Abstract—In this paper, we investigate the problem of designing compact-support interpolation kernels for a given class of signals. By using calculus of variations, we simplify the optimization problem from an infinite nonlinear problem to a finite dimensional linear case, and then find the optimum compact-support function that best approximates a given filter in the least square sense (\( \ell_2 \) norm). The benefit of compact-support interpolants is the low computational complexity in the interpolation process while the optimum compact-support interpolant guarantees the highest achievable Signal to Noise Ratio (SNR). Our simulation results confirm the superior performance of the proposed kernel compared to other conventional compact-support interpolants such as cubic spline.

Index Terms—Spline, Interpolation, Filter Design

I. INTRODUCTION

Due to the existence of powerful digital tools, nowadays it is very common to convert the continuous time signals into the discrete form, and after processing the discrete signal, we can convert the discrete signal back to the original domain. The conversion of the continuous signal into the discrete domain is usually called the sampling process; the common form of sampling consists of taking samples directly from the continuous signal at equidistant time instants (uniform sampling). Although the samples are uniquely determined by the continuous function, there are infinite number of continuous signals which produce the same set of samples. The reconstruction process is defined as selecting one of the infinite possibilities which satisfies certain constraints. For a given set of constraints, a proper sampling scheme is the one that establishes a one-to-one mapping between the discrete signals and the set of continuous functions that satisfy the constraints. One of the well-known constraints is the finite support in Fourier domain [1]. The theory of wavelets [2]–[4] introduced a generalized class of basis for representing continuous functions. In fact, any kind of such representation is equivalent to associating a countable infinite set of scalars (coefficients) to any given continuous function (similar to sampling). The one-to-one mapping of this association is achieved only if the continuous function belongs to a specific class. The reconstruction of the continuous function from the coefficients usually involves filter banks and interpolation. Multiresolution analysis [5], [6], self-similarity [7], [8], and singularity analysis [9] are inseparable from continuous-time interpolation. Theoretically, the optimum interpolations require interpolants with infinite support which are impractical from the implementational perspective. The common trend is to truncate the interpolate function or approximate it with a compact-support function.

In this field, polynomial splines, such as B-Splines, are particularly popular mainly due to their simplicity, compact-support, and excellent approximation capabilities compared to other methods. B-Spline interpolations have spread to various applications [10]–[12]. The cubic spline is of particular interest since it generates the function with minimum curvature passing through a given set of points [13]. Also fast methods for obtaining the spline coefficients of a continuous function is addressed in [14]; it is shown that the coefficients follow a recursive equation. For the asymptotic behaviour (as order increases) and approximation properties of the B-splines, the interested reader is referred to [15].

Many advantages of the B-splines arise from the fact that they are compact-support functions. However, there is no evidence that they are the best compact-support kernels for the interpolation process; i.e., it may be possible to improve the performance without compromising the desired property of the compact-supportedness. In this paper, we focus on the problem of designing compact-support interpolants that best resemble a given filter such as the ideal lowpass filter; more precisely, we aim to find a compact-support kernel that minimizes the least squared error when its cardinal is compared to a given function. The given filter may be any arbitrary function that reflects the properties and constraints of the class of signals that enter the sampling process. Different variations of this problem are previously studied in [16], [17]: the problem in [16] is to find the best one-sided (causal) kernel (not necessarily with compact-support) while in [17], the aim is to convert the required IIR filtering in the discrete domain for a given polynomial spline to an optimal causal filtering. The optimal B-spline interpolatants for hexagonal 2D signals are also derived in [18]. The main difference of the work in this paper from the aforementioned problems is that we do not restrict the kernel to be a polynomial spline. In fact, the optimality of the kernel is within the linear combination of the Dirichlet functions (see Def. 7 for the definition of Dirichlet functions); i.e., one cannot improve the least square error by modifying the resultant compact-support kernel with an additive Dirichlet function.

