This article describes the Key renewal theorem (KRT) and Blackwell's renewal theorem (BRT). These two limit theorems appear in different forms, but can be shown to be equivalent. For ease of presentation, we follow the terminology used in Karlin and Taylor [1] and refer to them collectively as the renewal theorem (RT). The RT is a very useful tool for characterizing the asymptotic behavior of a probabilistic quantity of interest in a renewal process (RP). To start, we briefly review renewal processes and the elementary renewal theorem (ERT).

An RP is a counting process with independent and identically distributed (iid) inter-event times. Mathematically, it can be defined as follows. Let $S_{n\geq 0}$ and $S_n$ be the occurrence time of the $n$th event, $n \geq 1$. Assume that

$$0 \leq S_1 \leq S_2 \leq S_3 \leq \cdots$$

Define

$$X_n = S_n - S_{n-1}, \quad n \geq 1.$$ 

Thus $(X_n, n \geq 1)$ is a sequence of inter-event times. Next define

$$N(t) = \sup \{ n \geq 0 : S_n \leq t \}.$$ 

Then $N(t)$ is called an RP generated by $(X_n, n \geq 1)$ if $(X_n, n \geq 1)$ is a sequence of nonnegative iid random variables. We refer the readers to the article titled Definition and Examples of Renewal Processes in this encyclopedia for a detailed introduction and more examples of RPs.

Suppose that the inter-event times $(X_n, n \geq 1)$ have a common cumulative distribution function (cdf) $G(\cdot)$, that is, $P(X_n \leq t) = G(t)$, $n \geq 1$. Let $\tau$ and $\sigma^2$ represent the mean and variance of $X_1$, respectively. To avoid triviality, we assume that $G(0^-) = 0$ and $G(0^+) = G(0) < 1$, which imply that $\tau > 0$. Let

$$M(t) = \mathbb{E}[N(t)], \quad t \geq 0.$$  

The function $M(t)$ is called the renewal function (RF), which is simply the expected number of events (renewals) up to time $t$. The ERT states that

$$\lim_{t \to \infty} \frac{M(t)}{t} = \frac{1}{\tau},$$  

which implies the following asymptotic relation $M(t) \sim t/\mu$ as $t \to \infty$. As a sample path version of the ERT (1), we have

$$\lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{\tau} \quad \text{with probability 1},$$  

which is a result from the strong law of large numbers. However, the ERT does not follow from Equation (2) directly since almost sure convergence does not imply convergence of expected values. A more intricate proof is needed to show the ERT. We refer the readers to the article titled Renewal Function and Renewal-Type Equations in this encyclopedia for an indepth discussion on the RF and ERT.

Blackwell’s renewal theorem, which was proposed by David Blackwell (Professor Emeritus of Statistics at the University of California, Berkeley), can be regarded as a refinement of this asymptotic relation. It asserts that for any $h > 0$

$$M(t+h) - M(t) \to \frac{h}{\tau}, \quad \text{as } t \to \infty,$$

under certain mild conditions on $G(\cdot)$. In words, the expected number of renewals during an interval of length $h$ is approximately
2 LIMIT THEOREMS FOR RENEWAL PROCESSES

$h/\tau$, provided that the process has already run for a long time. This result can be easily verified in a Poisson process, a special case of RPs.

In the next section, we first describe the KRT, which is equivalent to the BRT but of a different form. The KRT provides a convenient setting for the application of RT. Then we formally present the BRT, and briefly discuss its connection with the KRT.

THE RENEWAL THEOREM

The KRT reveals the asymptotic behavior of the solution to a renewal-type equation, which has the following form:

\[ H(t) = D(t) + \int_0^t H(t - x) dG(x), \tag{3} \]

where \( D(\cdot) \) is a known function, \( G(\cdot) \) is the cdf of inter-event times, and \( H(t) \) is to be determined. Renewal-type equations often arise when using the renewal argument, which is a method to derive probabilistic quantities in RPs by conditioning on the occurrence time of the first event, that is, \( S_1 \). We refer the readers to the article titled Renewal Function and Renewal-Type Equations in this encyclopedia for an introduction of the renewal-type equation and renewal argument. In this section, we focus on the asymptotic behavior of \( H(t) \).

