Large Dimensional Factor Analysis*

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Abstract

Econometric analysis of large dimensional factor models has been a heavily researched topic in recent years. This review surveys the main theoretical results that relate to static factor models or dynamic factor models that can be cast in a static framework. Among the topics covered are how to determine the number of factors, how to conduct inference when estimated factors are used in regressions, how to assess the adequacy of observed variables as proxies for latent factors, how to exploit the estimated factors to test unit root tests and common trends,

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and how to estimate panel cointegration models. The fundamental result that justifies these analyses is that the method of asymptotic principal components consistently estimates the true factor space. We use simulations to better understand the conditions that can affect the precision of the factor estimates.
An inevitable fact as we move forward in time and as information technology improves is that data will be available for many more series and over an increasingly long span. While the availability of more data provides the opportunity to understand economic phenomena and anomalies better, researchers can also suffer from an information overload without some way to organize the data into an easy to interpret manner. In recent years, the analysis of large dimensional data has received the attention of theoretical and empirical researchers alike. The early focus has primarily been on the use of factor models as a means of dimension reduction. But the volume of research, both at the empirical and theoretical levels, has grown substantially. Empirical researchers have found it useful to extract a few factors from a large number of series in many forecasting and policy exercises. Theoretical researchers have taken up the challenge to extend standard factor analysis to allow the size of both dimensions of a panel data set to increase. The theoretical implications of using estimated factors in both estimation and inference are now better understood. Factor analysis plays a role not just in forecasting. In recent years, the factor structure has been incorporated into regression analysis to deal with cross-sectionally correlated errors and endogeneity bias.
This review provides a survey of the main theoretical results for large dimensional factor models, emphasizing results that have implications for empirical work. We focus on the development of the static factor models, which are to be distinguished from dynamic factor models in ways to be made precise. Key results concerning large dynamic factor models are given in Forni et al. (2000, 2004, 2005). Results concerning the use of factors in forecasting are discussed in Stock and Watson (2006), Banerjee et al. (2006), and Giannone et al. (2007). Here, our focus will be on the use of estimated factors in subsequent estimation and inference. While we survey many of the analytical results that are of use to empirical researchers, a survey of empirical applications of large factor models will not be included. Surveys with heavier empirical focus can be found in Breitung and Eickmeier (2005) and Reichlin (2003). Suffice it to say that factor models have been used in forecasting of the conditional mean by Stock and Watson (2002b), Cristadoro et al. (2001), Artis et al. (2005), Marcellino et al. (2003), Schumacher (2005), Forni et al. (2001), den Reijer (2005), and many others. Boivin and Ng (2005) compared the use of dynamic and static factors in forecasting. Anderson and Vahid (2007) used the factor model to forecast volatility with jump components. A non-exhaustive list of policy analyses that adopt a factor approach includes Bernanke and Boivin (2003), Giannone et al. (2005a,b), Favero et al. (2005), Stock and Watson (2005), and Forni et al. (2003). Use of factors as conditioning information is discussed in the conditional risk-return analysis of Ludvigson and Ng (2007), and term structure analysis of Ludvigson and Ng (2005).

This survey, drawing heavily from our previous work, is organized to serve three purposes. First, the results are presented under a coherent and general set of assumptions. Situations that require stronger assumptions will be made clear as we go along. Second, results for stationary and non-stationary data are discussed separately, as they involve different assumptions and are used in different contexts. Third, consistent estimation of the factor space is fundamental to many of the results. We use simulations to study what are the main aspects of the data that affect the precision of the factor estimates.
We begin by setting up notation and making a distinction between a static and a dynamic factor model. Let $N$ be the number of cross-section units and $T$ be the number of time series observations. For $i = 1, \ldots, N, \ t = 1, \ldots, T,$ a static model is defined as