The remainder of the paper is organized as follows: The next section briefly describes the spline interpolation method. In section III, a novel scheme is proposed to produce new optimized kernels for interpolation regardless of the type of filtering. The performance of the proposed method is evaluated in section IV by comparing the interpolation results of the proposed method to those of well-known interpolation techniques. Section V concludes the paper.

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II. Preliminaries

We start by introducing some of the definitions required in the rest of the paper. The definitions and results are generic to the dimension of the space, therefore, instead of the 1D terms “continuous-time” and “discrete-time”, we use “continuous-space” and “discrete-space”, respectively. Furthermore, t represents the index for the continuous-space signals while n plays the same role for the discrete-space signals. To facilitate the reading of the paper, we have gathered all the notations in Table I.

**Definition 1.** For a continuous-space $k$-dimensional signal $x(t)$, the continuous-space signal $x_p(t)$ and the discrete-space signal $x_d(n)$ are defined as follows:

$$x_d[n_1, n_2, \ldots, n_k] \triangleq x(n_1T, n_2T, \ldots, n_kT)$$

$$x_p(t) \triangleq x(t)p(t) = \sum_{n \in \mathbb{Z}^k} x_d[n]\delta(t - Tn)$$

where $\delta(t)$ is the $k$-dimensional Dirac delta distribution centred at origin and $p(t) \triangleq \sum_{n \in \mathbb{Z}^k} \delta(t - Tn)$ is the $k$-dimensional periodic impulse train. The sampling period $T$ is normalized to 1 in all directions, without any loss of generality.

The sampling process is shown in Fig 1.

**Definition 2.** A Linear Time Invariant (LTI) filter with impulse response $h(t)$ is said to have the “interpolation property” if and only if,

$$h_p(t) = \delta(t).$$

In other words, the interpolation property implies that the impulse response vanishes at the integers or in general at the grid points. This is equivalent to the partition of unity in the 1D case.

**Definition 3.** A discrete-space signal $y_d[n]$ is called a “proper” signal if and only if it is bounded and has a unique, bounded inverse $y_d^{-1}[n]$.

It is not hard to check that a bounded discrete signal is proper if and only if it contains no zeros (and obviously no poles) on the unit circle.

**Definition 4.** For any continuous-space signal $y(t)$, if $y_d[n]$ is a proper signal, then $\tilde{y}(t)$ is defined as follows:

$$\tilde{y}(t) = ((y_p)^{-1} * y)(t)$$

**Corollary 1.** $\tilde{y}(t)$ in (4) is the impulse response of a filter with interpolation property, in other words:

$$\tilde{y}_p(t) = \delta(t)$$

**Proof:**

$$\tilde{y}_p(t) = \tilde{y}(p(t)) = \left[\left((y_p)^{-1} * y\right)(t)\right]p(t) = \left((y_p)^{-1} * y_p\right)(t) = \delta(t)$$

**Definition 5.** The polynomial B-Spline of degree $m$ is defined as follows:

$$\beta^m(t) \triangleq \sum_{n=0}^{m+1} (-1)^n \binom{m+1}{n} u^{m+1}(t - n)$$

**Definition 6.** According to the above definition, $c^m(t) \triangleq \tilde{\beta}^m(t)$ is defined as the cardinal spline of degree $m$ (Fig. 2).

III. Proposed Optimized Compact-Support Kernels

In many applications, it is desirable that the interpolation filter resembles an ideal filter, and there is no need for it either to be smooth or piecewise polynomial. In this section an optimized compact-support interpolation kernel will be introduced to emulate a desired filter.

**Definition 7.** Let $D^k$ denote the set of all $k$-dimensional continuous-space signals that satisfy the Dirichlet conditions, i.e, for any $y(t) \in D^k$.

1) $y(t)$ has a finite number of extrema in any given box,
2) \( y(t) \) has a finite number of discontinuities in any given box,
3) \( y(t) \) is absolutely integrable over a period,
4) \( y(t) \) is bounded.

