Thus far:

\[ CA = S - I \]

\( S = \) savings; We have considered adjustments in desired savings by private households due to income shocks and interest rate shocks.

\( I = \) investment in physical capital; We have considered adjustments in desired investment due to productivity and interest rate shocks.

This chapter:

Savings might change because of fiscal deficits. And more generally, how do fiscal imbalances affect the current account.
The General Idea

\[ CA = S - I \]
\[ S = S^p + S^g \]

\( S^p \) = private savings
\( S^g \) = government savings (i.e., fiscal surplus)

The twin deficits hypothesis says: if \( S^g \downarrow \), then \( CA \downarrow \)

But what if \( S^g \downarrow \) results in \( S^p \uparrow \)?

What do the data show?

What does theory say?
Let’s address the first question first.

What do the data show about the joint occurrence of current account and fiscal deficits?
The Genesis of the Twin Deficit Hypothesis

The Reagan Fiscal Imbalances and the Beginning of Current Account Deficits
• Large current account deficits open up in the early 1980s
... and at the same time the U.S. fiscal surplus, $S^g$, declines:
Putting these two graphs together and subtracting the respective 1981 values from $CA_t$ and $S^g_t$, we see that between 1981 and 1984 both fell by about $100$ billion.
The joint occurrence of fiscal deficits and current account deficits leads to the hypothesis that fiscal deficits cause current account deficits.

This hypothesis is known as the **Twin Deficit Hypothesis**.
We next ask whether this is a general regularity, that is, is it true that whenever we observed large changes in government savings they lead to similar changes in the current account.

Consider three important fiscal episodes:

1. The large fiscal deficits of World War II

2. The fiscal surpluses during the Clinton Era

3. The fiscal deficits during the Great Contraction of 2008
Twin Deficits During World War II? — No.

No Twin Deficits During WWII

- $S^g / GDP$
- $CA / GDP$
Twin Deficit Hypothesis During The Clinton Era Surpluses?  
— No.

No Twin Surpluses: The Clinton Era

- $S^g / GDP$
- $CA / GDP$


%
Twin Deficits During The Great Contraction? — No.
Twin Deficits: The Big Picture
The data seems to suggest that sometimes large changes in government saving are reflected in similar changes in the current account balance and sometimes not.

How can we understand this? Let’s turn back to our model.

To understand the link between government savings and the current account we introduce a government sector into the two-period endowment economy studied in chapter 3.
A Model of Current Account Determination with a Government Sector

- two-period, small open endowment economy
- $G_1, G_2$, government consumption in periods 1 and 2
- $T_1, T_2$, taxes in periods 1 and 2
- $B^g_t$, government asset holdings in periods $t = 0, 1, 2$

If $B^g_t < 0$, then the government is indebted, and if $B^g_t > 0$ the government is a creditor.
Government Budget Constraints

Uses of funds:
- government spending, $G_t$
- interest service on the debt, $-r_{t-1}B^g_{t-1}$

Sources of funds:
- tax revenues, $T_t$
- issuance of new debt, $-(B^g_t - B^g_{t-1})$
Period-1 budget constraint of the government:

\[ G_1 - r_0 B^g_0 = T_1 - (B^g_1 - B^g_0) \]

Period-2 budget constraint of the government:

\[ G_2 - r_1 B^g_1 = T_2 - (B^g_2 - B^g_1) \]

Borrowing limit: \( B^g_2 = 0 \)
Some definitions:

Primary fiscal deficit = $G_1 - T_1$.

Secondary fiscal deficit = $G_1 - T_1 - r_0 B_0^g = -S_1^g$.

The change in the secondary fiscal deficit is given by

$$\Delta(-S_1^g) = \Delta G_1 - \Delta T_1 - \Delta(r_0 B_0^g)$$

All else constant, an increase in government spending, $\Delta G_1 > 0$, or a tax cut, $\Delta T_1 < 0$, raise the fiscal deficit.
Combine the government’s period-by-period budget constraints to obtain a single present value budget constraint

\[ G_1 + \frac{G_2}{1 + r_1} = T_1 + \frac{T_2}{1 + r_1} + (1 + r_0)B_0^g \]

The LHS represents the present discounted value of government spending.