Before presenting the KRT, we need the following definition.

Definition 1 [Periodic Random Variables]. A nondefective random variable \( X \) (or its distribution) is called periodic (or arithmetic, or lattice) if there exists a \( d > 0 \) such that

\[ \sum_{k=0}^{\infty} P(X = kd) = 1. \]

The largest value of \( d \) for which the above equality holds is called the period (or span) of the random variable \( X \) (or its distribution).

If a random variable \( X \) is not periodic, it is called aperiodic (or non-arithmetic, or non-lattice). We refer the readers to Kulkarni [2] for more examples on periodic and aperiodic random variables. Note that all continuous random variables are aperiodic.

Theorem 1 [Key Renewal Theorem]. Let \( H(t) \) be a solution to the renewal-type equation (3). Suppose that \( D(t) \) is a monotonic function which is absolutely integrable in the sense that

\[ \int_0^\infty |D(t)| \, dt < \infty. \]

(i) If \( G(\cdot) \) is aperiodic with mean \( \tau > 0 \), then

\[ \lim_{t \to \infty} H(t) = \begin{cases} 1 & \text{if } \tau < \infty, \\ 0 & \text{if } \tau = \infty. \end{cases} \]

(ii) If \( G(\cdot) \) is periodic with period \( d \) and mean \( \tau > 0 \), then for \( 0 \leq x < \tau \),

\[ \lim_{k \to \infty} H(kd + x) = \begin{cases} \frac{d}{\tau} \sum_{k=0}^{\infty} D(kd + x) & \text{if } \tau < \infty, \\ 0 & \text{if } \tau = \infty. \end{cases} \]

We refer to Feller [3] for a proof of the KRT. In his proof, Feller assumes that the function \( D(t) \) is directly Riemann integrable. Feller's condition is technical, and not presented here. The condition we gave above is a simple and sufficient condition for Feller's, and suffices for most of the applications of the KRT. As noted in Tijms [4], a more general condition guaranteeing \( D(t) \) to be directly Riemann integrable is that \( D(t) \) can be written as a finite sum of monotone, integrable functions.

As mentioned earlier, another form of the KRT is the BRT, which is expressed more explicitly in terms of the renewal function \( M(t) \). We present the BRT below, and follow the convention that \( h/\tau = 0 \) if \( \tau = \infty \).

Theorem 2 [Blackwell’s Renewal Theorem]. Let \( h > 0 \) be a fixed number.
(i) If \( G(\cdot) \) is aperiodic with mean \( \tau > 0 \), then
\[
\lim_{t \to \infty} [M(t + h) - M(t)] = \frac{h}{\tau},
\]
(ii) If \( G(\cdot) \) is periodic with period \( d \) and mean \( \tau > 0 \), then
\[
\lim_{t \to \infty} [M(t + h) - M(t)] = \frac{h}{\tau},
\]
provided that \( h \) is a multiple of the period \( d \).

The BRT and the KRT can be shown to be equivalent. The proof of BRT using the KRT can be found in Karlin and Taylor [1] and Kulkarni [2]. As discussed in Karlin and Taylor [1] and Ross [5], the KRT can also be deduced from the BRT by approximating a directly Riemann integrable function with step functions.

The next section presents two classical applications of the KRT. We discuss the limiting distributions of recurrence times (including the current age, excess life, and total life) in an RP, and the asymptotic expansion of a renewal function. We briefly mention how the KRT can be used to find these results. Detailed derivations can be found in Karlin and Taylor [1], Kulkarni [2], and Ross [5].