First of all, an affine subspace of all \( k \)-dimensional signals that satisfy the Dirichlet conditions will be defined, and then the optimized solution will be obtained in this set by the calculus of variation.

**Definition 8.** Let \( y_d[n] \) be a proper signal that vanishes for all \( n \notin (0, m + 1)^k \), then \( \chi^m(y_d) \) is the set of all continuous-space signals \( y(t) \) that satisfy the following conditions,

1) \( y \in D^k \)
2) \( \forall n \in \mathbb{N}^k; \ y(n) = y_d[n] \)
3) \( \forall t \notin (0, m + 1)^k; \ y(t) = 0 \)

The use of \( y(t) \in \chi^m(y_d) \) as a kernel to interpolate \( x_d[n] \) is a linear time invariant process with the impulse response \( \hat{y}(t) \).

**Definition 9.** The error function \( e_x: \chi^m(y_d) \to \mathbb{R} \) is defined as follows:

\[
e_x(y) = \min_{y \in \chi^m(y_d)} e_x(y) = \frac{\|\hat{y} * x_p - x(t)\|_2^2}{\|\hat{y} * x_p\|_2^2}
\]

where \( F \) is defined as the \( k \)-dimensional continuous-space Fourier transform operator.

**Definition 10.** According to the above definition, if \( \rho_d^m \) is a proper signal that vanishes for all \( n \notin (0, m + 1)^k \), an optimized compact-support kernel \( \rho^m[x, \rho_d^m] \) is defined as follows:

\[
\rho^m[x, \rho_d^m] = \min_{y \in \chi^m(\rho_d^m)} e_x(y)
\]

Now, for a given proper discrete signal \( \rho_d^m \) with the required vanishing property, we employ the calculus of variations in order to find the optimized continuous interpolation kernel \( \rho^m \) that minimizes the error function \( e_x(\rho^m) \).

**Theorem 1.** Equation (9) has a unique solution that satisfies the following property,

\[
[x_p * \bar{x}_p * (\rho^m)^{-1} * (\bar{x}_p)^{-1}] * \rho^m = [(\rho^m)^{-1} * \bar{x}_p] * x \quad (10)
\]

for all \( t \in (0, m + 1)^k \), where \( \bar{y}(t) = y^*(t) \).

**Proof:** For \( \gamma \in \chi^m(0) \) and any \( \varepsilon > 0 \), we have \( \rho^m + \varepsilon \gamma \in \chi^m(\rho_d^m) \), and the variational derivation of \( e_x(\rho^m) \) with respect to \( \rho^m \) with \( \gamma \) as the test function is equal to

\[
(e_x(\rho^m), \gamma) \triangleq \lim_{\varepsilon \to 0} \frac{e_x(\rho^m + \varepsilon \gamma) - e_x(\rho^m)}{\varepsilon} = 2 \int_{\mathbb{R}^k} \Re\{\gamma(t)\} R\left\{\left[\mathcal{F}[x_p] \right]^* \left[\mathcal{F}[\rho^m]\right] - \mathcal{F}[x] \right\} dt
\]

\[
- 2 \int_{\mathbb{R}^k} \Im\{\gamma(t)\} \Re\left\{\left[\mathcal{F}[x_p] \right]^* \left[\mathcal{F}[\rho^m]\right] - \mathcal{F}[x] \right\} dt \quad (11)
\]

The proof of (11) is presented in (12). Since \( \chi^m(\rho_d^m) \) is boundless, in order to minimize \( e_x(\rho^m) \), \( e(\rho^m), \gamma \) should be zero for all \( \gamma \in \chi^m(0) \), which implies that the second term inside the integrals should be zero for \( t \in (0, m + 1)^k \), i.e,

\[
\mathcal{F}^{-1}\left\{ \left[\mathcal{F}[x_p] \right]^* \left[\mathcal{F}[\rho^m]\right] - \mathcal{F}[x] \right\} = 0 \quad (13)
\]

The above equation directly yields (10). Since we have a quadrature minimization problem subject to an affine feasible set \( \chi^m(\rho_d^m) \), the problem is convex and as a result, the solution must be unique.