The RHS represents the present discounted value of tax revenues plus initial government assets.
Households

This part of the model is based on the model without a government sector we studied in chapter 3. The only difference is that we replace the endowment, $Q_t$, with the after tax endowment (or disposable income), $Q_t - T_t$.

Period-1 budget constraint of the household:

$$C_1 + B_{11}^p - B_{10}^p = Q_1 - T_1 + (1 + r_0)B_{10}^p$$

Period-2 budget constraint of the household:

$$C_2 + B_{22}^p - B_{12}^p = Q_2 - T_2 + (1 + r_1)B_{12}^p,$$

where $B_{it}^p$ denotes bond holdings of private households at the end of period $t$.

The transversality condition is $B_{22}^p = 0$.
Combine the period-by-period budget constraint of the household to obtain a single present value budget constraint:

\[
C_1 + \frac{C_2}{1 + r_1} = Q_1 + \frac{Q_2}{1 + r_1} + (1 + r_0)B_0^p - T_1 - \frac{T_2}{1 + r_1}
\]

Notice that the only difference to the PVBC in an economy without a government sector is the term \( T_1 + T_2/(1 + r_1) \), which represents the present discounted value of taxes. In the space \((C_1, C_2)\), the intertemporal budget constraint of the household continues to be a straight downward sloping line with slope given by \(-(1 + r_1)\).
The household’s optimization problem is to

$$\max U(C_1, C_2)$$

subject to the intertemporal budget constraint derived above, that is,

$$C_1 + \frac{C_2}{1 + r_1} = Q_1 + \frac{Q_2}{1 + r_1} + (1 + r_0)B_0^p - T_1 - \frac{T_2}{1 + r_1}.$$ 

The household chooses a basket of consumption satisfying

$$U_1(C_1, C_2) = (1 + r_1)U_2(C_1, C_2)$$

As we discussed before, this condition states that at the optimal consumption choice the indifference curve has a slope $-(1 + r_1)$, the same slope as the intertemporal budget constraint.
Equilibrium

Combining the present value budget constraint of the government with the present value budget constraint of the household we obtain the following present value resource constraint of the country:

\[ C_1 + \frac{C_2}{1 + r_1} + G_1 + \frac{G_2}{1 + r_1} = Q_1 + \frac{Q_2}{1 + r_1} + (1 + r_0)(B_0^p + B_0^g) \]

It says that the present value of private and public consumption must equal the present value of the endowments plus initial foreign wealth of the country, \((1 + r_0)(B_0^p + B_0^g)\).
In equilibrium it must be true that:

\[ r_1 = r^*. \]

That is, the presence of a government sector does not alter the fact that the domestic interest rate must be equal to the world interest rate \( r^* \). It follows that in the small open economy fiscal deficits will not drive up interest rates.
Assume that $G_1$ and $G_2$ are exogenously given. Then an equilibrium in the small open endowment economy with a government are values for \{\(C_1, C_2, r_1, T_1, T_2\)\} satisfying

\[
C_1 + \frac{C_2}{1 + r_1} = Q_1 - T_1 + \frac{Q_2 - T_2}{1 + r_1} + (1 + r_0)B^p_0 \tag{1}
\]

\[
U_1(C_1, C_2) = (1 + r_1)U_2(C_1, C_2) \tag{2}
\]

\[
r_1 = r^* \tag{3}
\]

\[
G_1 + \frac{G_2}{1 + r_1} = T_1 + \frac{T_2}{1 + r_1} + (1 + r_0)B^g_0 \tag{4}
\]

given $G_1$, $G_2$, $r^*$, $B^p_0$, and $B^g_0$.

