Implementing Iskrev’s Identifiability Test

Stephanie Schmitt Grohé  Martín Uribe*

May 15, 2012

This document describes Iskrev’s (2010) test and the suite of MATLAB code we created to implement it.

Iskrev’s test consists in checking whether the derivative of the predicted autocovariogram of the vector of observables with respect to the vector of estimated parameters has a rank equal to the length of the vector of estimated parameters. Formally, let

\[ m(t) \equiv \frac{\partial \text{vec}E(d_t d_0')}{\partial \theta}, \]

for \( t = 0, \ldots T - 1 \), where \( d_t \) is the theoretical counterpart of the vector of observables used to estimate the model, \( \theta \) is a vector of model parameters whose identifiability the test establishes, and \( T \) is the sample size. Let

\[ M \equiv \begin{bmatrix} m(0) \\ \vdots \\ m(T - 1) \end{bmatrix}. \]

Then the estimated parameter \( \theta \) is identifiable if \( M \) has full column rank. The test is performed by our matlab code iskrev_test.m.

Using the notation in Schmitt-Grohé and Uribe (2004), we can write the solution of the DSGE model up to first order as

\[ y_t = g_x x_t \]

and

\[ x_{t+1} = h_x x_t + \eta \epsilon_{t+1}, \]

where \( y_t \) is a vector of endogenous controls, \( x_t \) is a vector of endogenous and exogenous states, and \( \epsilon_{t+1} \) is a white noise vector with identity variance/covariance matrix. The vectors \( y_t \) and \( x_t \) are deviations of the control and state variables of the model from their respective deterministic steady-state values \( \bar{y} \) and \( \bar{x} \). The elements of \( d_t \) are linear combinations of the elements of \( y_t \). The two vectors are related by an expression of the form

\[ d_t = D y_t, \]

where \( D \) is a matrix of known coefficients. This relation implies that

\[ \text{vec}(E d_t d_0') = (D \otimes D) \text{vec}(E y_t y_0'), \]

---

*Columbia University and NBER. E-mail: martin.uribe@columbia.edu.
and therefore
\[ m(t) = (D \otimes D) \frac{\partial \text{vec} E(y_t' y_0')}{\partial \theta}. \]

Given the structure of the solution of the linearized DSGE model, we can write
\[ E(y_t y_0') = g_x h_x' \Sigma_x g_x', \]
where \( \Sigma_x \equiv E x_t x_t'. \) Taking the derivative of \( \text{vec} E(y_t y_0') \) with respect to \( \theta \), we obtain
\[ \frac{\partial \text{vec}(g_x h_x' \Sigma_x g_x')}{\partial \theta} = (I_y \otimes g_x h_x' \Sigma_x) d g_x' + (g_x \otimes g_x h_x') d \Sigma_x + (g_x \Sigma_x \otimes g_x) d h_x' + (g_x \Sigma_x h_x' \otimes I_y) d g_x. \]

In this expression, the object \( d g_x \) denotes \( \partial \text{vec}(g_x) / \partial \theta \), and is a matrix of order \( n_y n_x \times n_\theta \), where \( n_y, n_x, \) and \( n_\theta \) are the lengths of \( y_t, x_t, \) and \( \theta \), respectively. Similar notation applies to other objects.

**Deriving \( d g_x \) and \( d h_x \)**

The equilibrium conditions of the class of DSGE models considered here take the form \( E_t f(y_{t+1} + \bar{y}, y_t + \bar{y}, x_{t+1} + \bar{x}, x_t + \bar{x}; \theta) = 0 \). Up to first order, this expression can be written as
\[ \begin{bmatrix} f_y' & f' x \end{bmatrix} E_t \begin{bmatrix} y_{t+1} \\ x_{t+1} \end{bmatrix} = - \begin{bmatrix} f_y & f_x \end{bmatrix} \begin{bmatrix} y_t \\ x_t \end{bmatrix}, \]
where \( f_y', f_y, f' x, f_x \), denote the partial derivatives of \( f \) with respect to \( y_{t+1}, y_t, x_{t+1}, \) and \( x_t \), respectively, evaluated at the steady steady point \( (\bar{y}, \bar{y}, \bar{x}, \bar{x}) \). Using the solution to the linearized model in the linearized equilibrium conditions, we obtain
\[ \begin{bmatrix} f_y g_x h_x & f' x h_x \end{bmatrix} \begin{bmatrix} x_t \\ x_t \end{bmatrix} = - \begin{bmatrix} f_y g_x \end{bmatrix} \begin{bmatrix} x_t \\ x_t \end{bmatrix}, \]
which implies that
\[ f_y g_x h_x + f' x h_x = -f_y g_x - f_x. \]