**A. Filter Estimation**

Another application of (10) is to approximate an ideal interpolation filter by an optimized compact-support kernel. In fact, these kernels are superior to FIR filters.

Now the goal is to design \( \rho^m \) such that \( \rho_d^m \) would be the best estimation of \( \hat{h} \), which denotes the impulse response of a filter that has the interpolation property.

**Lemma 1.** (Estimating a desired filter) Assume \( h(t) \) is the impulse response of a linear time invariant filter which satisfies the interpolation property and let \( \rho_d^m[n] \) be a proper discrete signal, then,

\[
\arg \min_{y \in \chi^m(\rho_d^m)} \|h - \hat{y}\|_2^2 = \rho^m[h, \rho_d^m] \quad (14)
\]
\[
\langle e_x(\rho^m + \varepsilon), \gamma \rangle = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\mathbb{R}^k} \left( \frac{\mathcal{F}(\rho^m + \varepsilon)}{\mathcal{F}(\rho^m + \varepsilon \rho^m_p)} \mathcal{F}(x_p) - \mathcal{F}(x) \right)^2 \, df - \int_{\mathbb{R}^k} \left( \frac{\mathcal{F}(\rho^m)}{\mathcal{F}(\rho^m_p)} \mathcal{F}(x_p) - \mathcal{F}(x) \right)^2 \, df 
\]

\[
= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\mathbb{R}^k} \left[ \Im \left\{ \left( \frac{\mathcal{F}(\rho^m + \varepsilon)}{\mathcal{F}(\rho^m + \varepsilon \rho^m_p)} \mathcal{F}(x_p) - \mathcal{F}(x) \right)^2 - \Im \left\{ \left( \frac{\mathcal{F}(\rho^m)}{\mathcal{F}(\rho^m_p)} \mathcal{F}(x_p) - \mathcal{F}(x) \right)^2 \right\} \right] \, df 
\]

\[
= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\mathbb{R}^k} \left[ \Re \left\{ \left( \frac{\mathcal{F}(\rho^m + \varepsilon)}{\mathcal{F}(\rho^m + \varepsilon \rho^m_p)} \mathcal{F}(x_p) - \mathcal{F}(x) \right)^2 - \Re \left\{ \left( \frac{\mathcal{F}(\rho^m)}{\mathcal{F}(\rho^m_p)} \mathcal{F}(x_p) - \mathcal{F}(x) \right)^2 \right\} \right] \, df 
\]

\[
= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\mathbb{R}^k} \left[ \Im \left\{ \left( \frac{\mathcal{F}(\rho^m + \varepsilon)}{\mathcal{F}(\rho^m + \varepsilon \rho^m_p)} \mathcal{F}(x_p) - \mathcal{F}(x) \right)^2 - \Im \left\{ \left( \frac{\mathcal{F}(\rho^m)}{\mathcal{F}(\rho^m_p)} \mathcal{F}(x_p) - \mathcal{F}(x) \right)^2 \right\} \right] \, df 
\]

Proof:

\[
\arg \min_{y \in \chi^m(h \rho^m_p)} ||h - \hat{y}||_2 = \arg \min_{y \in \chi^m(h \rho^m_p)} e_h(y) = \rho^m \quad (16)
\]

Corollary 2. For an impulse response \(h(t)\) with the interpolation property, if \(\rho^m \in \chi^m(h \rho^m_p)\) denotes the optimum compact-support kernel for which \(\rho^m\) best approximates \(h(t)\) (i.e., \(\hat{\rho}(t) = \arg \min_{y \in \chi^m(h \rho^m_p)} ||h - \hat{y}||_2\)), then \(\rho^m \in \chi^m(h \rho^m_p)\) satisfies:

\[
((\rho^m_p)^{-1} \ast (\rho^m_p)^{-1}) \ast \rho^m = (\rho^m_p)^{-1} \ast h 
\]

(17)
Fig. 3. The optimized compact-support interpolation kernel versus cubic B-spline. \( \rho^3 \{ h, \rho^3 \} \) is the optimized kernel built for estimating the ideal lowpass filter \( h(t) = \frac{\sin(\pi t)}{\pi t} \) with \( \rho^3 \{ z \} = 0.235z + 0.484z^2 + 0.235z^3 \).