Q: There are 5 unknowns, \{\(C_1, C_2, r_1, T_1, T_2\)\}, but only 4 equilibrium conditions, how can this be? Notice that the equilibrium conditions depend only on the present discounted value of taxes, $T_1 + \frac{T_2}{1 + r_1}$, and thus only the present discounted value of taxes is uniquely determined, but $T_1$ and $T_2$ individually are not.
Combining the present value budget constraints of the household and the government, equations (1) and (4) yields

\[ C_1 + \frac{C_2}{1 + r_1} = Q_1 - G_1 + \frac{Q_2 - G_2}{1 + r_1} + (1 + r_0)(B^p_0 - B^g_0) \]

The right-hand side of this expression is exogenously given. Let \( \tilde{Y} = Q_1 - G_1 + \frac{Q_2 - G_2}{1 + r_1} + (1 + r_0)(B^p_0 - B^g_0) \). Then the equilibrium conditions collapse to

\[ C_1 + \frac{C_2}{1 + r_1} = \tilde{Y} \]  
\[ U_1(C_1, C_2) = (1 + r_1)U_2(C_1, C_2) \]  
\[ r_1 = r^* \]

But these are the same equilibrium conditions as those of the endowment economy of chapter 3, with the only difference that the definition of the exogenous variable \( \tilde{Y} \) is different. This implies that we can use the same graphical approach as in chapter 3 to find the optimal consumption allocation.
Optimal consumption choice in the economy with a government sector

\[ C_2 = (1 + r^*) (Q_1 - C_1 - G_1) + Q_2 - G_2 \]

\[ (B_0^p + B_0^g = 0) \]
The figure depicts the equilibrium at point A.

Notice that the economy’s resource constraint depends only on $G_1$ and $G_2$ and is independent of $T_1$ and $T_2$. Hence the timing of taxes is irrelevant for the optimal allocation.

From here it follows that tax cuts that lead to an increase in the fiscal deficit, $G_1 - T_1$, will have no real effects and will not lead to a current account deterioration. That is, they do not give rise to the Twin Deficit phenomenon.

Let’s derive this result in more detail ...
The Effect of a Tax Cut on the Current Account

Experiment: A tax cut in period 1 combined with no change in government spending, that is, $\Delta T_1 < 0$, and $\Delta G_1 = \Delta G_2 = 0$.

Consider first the present value budget constraint of the government:

$$G_1 + \frac{G_2}{1 + r_1} = T_1 + \frac{T_2}{1 + r_1} + (1 + r_0)B^g_0$$

WLG assume $B^g_0 = 0$. Use equilibrium condition $r_1 = r^*$ and consider changes.
Then

\[ \Delta G_1 + \frac{\Delta G_2}{1 + r^*} = \Delta T_1 + \frac{\Delta T_2}{1 + r^*} \]

\[ 0 + 0 = \Delta T_1 + \frac{\Delta T_2}{1 + r^*} \]

This says that tax cut in period 1 must leave the present discounted value of taxes unchanged. This in turn requires that

\[ \Delta T_2 = -(1 + r^*) \Delta T_1 > 0 \]

That is, a tax cut in period 1 leads to a tax increase in period 2.
Now recall the household’s present value budget constraint

\[ C_1 + \frac{C_2}{1 + r_1} = Q_1 + \frac{Q_2}{1 + r_1} + (1 + r_0)B^p_0 - T_1 - \frac{T_2}{1 + r_1} \]

It depends only on the present discounted value of taxes, and hence the tax cut in period 1 has no effect on the present value budget constraint of the household. It follows that:

\[ \Delta C_1 = 0 \]

And hence from the budget constraint in period 1 that households will save the entire tax cut. From the definition of private savings \( S^p_1 = Q_1 - T_1 - C_1 + r_0B^p_0 \), it follows that

\[ \Delta S^p_1 = -\Delta T_1 > 0 \]
Why do households choose to save the entire tax cut, why don’t they consume at least some of it?

The reason is that households understand that taxes will increase in period 2. Thus they save more to avoid a cut in consumption in period 2.

The result that a tax cut in the current period that leaves government spending unchanged has no real effects, that is, leaves the consumption allocation unchanged, is known as **Ricardian Equivalence**.
What is the effect of the tax cut on the current account?

Find national savings:

National Saving \( = S_1 = S^p_t + S^g_1 \)

Recall that \( \Delta S^g_1 = \Delta T_1 \) and \( \Delta S^p_1 = -\Delta T_1 \).

\[ \Delta S_1 = \Delta S^p_1 + \Delta S^g_1 = -\Delta T_1 + \Delta T_1 = 0 \]
Use

\[ CA_1 = S_1 \]

so that

\[ \Delta CA_1 = \Delta S_1 = 0 \]
What have we shown?

