}/m > 0 \) if \( \hat{m} + 1 = m \). Otherwise the right-hand-side of (A-23) equals

\[
\frac{(m - \hat{m})}{m \hat{m}} \left[ (\hat{m} + 1)B_{\hat{m}+1} + \cdots + B_{m-1} \right] - \frac{1}{m} \sum_{j=\hat{m}+1}^{m-1} B_j \\
= \frac{1}{\hat{m}} \left[ (\hat{m} + 1)B_{\hat{m}+1} + \cdots + B_{m-1} \right] - \frac{1}{m} \left[ (\hat{m} + 1)B_{\hat{m}+1} + \cdots + B_{m-1} \right] - \frac{1}{m} \sum_{j=\hat{m}+1}^{m-1} B_j \\
= B_{\hat{m}+1} - \frac{1}{m} \left[ (\hat{m} + 1)B_{\hat{m}+1} + \cdots + B_{m-1} \right] \\
\geq B_{\hat{m}+1} - \frac{m-1}{m} B_{\hat{m}+1} = B_{\hat{m}+1}/m > 0.
\]

and so the result follows. \( \square \)
**Proof of Proposition 9:** Let \( f(B) \) be the producer’s optimal expected profits as a function of the retailers’ common budget. Then

\[
f(B) := \max_{w_r, \lambda \geq 0} \mathbb{E}_0^0 \left[ (w_r - c_r) \left( \frac{A_r - w_r (1 + \lambda)}{2} \right)^+ \right]
\]

subject to \( \mathbb{E}_0^0 \left[ w_r \left( \frac{A_r - w_r (1 + \lambda)}{2} \right)^+ \right] \leq B. \)

From (26) and (27) we see that the expected payoff, \( \Pi_p \), that the producer achieves by serving \( N \) competing retailers with individual budget \( B \) is therefore equal to

\[
\Pi_p = \frac{2N}{N+1} f \left( \frac{(N+1)}{2} B \right).
\]

When there is just one retailer with a budget of \( NB \), the producer’s expected profit, \( \bar{\Pi}_p \), say, satisfies \( \bar{\Pi}_p = f(NB) \). In order to prove the proposition we must therefore show that for all \( B \geq 0 \) and \( N \)

\[
f(NB) \leq \frac{2N}{N+1} f \left( \frac{(N+1)}{2} B \right).
\]

Since \( f(0) = 0 \), a sufficient condition for this inequality to hold is that \( f(B) \) is a concave function in \([0, \infty)\). We can use the results in Proposition 9 to rewrite \( f(B) \) as

\[
f(B) = \mathbb{E}_0^0 \left[ \frac{(A_r + (1 + \phi) c_r - 2 c_r) (A_r - (1 + \phi) c_r)^+}{8} \right] \tag{A-24}
\]

where \( \phi \) is the positive root of the equation \( \mathbb{E}_0^0 \left[ (\frac{A_r^2}{2} - (\phi c_r)^2) \right] = 8 B \). We will prove the concavity of \( f(B) \) by first proving it for a discrete approximation to \( \bar{A}_r \), and then using a convergence argument to prove it for \( \bar{A}_r \). Specifically, if we define \( A_\theta := \theta [\bar{A}_r / \theta] \) for an arbitrary \( \theta > 0 \), then \( A_\theta \) takes values in \( \{0, \theta, 2\theta, \ldots\} \) and

\[
\mathbb{Q}(A_\theta = k \theta) = \mathbb{Q}(k \theta \leq \bar{A}_r < (k+1) \theta) \quad \text{for} \quad k = 0, 1, 2 \ldots .
\]

We define the auxiliary function

\[
f_\theta(B) := \mathbb{E}_0^0 \left[ \frac{(A_\theta + (1 + \phi_\theta) c_r - 2 c_r) (A_\theta - (1 + \phi_\theta) c_r)^+}{8} \right] \tag{A-25}
\]

where \( \phi_\theta \) is the positive root of the equation \( \mathbb{E}_0^0 \left[ (A_\theta^2 - (\phi_\theta c_r)^2) \right] = 8 B \). Since \( \lim_{\theta \downarrow 0} A_\theta = \bar{A}_r \), we can apply the Dominated convergence Theorem\(^{35}\) to see that \( \lim_{\theta \downarrow 0} f_\theta(B) = f(B) \). And since the limit of concave functions is itself concave, it therefore suffices to prove the concavity of \( f_\theta(B) \). To show this, let us define \( \bar{B} \) such that \( \phi_\theta \geq 1 \) for all \( B \leq \bar{B} \). It follows that \( f_\theta(B) = f_\theta(\bar{B}) \) for all \( B \geq \bar{B} \). Hence, since \( f_\theta(B) \) is continuous and nondecreasing, we only need to prove that \( f_\theta(B) \) is concave in the domain \([0, \bar{B}]\).