APPLICATIONS OF THE KEY RENEWAL THEOREM

Limiting Distributions of Recurrence Times

The following three probabilistic quantities related to event times are of particular interest in the study of RPs:

\[
\begin{align*}
A(t) &= t - S_{N(t)}, \\
B(t) &= S_{N(t)+1} - t, \\
L(t) &= S_{N(t)+1} - S_{N(t)} \\
&= X_{N(t)+1} = A(t) + B(t),
\end{align*}
\]

where \( t \geq 0 \).

Physically, \( A(t) \) represents the time elapsed from the latest event before \( t \) to time \( t \), and \( B(t) \) represents the time duration between \( t \) and the next event immediately after \( t \). Their sum \( L(t) \) is the total inter-event time that contains \( t \). The quantities \( A(t), B(t), \) and \( L(t) \) are called the current age, remaining life (excess life), and total life at time \( t \), respectively. Sometimes \( A(t) \) is also called the backward recurrence time and \( B(t) \) is called the forward recurrence time. The stochastic processes \( \{A(t), t \geq 0\}, \{B(t), t \geq 0\}, \) and \( \{L(t), t \geq 0\} \) are called the age process, remaining life process, and total life process, respectively.

In this section, we discuss the asymptotic behavior of these three processes. Suppose that \( G(\cdot) \) is aperiodic and \( \tau < \infty \). Then the following results can be shown by using the KRT:

\[
\lim_{t \to \infty} \mathbb{P}(A(t) > x) = \lim_{t \to \infty} \mathbb{P}(B(t) > x) = \frac{1}{\tau} \int_{x}^{\infty} (1 - G(u)) \, du, \tag{5}
\]

\[
\lim_{t \to \infty} \mathbb{P}(A(t) > x, B(t) > y) = \frac{1}{\tau^2} \int_{x+y}^{\infty} (1 - G(u)) \, du,
\]

and

\[
\lim_{t \to \infty} \mathbb{P}(L(t) > x) = \frac{1}{\tau} \int_{x}^{\infty} u \, dG(u). \tag{6}
\]

To facilitate understanding, we present how to derive Equation (5) by applying the KRT, and other results can be shown in a similar way. We start with the derivation of the limiting distribution of \( B(t) \). Let \( H(t) = \mathbb{P}(B(t) > x) \). Then conditioning on the occurrence time of the first renewal event \( X_1 = u \), we have

\[
\mathbb{P}(B(t) > x | X_1 = u) = \begin{cases} 
H(t-u) & \text{if } 0 \leq u \leq t \\
0 & \text{if } t < u \leq x + t \\
1 & \text{if } u > x + t.
\end{cases}
\]

Unconditioning, we get

\[
H(t) = \int_{0}^{\infty} \mathbb{P}(B(t) > x | X_1 = u) \, dG(u)
= \int_{0}^{t} H(t-u) \, dG(u) + \int_{x+t}^{\infty} dG(u)
= \int_{0}^{t} H(t-u) \, dG(u) + 1 - G(x+t).
\]
4 LIMIT THEOREMS FOR RENEWAL PROCESSES

Now, for a fixed \( x > 0 \), \( 1 - G(x + t) \) is nonnegative and nonincreasing for \( t \geq 0 \), and
\[
\int_0^\infty (1 - G(x + t)) \, dt \leq \int_0^\infty (1 - G(t)) \, dt = \tau < \infty.
\]

Hence the conditions for the KRT are satisfied, from which it directly follows that
\[
\lim_{t \to \infty} \mathbb{P}(B(t) > x | X_1 = u) = \frac{1}{\tau} \int_0^\infty (1 - G(x + t)) \, dt = \frac{1}{\tau} \int_x^\infty (1 - G(t)) \, dt.
\]

To derive the limiting distribution for \( A(t) \), first notice that for \( t > x \), \( A(t) > x \) \( \iff \) \{No renewals in \([t-x, t]\)\} \( \iff \) \{\(B(t-x) > x\)\}. Thus,
\[
\lim_{t \to \infty} \mathbb{P}(A(t) > x) = \lim_{t \to \infty} \mathbb{P}(B(t-x) > x) = \lim_{t \to \infty} \mathbb{P}(B(t) > x).
\]

This completes the derivation of Equation (5).