$$x_{it} = \lambda_i' F_t + e_{it} = C_{it} + e_{it}. \quad (2.1)$$

In factor analysis, $e_{it}$ is referred to as the idiosyncratic error and $\lambda_i$ is referred to as the factor loadings. This is a vector of weights that unit $i$ put on the corresponding $r$ (static) common factors $F_t$. The term $C_{it} = \lambda_i' F_t$ is often referred to as the common component of the model. Factor models arise naturally in economics. For example, $x_{it}$ is the GDP growth rate for country $i$ in period $t$, $F_t$ is a vector of common shocks, $\lambda_i$ is the heterogenous impact of the shocks, and $e_{it}$ is the country-specific growth rate. In finance, $x_{it}$ is the return for asset $i$ in period $t$, and $F_t$ is vector of systematic risks (or factor returns)

\footnote{By letting $x_{it}^0$ be the observed data, the model can be generalized to $x_{it} = (1 - \rho_i(L)) x_{it}^0 = \lambda_i' F_t + e_{it}$.}
Factor Models

and \( \lambda_i \) is the exposure to the factor risks, and \( e_{it} \) is the idiosyncratic returns.

Let \( X_t = (x_{1t}, x_{2t}, \ldots, x_{Nt})' \), \( F = (F_1, \ldots, F_T)' \), and \( \Lambda = (\lambda_1, \ldots, \lambda_N)' \).

In vector form, we have

\[
X_t = \Lambda F_t + e_t. \tag{2.2}
\]

Let \( X = (X_1', \ldots, X_N') \) be a \( T \times N \) matrix observations. The matrix representation of the factor model is

\[
X = F \Lambda' + e, \tag{2.3}
\]

where \( e = (e_1', e_2', \ldots, e_N') \) is \( T \times N \) (The upper case \( E \) is reserved for the expectation operator). Although the model specifies a static relationship between \( x_{it} \) and \( F_t \), \( F_t \) itself can be a dynamic vector process that evolves according to

\[
A(L)F_t = u_t, \tag{2.4}
\]

where \( A(L) \) is a polynomial (possibly infinite order) of the lag operator. The idiosyncratic error \( e_{it} \) can also be a dynamic process. The assumptions to be stated below also permit \( e_{it} \) to be cross-sectionally correlated.

The static model is to be contrasted with a dynamic factor model, defined as

\[
x_{it} = \lambda_i'(L)f_t + e_{it},
\]

where \( \lambda_i(L) = (1 - \lambda_{i1}L - \cdots - \lambda_{is}L^s) \) is a vector of dynamic factor loadings of order \( s \). The term “dynamic factor model” is sometimes reserved for the case when \( s \) is finite, whereas a “generalized dynamic factor model” allows \( s \) to be infinite. In either case, the factors are assumed to evolve according to

\[
f_t = C(L)\varepsilon_t,
\]

where \( \varepsilon_t \) are iid errors. The dimension of \( f_t \), denoted \( q \), is the same as the dimension of \( \varepsilon_t \). We can rewrite the model as

\[
x_{it} = \lambda_i(L)'C(L)\varepsilon_t + e_{it}.
\]
In the literature, $q = \dim(\varepsilon_t)$ is referred to as the number of dynamic factors.

Both models have their origin in the statistics literature. Assuming $F_t$ and $e_t$ are uncorrelated and have zero mean, the covariance structure of the static model is given by

$$\Sigma = \Lambda \Lambda' + \Omega,$$

where $\Sigma$ and $\Omega$ are the $N \times N$ population covariance matrix of $X_t$ and $\varepsilon_t$, respectively; the normalization $E(F_t F_t') = I$, is assumed. If $\Omega$ is diagonal, (2.1) is referred to as a strict factor model, see Chamberlain and Rothschild (1983). In classical factor analysis, $F_t$ and $e_t$ in (2.1) are generally assumed to be serially and cross-sectionally uncorrelated.