For \( B \leq \bar{B} \) the budget constraint in the definition of \( f_\theta(B) \) is tight and so we can rewrite \( f_\theta(B) \) as

\[
f_\theta(B) = B - \frac{c_r}{4} H_\theta(B), \quad B \leq \bar{B},
\]

\(^{35}\)The expressions inside the expectations in (A-24) and (A-25) are dominated by \( \mathbb{E}_0^0 \left[ \frac{A_r^2}{8} \right] \) so that it is sufficient for convergence that \( A_r \) has a second moment.
where

$$H_\theta(B) := \mathbb{E}_0^Q [(A_\theta - \delta)^+] \quad \text{and} \quad \delta \geq 1 \text{ solves } \mathbb{E}_0^Q \left[ (A_\theta^2 - \delta^2)^+ \right] = 8B.$$ 

For a fixed $B \in [0, \bar{B}]$, let us define $\delta_\theta(B) \geq 0$ as the positive root in the budget constraint above and define a sequence of budgets $0 = B_0 \leq B_1 \leq \cdots \leq B_m = \bar{B}$ such that $k_i := \lceil \delta_\theta(B_i)/\theta \rceil$ for all $B \in [B_i, B_{i+1})$. It then follows that for all $B \in [B_i, B_{i+1})$,

$$\delta_\theta(B) = \sqrt{\sum_{k \geq k_i} (k \theta)^2 \mathbb{Q}(A_\theta = k \theta) - 8B \over \mathbb{Q}(A_\theta \geq k_i \theta)} \quad \text{and} \quad H_\theta(B) = \sum_{k \geq k_i} k \theta - \delta_\theta(B) \mathbb{Q}(A_\theta \geq k_i \theta).$$

We therefore see that $H_\theta(B)$ is convex in each interval, $[B_i, B_{i+1})$, since $\delta_\theta(B)$ is concave in these intervals. To complete the proof, it suffices to show that $H_\theta(B)$ is continuously differentiable in $[0, \bar{B}]$ which is equivalent to showing that $H_\theta(B)$ is differentiable at each $B_i$, $i = 1, \ldots, m$. Since $\delta_\theta(B)$ is continuous in $B$ by construction, the continuously differentiability of $H_\theta(B)$ follows by observing that the derivative of $\delta_\theta(B)$ with respect to $B$ is proportional to $1/\delta_\theta(B)$. □

### B Appendix: Further Details on Extensions to Basic Model

#### B.1 Martingale Pricing with Foreign Assets

Martingale pricing theory states that the time 0 value, $G_0$, of a security that is worth $G_\tau$ at time $\tau$ and does not pay any intermediate cash-flows, satisfies $G_0/N_0 = \mathbb{E}_0^Q[G_\tau/N_\tau]$ where $N_t$ is the time $t$ price of the numeraire security and $\mathbb{Q}$ is an equivalent martingale measure (EMM) associated with that numeraire. It is common to take the cash account as the numeraire security and this is the approach we have followed in most of this paper. With the exception of Section 5.3, however, the value of the cash account at time $t$ was always $\$1$ since we assumed interest rates were identically zero. We therefore had $G_0 = \mathbb{E}_0^Q[G_\tau]$ and since we insisted $G_0 = 0$ we obtained $\mathbb{E}_0^Q[G_\tau] = 0$. When interest rates are non-zero we still have $N_0 = 1$ but now $N_\tau = \exp \left( \int_0^\tau r_s \, ds \right)$ and so, for example, we have (41). In the main text we take $D_t = N_t^{-1}$. See Duffie (2004) for a development of martingale pricing theory.

**Martingale Pricing with Foreign Assets**

Suppose now that there is a domestic currency and a foreign currency with $Z_t$ denoting the exchange rate between the two currencies at time $t$. In particular, $Z_t Y_t$ is the time $t$ domestic currency value of a foreign asset that has a time $t$ foreign currency value of $Y_t$. Let $B_t^{(f)}$ denote the time $t$ value of the foreign cash account and let $\mathbb{Q}$ denote the EMM of a foreign investor taking the foreign cash account as numeraire. This implies

$$\mathbb{E}_t^\mathbb{Q} \left[ {Y_T \choose B_T^{(f)}} \right] = {Y_t \choose B_t^{(f)}} \quad \text{(B-26)}$$

for all $t \leq T$ and where we assume again that the asset with foreign currency value $Y_t$ at time $t$
does not\footnote{If it did pay intermediate cash-flows between $t$ and $T$ then they would have to be included inside the expectation in (B-26).} pay any intermediate cash flows. But equation (B-26) can be re-written as

\[
E_t^Q \left[ \frac{Z_T Y_T}{Z_T B_T^{(f)}} \right] = \frac{Z_t Y_t}{Z_t B_t^{(f)}}. \tag{B-27}
\]

Note that $Z_t Y_t$ is the domestic currency value of the foreign asset and $Z_t B_t^{(f)}$ is the domestic currency value of the foreign cash account. Note also that if $V_t$ is the time $t$ price of a domestic asset then $V_t/Z_t$ is the foreign currency value of the asset at time $t$. We therefore obtain by martingale pricing that

\[
E_t^Q \left[ \frac{V_T/Z_T}{B_T^{(f)}} \right] = \frac{V_t/Z_t}{B_t^{(f)}}
\]

or equivalently,

\[
E_t^Q \left[ \frac{V_T}{Z_T B_T^{(f)}} \right] = \frac{V_t}{Z_t B_t^{(f)}}. \tag{B-28}
\]

We can therefore conclude from (B-27) and (B-28) that $Q$ is also the EMM of a domestic investor, but now with the \textit{domestic} value of the foreign cash account as the corresponding. We use these observations in Section 5.2.