Applying the KRT, one can further show that
\[
\lim_{t \to \infty} \mathbb{E}(A(t)) = \lim_{t \to \infty} \mathbb{E}(B(t)) = \frac{\sigma^2 + \tau^2}{2\tau}, \tag{7}
\]
and
\[
\lim_{t \to \infty} \mathbb{E}(L(t)) = \frac{\sigma^2 + \tau^2}{\tau}. \tag{8}
\]

However, we should remind the readers that Equations (7) and (8) cannot be directly deduced from Equations (5) and (6), respectively, since convergence in distribution does not imply convergence of expected values. Hence, one should resort to the KRT for a proof. See, for example, Kulkarni [2], for a detailed derivation.

Below we introduce an interesting phenomenon called inspection paradox in relation to Equations (4) and (8). Recall from Equation (4) that \( L(t) = X_{N(t)+1} \), but Equation (8) says that
\[
\lim_{t \to \infty} \mathbb{E}(X_{N(t)+1}) = \frac{\sigma^2 + \tau^2}{\tau} \geq \tau = \mathbb{E}(X),
\]
that is, for a large \( t \), the inter-event time that covers \( t \) is on average longer than a generic inter-event interval. This counterintuitive phenomenon is known as the inspection paradox. In fact, one can further show that, for a fixed \( t \geq 0 \),
\[
\mathbb{P}(X_{N(t)+1} > x) \geq \mathbb{P}(X_1 > x). \tag{9}
\]

In words, the length of the renewal interval that contains time \( t \) is stochastically larger than the first renewal interval. This immediately implies that
\[
\mathbb{E}(X_{N(t)+1}) \geq \mathbb{E}(X_1).
\]

To understand this paradox, let us first fix an arbitrary point \( t \). We measure the length of the interval containing \( t \). However, it seems plausible that the longer the duration of an interval, the larger the likelihood that it would be experienced and measured. Such samplings are known as length-biased sampling, and occur in many sampling situations. Stein and Dattenero [6] provide introductory examples to illustrate such sampling bias and the inspection paradox. For a mathematical proof of Equation (9), the readers are referred to Angus [7] and Ross [8].

Now, let us consider a concrete example. Let \( \{N(t), t \geq 0\} \) be a Poisson process with mean inter-event time \( 1/\lambda \). It can be shown that (see, for example, Chapter 5.3 of Karlin and Taylor [11])
\[
\mathbb{E}(L(t)) = \frac{1}{\lambda} + \frac{1}{\lambda} (1 - e^{-\lambda t}) \geq \frac{1}{\lambda} = \mathbb{E}(X_1).
\]

We see that, as \( t \to \infty \), the mean total life doubles the mean inter-event time!

Asymptotic Expansion of the Renewal Function

Recall that the ERT implies that
\[
M(t) = \frac{t}{\tau} + o(t),
\]
where \( o(t) \) represents a function \( f(t) \) such that \( f(t)/t \to 0 \) as \( t \to \infty \). We are now interested in refining this asymptotic approximation of \( M(t) \). To do this, we study the function \( o(t) \), and give the second term of the asymptotic
expansion of $M(t)$. Suppose that $\sigma^2 < \infty$ and $G(\cdot)$ is aperiodic. To start, we define

$$H(t) = M(t) - \frac{t}{\tau},$$

and then apply the renewal argument, which leads to a renewal-type equation with $H(t)$ as its solution. From this renewal-type equation, one is able to identify the function $D(t)$, and verify that $D(t)$ can be written as a sum of two monotone functions both of which are absolutely integrable. Thus, the KRT is directly applicable, and it follows that