Taking derivative with respect to \( \theta \), we obtain
\[ (I_x \otimes f_y g_x) d h_x + (h_x' \otimes f_y') d g_x + (h_x' g_x' \otimes I_n) df_y' + (I_x \otimes f' x') d h_x + (h_x' \otimes I_n) df_x' = - (I_x \otimes f_y) d g_x - (g_x' \otimes I_n) df_y - df_x. \]

Let
\[ A \equiv (h_x' \otimes f_y') + (I_x \otimes f_y), \]
\[ B \equiv (I_x \otimes f_y g_x) + (I_x \otimes f' x'), \]
and
\[ C \equiv -(h_x' g_x' \otimes I_n) df_y' - (h_x' \otimes I_n) df_x' - (g_x' \otimes I_n) df_y - df_x. \]

Then, we can write
\[ \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} d g_x \\ d h_x \end{bmatrix} = C, \]
which can be solved to obtain

\[
\begin{bmatrix}
    dg_x \\
    dh_x
\end{bmatrix}
= \left[ \begin{array}{cc}
    A & B \\
\end{array} \right]^{-1} C.
\]

We now explain how to obtain the objects \(df_x, df_{x'}, df_y,\) and \(df_{y'}\). We explain in detail how to obtain \(df_x\), the other derivations follow similar steps. We view \(f_x\) as a function of the parameter vector \(\theta\) and of the vector \(z(\theta) \equiv [\bar{y}(\theta)\bar{x}(\theta)]'\), which is the vector of steady-state values of controls and states, respectively. Thus, we write \(f_x(z(\theta); \theta)\). Then, we have

\[
df_x = \frac{\partial f_x}{\partial \theta} + \frac{\partial f_x}{\partial z} \frac{\partial z(\theta)}{\partial \theta}.
\]

The objects \(\frac{\partial f_x}{\partial \theta}\) and \(\frac{\partial f_x}{\partial z}\) are produced analytically by our Matlab code `iskrev_anal_deriv.m`. To facilitate the numerical evaluation of these symbolic expressions, the code writes these derivatives to a Matlab script file called `filename_iskrev_anal_deriv.m`, where the prefix `filename` is an input of `iskrev_anal_deriv.m` chosen by the user.

To obtain \(\frac{\partial z(\theta)}{\partial \theta}\) note that one can write the steady state of the model as \(f(z; \theta) = 0\), which implicitly defines \(z(\theta)\). Differentiating we get

\[
\frac{\partial f(z(\theta); \theta)}{\partial \theta} + \frac{\partial f(z(\theta); \theta)}{\partial z} \frac{\partial z(\theta)}{\partial \theta} = 0,
\]

which can be solved to obtain

\[
\frac{\partial z(\theta)}{\partial \theta} = - \left[ \frac{\partial f(z(\theta); \theta)}{\partial z} \right]^{-1} \frac{\partial f(z(\theta); \theta)}{\partial \theta}.
\]

The Matlab code `iskrev_anal_deriv.m` writes this formula into the Matlab script `filename_iskrev_anal_deriv.m`.