Properties of such a model, under the assumptions that (i) $e_t$ is iid over $t$; (ii) $N$ is fixed as $T$ tends to infinity (or vice versa); and (iii) both $F_t$ and $e_t$ are normally distributed, are well documented; see Lawley and Maxwell (1971), Anderson and Rubin (1956), and Anderson (1984), Classical factor analysis estimates $\Lambda$ and the diagonal elements of $\Omega$, with which factor scores $F_t$ can also be estimated. The estimated score cannot be consistent since $N$ is fixed. The limiting distribution is based on asymptotic normality for an estimator of $\Sigma$ (e.g., the sample covariance matrix). For large $N$, this method of analysis is not applicable since $\Sigma$ $(N \times N)$ is not consistently estimable.

Classical factor models have been widely used in psychology and other disciplines of the social sciences but less so in economics, perhaps because the assumption that the factors and errors are serially and cross-sectionally correlated do not match up well with economic data. The dynamic classical factor model maintains the assumption that the errors are independent across $i$ but explicitly recognizes the fact that the data being analyzed are serially correlated. Sargent and Sims (1977) and Geweke (1977) were amongst the first to apply the dynamic factor approach to macroeconomic analysis.

A dynamic factor model with $q$ factors can be written as a static factor model with $r$ factors, where $r$ is finite. However, the dimension of $F_t$ will in general be different from the dimension of $f_t$ since $F_t$ includes the leads and lags of $f_t$. More generally, if we have $q$ dynamic factors, we will end up with $r = q(s + 1) \geq q$ static factors. Although
knowledge of the dynamic factors is useful in some analysis such as precisely establishing the number of primitive shocks in the economy, it turns out that many econometric methods can be developed within the static framework. Consequently, the properties of the estimated static factors are much better understood from a theoretical standpoint. Empirically, the static and the dynamic factor estimates produce rather similar forecasts. From a practical perspective, the primary advantage of the static framework is that it is easily estimated using time domain methods and involves few choices of auxiliary parameters. Dynamic factor models are estimated using tools of frequency domain analysis, and the proper choice of the auxiliary parameters remains an issue requiring further research.

An important characteristic of a static model with $r$ factors is that the largest $r$ population eigenvalues of $\Sigma$ increase with $N$, while the remaining eigenvalues of $\Sigma$, as well as all eigenvalues of $\Omega$, are bounded. Intuitively, the information of the common component accumulates as we sum up the observations across $i$ and therefore the eigenvalues of the population covariance matrix of the common component will increase with $N$. In contrast, the $e_{it}$ are unit-specific errors and summing the errors across $i$ does not lead to the same accumulation of information. In other words, the eigenvalues of $\Omega$ cannot increase without bound, as $N$ increases. It is this difference in the property of the eigenvalues that distinguishes the common from the idiosyncratic component. If the eigenvalues of the common component increases with $N$, so will the population eigenvalues of $\Sigma$.

The large dimensional static factor model we consider differs from the classical factor model by relaxing the three mentioned assumptions. Work in this direction was initiated by Stock and Watson in the late 1990s. Around that same time, assumptions of the classical dynamic factor model were also relaxed, notably by Forni, Hallin, Lippi, and Reichlin. The efforts of these researchers were instrumental in advancing the theory and use of large dimensional dynamic factor models. Collectively, the new generation of factor models has come to be known as “large dimensional approximate factor models.” By “large”, we mean that the sample size in both dimensions tends to infinity in the asymptotic theory. By an “approximate” factor
structure, we mean that the idiosyncratic errors are allowed to be “weakly” correlated across $i$ and $t$ in a sense to be explained.

The only quantities that are observed in factor analysis are the data, $x_{it}$. Neither the factors, their loadings, nor the idiosyncratic errors are observed, and the factors and the loadings are not even separately identifiable. Even estimation of classical factor models with the sample size fixed in one dimension can pose difficulties if one allows for heterogeneous variables, for example. Large dimensional factor models pose additional statistical problems that need to be solved. Whereas classical (static or dynamic) factor models can be estimated consistently by methods that rely on sample moments converging to population moments of fixed dimensions, this approach is no longer appropriate when the dimensions of these moment matrices are themselves increasing. The theory we explore below surrounds the new estimation and inferential results that are developed specifically for the large $N$ and $T$ environment. Results are presented for the principal components estimator, which is easy to compute.