We have shown that a tax cut in period 1 that leaves government spending unchanged:
– leads to a fiscal deficit in period 1, $\Delta S^g_1 = \Delta T_1 < 0$
– but does not result in a decline in national savings because private savings increase one for one with the tax cut.
– does not lead to a current account deficit

$\Rightarrow$ in this model the Twin Deficit Hypothesis does not hold.

We thus have the following important result:

**When Ricardian Equivalence holds and the fiscal deficit is the result of a tax cut, then the Twin Deficit Hypothesis fails.**
Taking stock:

If the model of Ricardian Equivalence represents an adequate description of how the economy works and if the main cause of the fiscal deficits of the 1980s was the Reagan tax cuts, then what we should have observed is a decline in public savings, an offsetting increase in private savings, and no change either in national savings or the current account.

What does the data show? In the 1980s there was a significant cut in taxes. As predicted by theory, the tax cuts were accompanied by a significant decline in public savings.
However, contrary to the predictions of Ricardian Equivalence, private savings did not increase by the same amount as the decline in public savings and as a result both national savings and the current account plummeted.

We therefore conclude that either the fiscal deficits of the 1980s were caused by factors other than the tax cuts, such as increases in government spending, or Ricardian Equivalence does not hold, or both.
Assume now that the fiscal deficit is not the result of a tax cut, but instead is brought about by an increase in government spending.

Experiment: $G_1 \uparrow$ and $\Delta G_2 = 0$. 
Adjustment to a temporary increase in government purchases

\[ C_2 = (1 + r^*)(Q_1 - C_1 - G_1) + Q_2 - G_2 \]
According to the analysis of this graph, consumption falls in response to a temporary increase in government spending, but by less than the increase in government spending,

$$\Delta G_1 > 0 \Rightarrow 0 > \Delta C_1 > -\Delta G_1.$$ 

This means that the trade balance, which is given by \(Q_1 - C_1 - G_1\) deteriorates but by less than the increase in government spending.

$$\Delta TB_1 = \Delta Q_1 - \Delta (G_1 + C_1) = 0 - \Delta (G_1 + C_1) < 0.$$ 

Also, the current account, which is given by \(CA_1 = TB_1 + r_0B_0^*\) deteriorates but by less than the increase in government spending.
Let’s see if we could explain the size in the decline in the current account in the early 1980s by the observed size of the increase in government spending.

Reagan’s military buildup of the early 1980s represented an increase in gov’t spending of about 1.5% of GDP. ($\Delta G_1 = 0.015 \times GDP$)

Theory tells us that this should be associated with a deterioration of the current account. But the theory also tells us that the deterioration of the current account should be less than the increase in gov’t spending, i.e., less than 1.5% of GDP.

Yet during this period the current account deteriorated by 3% of GDP. Thus, there is at least 1.5% of GDP of current account deficit that is not accounted by the military build up.

We saw that if Ricardian equivalence holds, the Reagan tax cut of the early 1980s, which did amount to about 1.5% of GDP,
cannot explain the 1.5% deterioration in the current account that the military buildup can’t explain.

But what if Ricardian equivalence doesn’t hold? Could it be that in this case the 1.5%-of-GDP Reagan tax cut explain the 1.5% current account deterioration that the military buildup leaves unexplained?

We turn to this issue next.
Failure Of Ricardian Equivalence

Three Reasons Why It May Fail

1. Borrowing Constraints

2. Intergenerational Effects

3. Distortionary Taxation
1.) Borrowing Constraints

Assume that private households face borrowing constraints

Period 1 budget constraint: \( C_1 + B^p_1 = Q_1 - T_1 \).

Period 1 borrowing constraint: \( B^p_1 \geq 0 \).

Assume further that in period 1 the borrowing constraint is binding, that is, assume that \( B^p_1 = 0 \).

Consider now a tax cut, \( \Delta T_1 < 0 \). Then \( \Delta C_1 = -\Delta T_1 > 0 \) and \( \Delta S^p_1 = \Delta Q_1 - \Delta T_1 - \Delta C_1 = 0 \), that is, households consume the tax cut rather than save it (which is what they would do in the absence of binding borrowing constraints).
Adjustment to a temporary tax cut when households are liquidity constrained
What is the effect on national savings? \( S_1 = S^g_1 + S^p_1 \), and we have \( \Delta S_1 = \Delta S^g_1 + \Delta S^p_1 = \Delta T_1 < 0 \). National savings fall.