**Deriving \(dg'_x\) and \(dh'_x\)**

Let \(R_h\) be a matrix such that

\[
\text{vec}(h'_x) = R_h \text{vec}(h_x)
\]

The matrix \(R_h\) is a permutation matrix of order \(n_x^2\). Its unitary elements are located in row \(i\) column \(\text{fix}((i-1)/n_x) + 1 + \text{rem}(i-1, n_x)n_x\), for \(i = 1, \ldots, n_x^2\). Then we have that

\[
dh'_x = R_h dh_x
\]

Similarly, we can deduce that

\[
dg'_x = R_g dg_x,
\]

where the matrix \(R_g\) is a permutation matrix (i.e., a square matrix with only one element equal to unity per row and per column and all remaining elements equal to zero) of order \(n_xn_y\). Its unitary elements are located in row \(i\) column \(\text{fix}((i-1)/n_x) + 1 + \text{rem}(i-1, n_x)n_y\), for \(i = 1, \ldots, n_xn_y\).
Deriving $d\Sigma_x$

From (1), we have that the matrix $\Sigma_x \equiv E x_t x_t'$ satisfies

$$\Sigma_x = h_x \Sigma_x h_x' + \eta \eta'.$$

The derivative of $\Sigma_x$ with respect to $\theta$ must then satisfy

$$d\Sigma_x = (h_x \otimes h_x)d\Sigma_x + (h_x \Sigma_x \otimes I_x)dh_x + (I_x \otimes h_x \Sigma_x)dh_x' + d(\eta \eta').$$

Solving for $d\Sigma_x$, we obtain

$$d\Sigma_x = [I_{n_x^2} - (h_x \otimes h_x)]^{-1}[(h_x \Sigma_x \otimes I_x)dh_x + (I_x \otimes h_x \Sigma_x)dh_x' + d(\eta \eta')].$$

The object $d(\eta \eta')$ is produced symbolically by `iskrev_anal_deriv.m` and then written to the script file `filename_iskrev_anal_deriv.m`.

Deriving $dh_t^x$

For $t = 1$, it is $dh_x$, which we already derived. For $t \geq 2$, we proceed iteratively, noticing that $h_t^x = h_{t-1}^x h_x$, whose derivative is given by

$$dh_t^x = (I_x \otimes h_{t-1}^x)dh_x + (h_{t-1}^x \otimes I_x)dh_{t-1}^x.$$

What if M Is Not Full Column Rank

Suppose $M$ is less than full column rank at a parameter value $\theta_0$. Then, we conclude that with the selected observables and sample size, the parameter $\theta$ is not identifiable in the vicinity of $\theta_0$. This essentially means that in this case there will be an infinite number of parameter vectors $\theta$ that will give rise to the same autocovariogram as $\theta_0$. When $\theta$ is not identifiable, we can establish what linear combinations of the elements of $\theta$ will deliver the same autocovariogram as $\theta_0$.

Let $V(\theta, T)$ be the vectorized covariogram of the vector of observables, $d_t$, of order $T$. That is,

$$V(\theta) = \begin{bmatrix} \text{vech}(Ed_0 d_0') \\ \vdots \\ \text{vec}(Ed_{T-1} d_{T-1}') \end{bmatrix}.$$

Then, Taylor-expanding around $\theta_0$ up to first order, we obtain

$$V(\theta) \approx V(\theta_0) + M(\theta_0)(\theta - \theta_0).$$

If $M(\theta_0)$ has full column rank, then $V(\theta) = V(\theta_0)$ if and only if $\theta = \theta_0$ in the neighborhood of $\theta_0$. If, on the other hand, $M(\theta_0)$ is rank deficient, then there exists an infinite number of vectors $\theta$ in the vicinity of $\theta_0$ satisfying $V(\theta) = V(\theta_0)$. To obtain these vectors, perform a singular value decomposition of $M(\theta_0)'$. That is, find matrices $U, S, V$ such that

$$M(\theta)U = VS'.$$
where $U$ and $V$ are unitary (i.e., $UU' = I$ and $VV' = I$) and $S$ is diagonal with its diagonal elements nonnegative and decreasing. The matrix $S$ has as many rows as $M(\theta)$ and as many columns as the length of $\theta$. Now partition the matrix $U$ as $[U^1 \ U^2]$, where $U^2$ has as many columns as $S$ has zero diagonal elements. Then, we have that any vector $\theta$ of the form

$$\theta = \theta_0 + u^2 \alpha$$

delivers the same autocovariogram as $\theta_0$ for any (small) scalar $\alpha$ and any vector $u^2$ taken from the columns of $U^2$.

References