\footnote{Connor et al. (2007) restrict the factor loadings to be unknown functions of some observable variables such that $\lambda_{ij} = g_j(z_i)$ ($j = 1, 2, \ldots, r$). There are $r$ unknown functions to be estimated. The estimation of this model will not be considered in this survey.}
Under large $N$ and large $T$, it is possible to estimate $\Lambda$ and $F$ simultaneously. That is, both $\Lambda$ and $F$ are treated as parameters. After obtaining $\Lambda$ and $F$, the residual matrix $e$ is also obtained from $e = X - FA'$. In contrast, classical factor analysis estimates $\Lambda$ (under fixed $N$) and the covariance matrix of $e_t$, which is assumed to be diagonal. Given $\Lambda$, factor scores $F_t$ are then estimated at the second stage. The estimate for $F_t$ is not consistent under fixed $N$. Consider estimating (2.3),

$$X = FA' + e.$$ 

Clearly $F$ and $\Lambda$ are not separately identifiable. For an arbitrary $r \times r$ invertible matrix $A$, $FA' = FAA^{-1}A' = F^*\Lambda^*$, where $F^* = FA$ and $\Lambda^* = \Lambda A^{-1}$, the factor model is observationally equivalent to $X = F^*\Lambda^* + e$. Hence restrictions are needed to uniquely fix $F$ and $\Lambda$. Since an arbitrary $r \times r$ matrix has $r^2$ free parameters, we need $r^2$ number of restrictions. The normalization

$$F'F/T = I_r$$

provides $r(r + 1)/2$ restrictions. The requirement of $\Lambda'\Lambda$ being diagonal gives $r(r - 1)/2$ additional restrictions.\(^1\) Combining normalization and

\(^1\)Diagonality in fact imposes $r(r - 1)$ restrictions. Within the class of symmetric matrices, diagonality only imposes half as many restrictions.
diagonality we have $r^2$ restrictions, which will uniquely fix $F$ and $\Lambda$ (still up to a column sign change) given the product $F\Lambda'$. Alternatively, the normalization $\Lambda'\Lambda/N = I_r$ and $F'F$ being diagonal can be used. These restrictions are used in the principal components method. When $k$ factors are estimated, $k^2$ restrictions are required.

The method of asymptotic principal components was first considered by Connor and Korajczyk (1986) and Connor and Korajczyk (1998) as an estimator of the factors in a large $N$, fixed $T$ setup. For any given $k$ not necessarily equal to the true number of factors $r$, the method of principal components (PC) constructs a $T \times k$ matrix of estimated factors and a corresponding $N \times k$ matrix of estimated loadings by solving the optimization problem

$$\min_{\Lambda^k, F^k} S(k), \quad \text{with} \quad S(k) = (NT)^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} (x_{it} - \lambda_i^k F_i^k)^2$$

subject to the normalization that $\Lambda^k\Lambda^k/N = I_k$ and $F^kF^k/T = I_k$ and $\Lambda^k\Lambda^k$ being diagonal.

Mechanically speaking, the estimates can be obtained in one of two ways. The first solution obtains if we concentrate out $\Lambda^k$. The problem is then identical to maximizing $\text{tr}(F^k(X'X)F^k)$. The estimated factor matrix, $\tilde{F}^k$, is $\sqrt{T}$ times the eigenvectors corresponding to the $k$ largest eigenvalues of the $T \times T$ matrix $XX'$. Using the normalization $\tilde{F}^k\tilde{F}/T = I_k$ yields $\tilde{\Lambda}^k = \tilde{F}^kX/T$. Note that in this case, $\tilde{\Lambda}^k\tilde{\Lambda}^k$ being diagonal is automatically fulfilled. The solution for $F^k$ that maximizes $\text{tr}(F^k(X'X)F^k)$ is not unique under the restriction $F^kF^k/T = I_k$. Any orthogonal rotation of a solution is also a solution. Choosing $F^k$ to be the eigenvector fixes this rotational indeterminacy and, at the same time, makes $\Lambda^k\Lambda^k$ diagonal.