What is the effect of the tax cut on the current account?

\[
\Delta CA_1 = \Delta S_1 = \Delta T_1 < 0
\]

It deteriorates, and we observe a Twin Deficit.

Note that for a tax cut of $100 to lead to a current account deterioration of the same magnitude, we need that 100% of households benefiting from the tax cut are borrowing constraint.
2.) Intergenerational Effects

Generation that benefits from the tax cut not the same as the one that pays for the future tax increases.

Assume that households are one-period lived.

Generation alive in period 1:
\[ C_1 = Q_1 - T_1 \quad \rightarrow \quad \Delta C_1 = -\Delta T_1 \]

Generation alive in period 2:
\[ C_2 = Q_2 - T_2 \quad \rightarrow \quad \Delta C_2 = -\Delta T_2 \]

From the government’s budget constraint:
\[ G_1 + \frac{G_2}{1 + r_1} = T_1 + \frac{T_2}{1 + r_1} \]
Policy Experiment:

Tax cut in period 1: $\Delta T_1 < 0$ and no change in government spending: $\Delta G_1 = \Delta G_2 = 0$

By govt budget constraint: $\Delta T_1 = -\frac{\Delta T_2}{1 + r_1}$

In period 1:
$\Delta S^p_1 = \Delta Q_1 - \Delta T_1 - \Delta C_1$
$\Delta S^g_1 = \Delta T_1$
$\Rightarrow \Delta S_1 = \Delta S^p_1 + \Delta S^g_1 = \Delta T_1 < 0$, national savings declines same as gov’t savings

and hence
$\Delta CA_1 = \Delta S_1 = \Delta S^g_1 < 0$

The tax cut gives rise to a fiscal deficit and a current account deterioration of the same magnitude, generating a **Twin Deficit**.
3.) Distortionary Taxation

2-period endowment economy
free capital mobility

New: proportional consumption taxes in period 1

HH budget constraint in period 1:
\[(1 + \tau_1)C_1 + B_1^p = Q_1\]

HH budget constraint in period 2:
\[(1 + \tau_2)C_2 = Q_2 + (1 + r_1)B_1^p\]

HH present value budget constraint:
\[
(1 + \tau_1)C_1 + \frac{1 + \tau_2}{1 + r_1}C_2 = Q_1 + \frac{Q_2}{1 + r_1}
\] (8)
HH problem

\[
\max_{\{C_1, C_2\}} U(C_1, C_2)
\]

subject to (8) taking as given \(Q_1, Q_2, \tau_1, \tau_2, \) and \(r_1\).

Optimality conditions:

\[
(1 + \tau_1)C_1 + \frac{1 + \tau_2}{1 + r_1}C_2 = Q_1 + \frac{Q_2}{1 + r_1} \tag{8}
\]

\[
\frac{U_1(C_1, C_2)}{U_2(C_1, C_2)} = \left(\frac{1 + \tau_1}{1 + \tau_2}\right)(1 + r_1) \tag{9}
\]

Notice that now there is a wedge, \(\frac{1 + \tau_1}{1 + \tau_2}\), between the MRS and the interest rate.
If \( \left( \frac{1 + \tau_1}{1 + \tau_2} \right) \downarrow \), then all else equal, household should increase \( C_1 \) and decrease \( C_2 \). It follows that a cut in \( \tau_1 \) most likely leads to a decline in the trade balance and hence the current account in period 1 implying that Ricardian equivalence fails.

To see that the tax cut, \( \tau_1 \downarrow \), does not result in an income effect in equilibrium, we need to derive the economy wide resource constraint. To obtain this, first consider the budget constraint of the government.
Government Budget Constraints:

Gov budget constraint in period 1:
\[ \tau_1 C_1 = G_1 + B^g_1 \]

Gov budget constraint in period 2:
\[ \tau_2 C_2 = G_2 + (1 + r_1)B^g_1 \]

Gov present value budget constraint:
\[ \tau_1 C_1 + \frac{\tau_2}{1 + r_1} C_2 = G_1 + \frac{G_2}{1 + r_1} \]  \hspace{1cm} (10)

We assume that government spending \( G_1 \) and \( G_2 \) are exogenously given, and the government must choose \( \tau_1 \) and \( \tau_2 \) so as satisfy its budget constraint (10).