The proof follows directly from (10) and the fact that \( h(t) \) has the interpolation property.

Figure 3 shows the cubic B-spline and the optimized interpolation kernel designed for estimating the ideal lowpass filter \( h(t) = \text{sinc}(t) \) with \( \rho^3 \{ z \} = 0.235z + 0.484z^2 + 0.235z^3 \), while Fig. 4 shows \( \rho^3[h, \rho^3](t) \) in comparison to \( c^3(t) \).

### B. Details of implementing in 1D

Thus, it is proven that the optimized interpolation kernel which gives the best estimation of \( x \), should satisfy (10). By defining

\[
\begin{align*}
v_p & \triangleq (x_p \ast \overline{\alpha}_p) \ast [(\rho^m_p)^{-1} \ast (\overline{\rho^m_p})^{-1}] \\
w & \triangleq [(\overline{\rho^m_p})^{-1} \ast \overline{x}, \overline{x}],
\end{align*}
\]

for \( k = 1 \), we can rewrite (10) as \( (v_p \ast \rho^m)(t) = w(t)|_{t\in(0,m+1)} \). Since this equation is only valid in a particular interval, \( (v_p)^{-1} \) cannot be used to obtain \( \rho^m \). However, since \( v_p \) is an impulse train (convolution of four impulse trains) we will show that the continuous functional equation in (10) boils down to solving a finite Hermitian Toeplitz system of linear equations. For this purpose, we first propose a notion for convolution matrix and then define two vectors containing functions which are supported only on a unit-length interval:

**Definition 11.** For a discrete-space one-dimensional signal \( x_d[n] \), and any \( a, b \in \mathbb{Z} \), matrix \( M^{a,b}_{x_d} \triangleq [x_d[i-j+a], i=1, \ldots, b+1; j=1, \ldots, b+1] \) is defined as a convolution matrix:

\[
M^{a,b}_{x_d} = \begin{bmatrix}
x_d[a] & x_d[a-1] & \ldots & x_d[a-b] \\
x_d[a+1] & x_d[a] & \ldots & x_d[a-b] \\
\vdots & \vdots & \ddots & \vdots \\
x_d[a+b] & x_d[a+b-1] & \ldots & x_d[a]
\end{bmatrix}
\]

\[(20)\]

**Definition 12.** For a continuous-space one-dimensional signal \( x(t) \), and any \( a, b \in \mathbb{Z} \), convolution vector is defined as \( V^{a,b}_x = [x_a, x_{a+1}, \ldots, x_{a+b}]^T \) where for any \( n \in \mathbb{Z} \),

\[
x_a(t) = \begin{cases} 
  x(t + n) & 0 \leq t < 1 \\
  0 & \text{o.w.}
\end{cases}
\]

\[(21)\]

Now, here is (10) in the matrix form:

\[
M^{0,m}_{v_d} V^{0,m}_\rho = V^{0,m}_w
\]

\[(22)\]

Since \( v_d[-n] = v_d^*[n] \), the above matrix \( M^{0,m}_{v_d} \) is also Hermitian Toeplitz. An efficient recursive method for solving this kind of linear systems is presented in [19]. Now by solving (22), \( \{ \rho^m_n \}_{n=0} \) is derived and thus the optimized compact-support kernel is given as

\[
\rho^m(t) = \sum_{n=0}^{m} \rho^m_n (t - n)
\]

\[(23)\]

This optimized compact-support interpolation kernel minimizes the mean squared error of interpolation. On the other hand, smoothness of the kernel and causality of the prefilter can be achieved by adjusting \( \rho_d \).