The second solution obtains if we concentrate out $F^k$. Then the matrix of factor loadings $\hat{\Lambda}^k$ is $\sqrt{N}$ times the eigenvectors corresponding to the $k$ largest eigenvalues of the $N \times N$ matrix $X'X$. Using the normalization that $\hat{\Lambda}^k\hat{\Lambda}^k/N = I_k$ yields $\hat{F}^k = X\hat{\Lambda}^k/N$. The second set of calculations is computationally less costly when $T > N$ while the first is less intensive when $T < N$. In all except one case that will be stated below, our results do not depend on which of the two methods is used.
Let $\bar{V}^k$ denote the $k \times k$ diagonal matrix consisting of the first $k$ largest eigenvalues of the matrix $XX'/(TN)$, arranged in decreasing order. Note that the matrices $XX'$ and $X'X$ have identical nonzero eigenvalues. We first explore some identities among different estimators. First, we have

$$\begin{align*}
\bar{F}^k' \bar{F}^k &= \bar{V}^k \\
\bar{\Lambda}^k &= \bar{V}^k\bar{\Lambda}^k
\end{align*}$$

(3.1)

To see this, by the definition of the eigenvalues and eigenvectors, $(NT)^{-1}X'X\bar{\Lambda}^k = \bar{\Lambda}^k\bar{V}^k$. Left multiplying the transpose of $\bar{\Lambda}^k$, and using $\bar{\Lambda}^k\bar{\Lambda}^k = N$ and $\bar{F}^k = X\bar{\Lambda}^k/N$, the first equality is obtained. The second equality follows from a similar argument (or simply by symmetry). Additional useful identities are

$$\begin{align*}
\bar{F}^k &= \bar{F}^k(\bar{V}^k)^{1/2} \\
\bar{\Lambda}^k &= \bar{\Lambda}^k(\bar{V}^k)^{1/2}.
\end{align*}$$

(3.2)

To see this, by the definition of eigen relationship,

$$\left(\frac{1}{NT}X'X\right) \bar{\Lambda}^k = \bar{\Lambda}^k \bar{V}^k.$$

Left multiplying $X/N$ on each side of above and noting $\bar{F}^k = X\bar{\Lambda}^k/N$, we have

$$\left(\frac{1}{NT}XX'\right) \bar{F}^k = \bar{F}^k \bar{V}^k.$$

This means that $\bar{F}^k$ is a $T \times k$ matrix consisting of the eigenvectors of the matrix $XX'$. By the uniqueness of eigenvector with probability 1 (up to a scale), each column of $\bar{F}^k$ is a scalar multiple of the corresponding column of $\bar{F}^k$ because $\bar{F}^k$ is also an eigenvector matrix. The squared length of each column in $\bar{F}^k$ is equal to the corresponding eigenvalue (see Equation (3.1)), and the columns of $\bar{F}^k$ have unit length, it follows that $\bar{F}^k = \bar{F}^k(\bar{V}^k)^{1/2}$. The second equality in (3.2) follows from symmetry.
Throughout this survey, when $k$ is equal to $r$ (the true number of factors), we simply drop the subscript $k$ to write $\tilde{F}$, $\tilde{\Lambda}$, and $\tilde{V}$. Similarly, the superscript $k$ will be dropped for the “bar” version of the estimates when $k = r$. The matrix $\tilde{V}$ is identical to its bar version (does not depend on the estimation method).
To study the properties of this estimator, we let $F_0^t$ and $\lambda^0_i$ denote the true factors and the loadings, respectively. Let $M$ be a generic constant. For stationary data, the following assumptions underlie much of the analysis:

**Assumption F(0):** $E\|F_0^t\|^4 \leq M$ and $\frac{1}{T} \sum_{t=1}^{T} F_0^t F_0^{t'}^\prime \xrightarrow{p} \Sigma_F > 0$ for an $r \times r$ non-random matrix $\Sigma_F$.