Notice that the government cannot pick \( \tau_1 \) and \( \tau_2 \) freely. Given \( G_1 \) and \( G_2 \) once the government settles on the value of \( \tau_1 \) it must set \( \tau_2 \) to ensure satisfaction of (10).
Combining the PVBC of the government, (10), with that of the household, (8), we obtain the intertemporal resource constraint of the economy

\[ C_1 + \frac{1}{1 + r_1} C_2 = Q_1 - G_1 + \frac{Q_2 - G_2}{1 + r_1} \]

It follows from here that a tax cut, \( \tau_1 \downarrow \), does not change the economy wide resource constraint. And therefore, the tax cut will lead to an increase in period 1 consumption causing a trade deficit and a current account deficit. We thus have shown what we set out to show, namely, that Ricardian equivalence fails when taxes are distortionary.
So if Ricardian Equivalence fails, then fiscal deficits can lead to current account deficits.

Let’s go back to the Reagan deficits of the early 1980s. And see if we can say anything further whether the U.S. current account deficits were indeed linked to a decline in desired national savings in the U.S.
Testable Implications of the Twin Deficit Hypothesis

In the early 1980s there were 2 prevailing views regarding the emergence of large U.S. current account deficits. One was that the CA deficits were the consequence of factors external to the United States and the other was the Twin Deficit hypothesis.

View 1: The rest of the world wanted to save more. That is, the CA schedule of the rest of the world shifted to the left.

Why did RW want to send funds?

- U.S. safe heaven; capital flight from Latin America
- Developing country debt crisis drastically reduced demand for international borrowing
- Financial deregulation in foreign countries, e.g., Japan, made it easier for them to hold U.S. assets
The U.S. current account in the 1980s: View 1 The rest of the world wanted to save more, the $CA^{RW}$ shifts to the left to $CA^{RW'}$. 
View 2: Twin Deficit Hypothesis: the United States wanted to borrow more from the rest of the world

The CA schedule of the U.S. shifts up and to the left

Why?

• Reagan deficits lead to a decline in national savings
The U.S. current account in the 1980s: View 2
How can we tell view 1 and view 2 apart?

Note that under view 1 the interest rate declines whereas under view 2 the interest rate increases.

So a natural candidate to determine which hypothesis is more plausible empirically is the interest rate.

The next slide shows the behavior of the interest rate.
Interest Rates in the United States 1977-1985

Note: The real interest rate is measured as the difference between the 1-year constant maturity Treasury rate and ex post inflation.
How to construct the real interest rate?

\[ r_t = \text{real rate between period } t \text{ and } t + 1 \]

\[ i_t = \text{nominal interest rate between } t \text{ and } t + 1 \]

\[ \pi_{t+1} \equiv \frac{P_{t+1}}{P_t} = \text{gross rate of inflation between } t \text{ and } t + 1 \]

Use Fisher equation:

\[ 1 + r_t = \frac{(1 + i_t)}{E_t \pi_{t+1}} \]

How to measure expected inflation, \( E_t \pi_{t+1} \)? Here we just use actual inflation, that is, we approximate \( E_t \pi_{t+1} \) with \( \pi_{t+1} \).

Recall in savings glut lecture we used the forecast of the survey of professional forecasters to proxy expected inflation.

Alternatively, one could run a regression of \( 1/\pi_t \) on its own lags.
This evidence seems to vindicate view 2 lending support to the twin deficit hypothesis as an explanation of current account dynamics in the early 1980s in the United States.
Ramsey Optimal Tax Policy

We close this chapter with a discussion of how to set \( \tau_1 \) and \( \tau_2 \) optimally. By optimally we mean setting taxes in a welfare maximizing way.

Notice that there is more than one possible choice for the pair \((\tau_1, \tau_2)\) such that the government can finance its expenditures \( G_1 \) and \( G_2 \). How should the government pick its tax rates? Suppose the government is benevolent, that is, it sets \((\tau_1, \tau_2)\) to bring about that equilibrium in which — given \( G_1 \) and \( G_2 \) — utility is highest. The problem of the government in this case is known as the Ramsey problem. How can we find that equilibrium?
Let’s start by stating what an equilibrium is in our economy.