### C. Details of implementing in 2D

To derive the optimized compact-support interpolation kernel \( \rho^m(t_1, t_2) \) from equation (10) in 2D case, we have an \((m+1)^2\) by \((m+1)^2\) Toeplitz-block-Toeplitz linear system to solve:

\[
MP = W
\]

\[(24)\]

where,

\[
M = \begin{bmatrix}
M^{0,m}_{v_d[0,0]} & M^{0,m}_{v_d[0,-1]} & \ldots & M^{0,m}_{v_d[-m,0]} \\
M^{0,m}_{v_d[0,1]} & M^{0,m}_{v_d[0,0]} & \ldots & M^{0,m}_{v_d[-1,0]} \\
\vdots & \vdots & \ddots & \vdots \\
M^{0,m}_{v_d[m,0]} & M^{0,m}_{v_d[m-1,0]} & \ldots & M^{0,m}_{v_d[0,0]}
\end{bmatrix}
\]

\[(25)\]

and

\[
P = \begin{bmatrix}
V^{0,m}_x \rho^m_{(0,0)} & V^{0,m}_x \rho^m_{(0,-1)} & \ldots & V^{0,m}_x \rho^m_{(0,-m)} \\
V^{0,m}_x \rho^m_{(1,0)} & V^{0,m}_x \rho^m_{(1,-1)} & \ldots & V^{0,m}_x \rho^m_{(1,-m)} \\
\vdots & \vdots & \ddots & \vdots \\
V^{0,m}_x \rho^m_{(m,0)} & V^{0,m}_x \rho^m_{(m,-1)} & \ldots & V^{0,m}_x \rho^m_{(m,-m)}
\end{bmatrix}, W = \begin{bmatrix}
V^{0,m}_w(0,0) & V^{0,m}_w(0,1) & \ldots & V^{0,m}_w(0,-m) \\
V^{0,m}_w(1,0) & V^{0,m}_w(1,1) & \ldots & V^{0,m}_w(1,-m) \\
\vdots & \vdots & \ddots & \vdots \\
V^{0,m}_w(m,0) & V^{0,m}_w(m,1) & \ldots & V^{0,m}_w(m,-m)
\end{bmatrix}
\]

\[(26)\]

According to [20] the linear system (24) can be solved by \( O(m^3) \) steps and the solution can be obtained from the derived vector as follows:
\[ p^m(t_1,t_2) = \sum_{n_1=0}^{m} \sum_{n_2=0}^{m} P_{n_1(m+1)+(n_2+1)}(t_1-n_1,t_2-n_2) \]

(27)

IV. SIMULATION RESULTS

To compare the proposed method with the existing interpolation techniques, we have performed various simulations. The cubic B-spline, due to its short time support and relatively high accuracy in approximating the ideal lowpass filter, is the most common technique for interpolating 1-D lowpass signals. For the purpose of comparison, we have optimized a kernel with the same time support as an ideal lowpass filter. Figures 3 and 4 show the shape of the obtained kernel and the interpolating spline, respectively. Figure 3 shows that the energy is more concentrated in the middle of the cubic B-spline while the optimized kernel has slower decaying rate of energy at the sides. The resultant interpolation kernels, as depicted in Fig. 4, reveal that the main advantage of the optimized kernel (\( \hat{\rho}^3 \)) compared to the cubic spline (\( c^3 \)), is the smaller error in the first side-lobe. The SNR values of \( \hat{\rho}^3 \) and \( c^3 \) with respect to the sinc function are 20.39 and 13.15 dBs, respectively.

For a more realistic comparison, we have applied different interpolation techniques on standard test images. For this purpose, the original images, with or without applying the anti-aliasing filter (ideal lowpass filter), are down-sampled by a factor 2 in each direction (25% of the original pixels) and then they are enlarged (zooming) using the interpolation techniques. The comparison is made with the following interpolation methods: 1) bilinear interpolation, 2) bicubic interpolation, 3) wavelet-domain zero padding cycle-spinning [21], and 4) soft-decision estimation technique for adaptive image interpolation [22]. Also for our proposed method, two different scenarios are implemented: the ideal filter for which we are optimizing the kernel function is first taken as a sinc filter, and first the spectrum of the original image.