$$\lim_{t \to \infty} \left[ M(t) - \frac{t}{\tau} \right] = \frac{1}{\tau} \int_0^\infty D(t) \, dt = \frac{\sigma^2 - \tau^2}{2\tau^2},$$

or equivalently

$$M(t) = \frac{t}{\tau} + \frac{\sigma^2 - \tau^2}{2\tau^2} + o(1),$$

where $o(1)$ is a function of $t$ which approaches $0$ as $t \to \infty$. This expansion gives a more accurate estimate of $M(t)$ for large $t$, and implies that $t/\tau$ overestimates $M(t)$ for large $t$ if the coefficient of variation $\sigma^2/\tau^2 < 1$. Consider a Poisson process, where $\sigma^2 = \tau^2$, and in fact $M(t) = t/\tau$, verifying Equation (10) in this special case of RPs.

### FURTHER READINGS

Further details on these two limit theorems for RPs, namely, the KRT and the BRT, can be found in classical textbooks including Karlin and Taylor [1], Kulkarni [2], and Ross [5]. Two seminal papers that prove the BRT and the KRT are Blackwell [9] and Smith [10], respectively. To track the theoretical development of these limit theorems and in general the renewal theory, we refer the readers to Smith [11].

These limit theorems are very useful tools in the study of stochastic processes. They are used in the proof of those key results in alternating renewal processes and regenerative processes [2,5,12]. These two processes are discussed in the article titled Alternating Renewal Processes and Regenerative Processes in this encyclopedia, respectively. The two limit theorems have also been extensively used in the study of queuing theory, inventory control, and warranty management. Literature on these applications is enormous. We provide a few examples in relevance to those applications of the KRT discussed above.

The remaining life in RPs is widely used in stochastic modeling. In queuing theory, it is the remaining service time; in the context of inventory control, it is the undershoot of the reorder point. By considering only those times when the server is busy, the remaining service time seen by an arriving customer in an M/G/1 queue is equivalent to the remaining life in a RP generated by a nonstopping service process [13,14]. Using this result, one can construct a proof for the famous Pollaczek–Khinchine formula, and study the distribution of waiting times in an M/G/1 queue [15]. Karlin [16] discusses the application of renewal theory to the study of inventory control. In particular, he applies the excess life in the study of a periodic review inventory system managed by an $(s,S)$ policy. Green [17] extends the study of limiting distribution of recurrence times. She shows that the asymptotic joint distribution of the remaining life and total life of an subinterval is that of an RP generated by these subintervals only, if attention is restricted to these subintervals.

The RF plays an important role in the cost analysis of warranty management. Many asymptotic results are available for approximating RFS involved in such cost computations. Feller [3] and Cox and Miller [18] provide asymptotic expansions for an RF when inter-event times are periodic. General bounds for an RF are discussed in Lorden [19] and Barlow and Proschan [20]. More recent development of these bounds can be found in Politis Koutras [21]. For an introduction and review on warranty management and its connection with the renewal theory, the readers are referred to Blischke and Murthy [22] and Thomas ans Rao [23].
REFERENCES

Please note that the abstract and keywords will not be included in the printed book, but are required for the online presentation of this book which will be published on Wiley’s own online publishing platform.

If the abstract and keywords are not present below, please take this opportunity to add them now.
The abstract should be a short paragraph upto 200 words in length and keywords between 5 to 10 words.

Abstract: This article describes the Key renewal theorem and the Blackwell’s renewal theorem. These two limit theorems for renewal processes are equivalent but of different forms. They are particularly useful for characterizing the asymptotic behavior of a probabilistic quantity of interest in a renewal process. We present two applications of these limit theorems: the limiting distributions of recurrence times in a renewal process and the asymptotic expansion of a renewal function. Further readings on the theoretical development of these limit theorems and their applications in different areas are also provided.

Keywords: renewal process; Key renewal theorem; Blackwell’s renewal theorem