An equilibrium, is an allocation, a tax policy, and an interest rate such that households maximize, the government satisfies its budget constraint, and there are no arbitrage opportunities between domestic and international financial markets.
Formally, we define an equilibrium as values for $C_1$, $C_2$, $\tau_1$, $\tau_2$, and $r_1$ that satisfy

$$
(1 + \tau_1)C_1 + \frac{1 + \tau_2}{1 + r_1}C_2 = Q_1 + \frac{Q_2}{1 + r_1}
$$

(8)

$$
\frac{U_1(C_1, C_2)}{U_2(C_1, C_2)} = \left(\frac{1 + \tau_1}{1 + \tau_2}\right)(1 + r_1)
$$

(9)

$$
\tau_1 C_1 + \frac{\tau_2}{1 + r_1}C_2 = G_1 + \frac{G_2}{1 + r_1}
$$

(10)

and

$$
r_1 = r^*
$$

(11)

given $G_1$, $G_2$, $r^*$, $Q_1$, and $Q_2$. 
... and the Ramsey problem is to pick $C_1$, $C_2$, $\tau_1$, $\tau_2$, and $r_1$ to

$$\max \quad U(C_1, C_2)$$

subject to

(8),
(9),
(10),
and (11),
given $Q_1$, $Q_2$, $G_1$, $G_2$, and $r^*$. 
This is a maximization problem in 5 variables and 4 constraints. Let’s see if we can simplify this problem. First use (11) to eliminate $r_1$ from the constraints. Then combine (8) with (10) to obtain

\[ C_1 + \frac{C_2}{1 + r^*} = Q_1 - G_1 + \frac{Q_2 - G_2}{1 + r^*} \]  

(12)
Now consider the less constrained problem of

$$\max_{C_1, C_2} U(C_1, C_2)$$

subject to (12).

Why is this a less constrained problem? Because we are imposing only that (11) and (12) hold but we are not imposing that (8), (9), and (10) hold. In addition, notice that if \((C_1, C_2)\) satisfy (8), (9), and (10), and (11) then this basket also satisfies (11) and (12), which implies that we are not imposing more restrictions by asking the basket to satisfy (12).
The solution to this less constrained problem is familiar to us. Let
\[ Y \equiv Q_1 - G_1 + \frac{Q_2 - G_2}{1 + r^*} \]
and solve (12) for \( C_2 \)
\[ C_2 = (1 + r^*)(Y - C_1) \]
Use this expression to eliminate \( C_2 \) from the objective function, and our maximization problem becomes
\[
\max_{C_1} U(C_1, (1 + r^*)(Y - C_1))
\]
This is a problem in one unknown, just like the one studied ever since chapter 3.
The first-order optimality condition is

\[ U_1(C_1, C_2) = (1 + r^*) U_2(C_1, C_2) \]  \hspace{1cm} (13)

Now we claim that any pair \((C_1, C_2)\) that satisfies (12) and (13), also is a solution to the Ramsey problem introduced earlier.

To show this claim, pick \(r_1 = r^*\). Then (11) holds.

If we pick \(\tau_1 = \tau_2\)

then (13) implies that constraint (9) holds.
And if we then set $\tau_1$ to

$$\tau_1 = \frac{Q_1 + \frac{Q_2}{1+r^*}}{C_1 + \frac{C_2}{1+r^*}} - 1$$

we guarantee that constraint (8) holds.

What remains to be shown is that (10) holds. But this follows immediately from combining (12) with (8).
We have therefore shown that the solution to the less constrained problem satisfies all four constraints of the original Ramsey problem, that is, (8), (9), (10), and (11).

Therefore, we have shown that the solution to the less constrained problem also is a solution to the original Ramsey problem, which is what we set out to show.
Finally, notice that the less constrained problem is identical to the problem of a social planner. It follows from here that the Ramsey optimal allocation in an economy in which the government has access to distortionary consumption taxes and chooses them optimally is the same as the real allocation in an economy in which the government has access to non-distorting lump-sum taxes.