In order to evaluate the quality of the interpolated images, we have considered the Peak Signal-to-Noise Ratio (PSNR) criterion. Table II indicates the resultant PSNR values when the original image is subject to the anti-aliasing filter before down-sampling while the error is calculated based on the image without applying the filter. Table III contains similar values while the basis for the error calculation is the anti-aliased image. In both cases, the PSNR criterion the proposed optimized kernel for the ideal lowpass (sinc) filter. On the average, the proposed method outperforms the other standard interpolating methods by 0.74 dB in Table II and 4.19 dB in Table III.

To exclude the effect of the anti-aliasing filter, the simulations are repeated without applying it and the results are presented in Table IV. As expected, the optimized kernel which is matched to the spectrum of the original image outperforms other competitors in all cases. It should be mentioned that the function \( p^m[x, \hat{\rho}^3] \) is not a universal filter in this case and depends on the choice of the image.

Although the PSNR value is a good measure of global quality of an image, it does not reflect the local properties. In order to present a qualitative view of various interpolation methods, we have plotted the enlarged images for a segment of the Lena test image in Fig. 5. To highlight the differences, one could compare the texture on the top and the sharpness on the bottom edge of the hat.

V. CONCLUSION

The interpolation problem using uniform knots is a well studied subject. In this paper, we considered the problem of optimizing the interpolation kernel for a given class of signals (represented by a filter). Although functional optimization in the continuous domain is often very difficult, we have demonstrated the equivalency of this problem with a finite dimensional linear problem which can be easily solved using linear algebra. As a special case, we considered the class of lowpass signals which is associated with the sinc function as the optimum interpolation kernel. For the optimum compact-support interpolant, we compared our function with the conventional cubic B-Spline; the simulation results indicate 1dB improvement in the SNR of the interpolated signal (on the average) using the introduced function, and 7dB improvement compared to the cardinal spline itself (compared to the sinc function) (Fig. 4).

ACKNOWLEDGMENT

The authors would like to thank Prof. M. Unser from EPFL and Dr. R. Razvan from the mathematical sciences department of Sharif university for their helpful comments.

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Fig. 5. Comparison of different methods for the Lena image: (a) The original image, (b) bilinear interpolation, (c) bicubic Interpolation. (d) WZP Cycle-Spinning [21], (e) SAI [22], and (f) the proposed method.

TABLE II
PSNR (dB) RESULTS OF THE RECONSTRUCTED IMAGES BY VARIOUS METHODS, THE ORIGINAL IMAGE WAS ANTI-ALIASED BEFORE SAMPLING AND THE RESULTS ARE COMPARED TO THE ORIGINAL IMAGE (IMAGE ENLARGEMENT FROM 256 × 256 TO 512 × 512)

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<td>27.54</td>
<td>27.43</td>
<td>28.14</td>
<td>28.35</td>
<td>28.88</td>
</tr>
</tbody>
</table>

TABLE III
PSNR (dB) RESULTS OF THE RECONSTRUCTED IMAGES BY VARIOUS METHODS, THE ORIGINAL IMAGE WAS ANTI-ALIASED BEFORE SAMPLING AND THE RESULTS ARE COMPARED TO THE ANTI-ALIASED IMAGE (IMAGE ENLARGEMENT FROM 256 × 256 TO 512 × 512)

<table>
<thead>
<tr>
<th>Images</th>
<th>Bilinear</th>
<th>Bicubic [23]</th>
<th>WZP–CS [21]</th>
<th>SAI [22]</th>
<th>Optimized kernel for image $\rho^3[x, \rho^3_{[d]}]$</th>
<th>Optimized kernel for the ideal lowPass $\rho^3[sinc(t), \rho^3_{[d]}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lena</td>
<td>31.56</td>
<td>31.72</td>
<td>31.62</td>
<td>32.45</td>
<td>35.28</td>
<td>35.21</td>
</tr>
<tr>
<td>Baboon</td>
<td>26.88</td>
<td>27.26</td>
<td>26.89</td>
<td>28.48</td>
<td>26.33</td>
<td>35.70</td>
</tr>
<tr>
<td>Barbara</td>
<td>30.61</td>
<td>30.74</td>
<td>30.65</td>
<td>31.86</td>
<td>30.40</td>
<td>35.72</td>
</tr>
<tr>
<td>Peppers</td>
<td>31.62</td>
<td>31.82</td>
<td>31.77</td>
<td>32.14</td>
<td>37.41</td>
<td>35.02</td>
</tr>
<tr>
<td>Girl</td>
<td>34.08</td>
<td>34.24</td>
<td>34.25</td>
<td>33.10</td>
<td>35.89</td>
<td>34.28</td>
</tr>
<tr>
<td>Fishing bout</td>
<td>29.91</td>
<td>30.19</td>
<td>29.93</td>
<td>30.92</td>
<td>32.00</td>
<td>34.87</td>
</tr>
<tr>
<td>Couple</td>
<td>29.81</td>
<td>30.10</td>
<td>29.84</td>
<td>29.77</td>
<td>31.44</td>
<td>34.28</td>
</tr>
<tr>
<td>Overall Average</td>
<td>30.64</td>
<td>30.87</td>
<td>30.71</td>
<td>31.25</td>
<td>32.68</td>
<td>35.44</td>
</tr>
</tbody>
</table>

TABLE IV
PSNR (dB) RESULTS OF THE RECONSTRUCTED IMAGES BY VARIOUS METHODS, THE ORIGINAL IMAGE WAS NOT ANTI-ALIASED AND THE RESULTS ARE COMPARED TO THE ORIGINAL IMAGE 256 × 256 TO 512 × 512

<table>
<thead>
<tr>
<th>Images</th>
<th>Bilinear</th>
<th>Bicubic [23]</th>
<th>WZP–CS [21]</th>
<th>SAI [22]</th>
<th>Optimized kernel for image $\rho^3[x, \rho^3_{[d]}]$</th>
<th>Optimized kernel for the ideal lowPass $\rho^3[sinc(t), \rho^3_{[d]}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lena</td>
<td>30.21</td>
<td>30.13</td>
<td>30.05</td>
<td>30.88</td>
<td>32.29</td>
<td>30.95</td>
</tr>
<tr>
<td>Baboon</td>
<td>21.67</td>
<td>21.34</td>
<td>21.70</td>
<td>22.09</td>
<td>22.50</td>
<td>21.63</td>
</tr>
<tr>
<td>Barbara</td>
<td>23.90</td>
<td>23.32</td>
<td>23.88</td>
<td>23.71</td>
<td>25.10</td>
<td>22.58</td>
</tr>
<tr>
<td>Peppers</td>
<td>28.82</td>
<td>28.61</td>
<td>26.93</td>
<td>28.91</td>
<td>30.64</td>
<td>29.77</td>
</tr>
<tr>
<td>Girl</td>
<td>30.41</td>
<td>29.97</td>
<td>30.20</td>
<td>29.94</td>
<td>30.90</td>
<td>29.20</td>
</tr>
<tr>
<td>Fishing bout</td>
<td>27.10</td>
<td>26.93</td>
<td>27.07</td>
<td>27.63</td>
<td>28.50</td>
<td>27.66</td>
</tr>
<tr>
<td>Couple</td>
<td>26.92</td>
<td>26.73</td>
<td>26.86</td>
<td>26.93</td>
<td>27.91</td>
<td>27.08</td>
</tr>
<tr>
<td>Overall Average</td>
<td>27.00</td>
<td>26.72</td>
<td>26.67</td>
<td>27.16</td>
<td>29.12</td>
<td>26.98</td>
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</tbody>
</table>