**Assumption L:** $\lambda^0_i$ is either deterministic such that $\|\lambda^0_i\| \leq M$, or it is stochastic such that $E\|\lambda^0_i\|^4 \leq M$. In either case, $N^{-1} \Lambda^0 \Lambda^0^\prime \xrightarrow{p} \Sigma_\Lambda > 0$ for an $r \times r$ non-random matrix $\Sigma_\Lambda$, as $N \to \infty$.

**Assumption E:**

(a) $E(e_{it}) = 0$, $E|e_{it}|^8 \leq M$.

(b) $E(e_{it}e_{js}) = \sigma_{ij,ts}$, $|\sigma_{ij,ts}| \leq \bar{\sigma}_{ij}$ for all $(t,s)$ and $|\sigma_{ij,ts}| \leq \tau_{ts}$ for all $(i,j)$ such that $\frac{1}{N} \sum_{i,j=1}^{N} \bar{\sigma}_{ij} \leq M$, $\frac{1}{T} \sum_{t,s=1}^{T} \tau_{ts} \leq M$, and $\frac{1}{NT} \sum_{i,j,t,s=1}^{\infty} |\sigma_{ij,ts}| \leq M$.

(c) For every $(t,s)$, $E|N^{-1/2} \sum_{i=1}^{N} [e_{is}e_{it} - E(e_{is}e_{it})]|^4 \leq M$. 

102
(d) For each \(t\), \(\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \lambda_i e_{it} \xrightarrow{d} N(0, \Gamma_t)\), as \(N \to \infty\) where
\[
\Gamma_t = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} E(\lambda_i \lambda_j' e_{it} e_{jt}).
\]

(e) For each \(i\), \(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} F_te_{it} \xrightarrow{d} N(0, \Phi_i)\) as \(T \to \infty\) where
\[
\Phi_i = \lim_{T \to \infty} T^{-1} \sum_{s=1}^{T} \sum_{t=1}^{T} E(F_t^0 F_{s}^0 e_{is} e_{it}).
\]

**Assumption LFE:** \(\{\lambda_i\}, \{F_t\},\) and \(\{e_{it}\}\) are three mutually independent groups. Dependence within each group is allowed.

**Assumption IE:** For all \(t \leq T, \ i \leq N, \ \sum_{s=1}^{T} |\tau_{s,t}| \leq M,\) and \(\sum_{i=1}^{N} |\bar{\sigma}_{ij}| \leq M.\)

Assumptions \(\text{F}(0)\) (stationary factors) and \(\text{L}\) are moment conditions on the factors and the loadings, respectively and are standard in factor models. They ensure that the factors are non-degenerate and that each factor has a nontrivial contribution to the variance of \(X_t\). Assumption \(\text{E}\) concerns the idiosyncratic errors. Part (b) allows for “weak” time series and cross-section dependence in the idiosyncratic errors so that the model has an approximate instead of a strict factor structure. The notion of an approximate factor model is due to Chamberlain and Rothschild (1983) who showed in the context of asset returns that so long as the largest eigenvalue of \(\Omega\) is bounded, the idiosyncratic errors are allowed to be mildly cross-sectionally correlated. A similar assumption was used by Connor and Korajczyk (1986). Our restrictions on the errors allow weak correlation not just cross-sectionally, but also serially; heteroskedasticity is also allowed. Now under covariance stationarity with \(E(e_{it} e_{jt}) = \sigma_{ij}\) for all \(t\), the largest eigenvalue of \(\Omega\) is bounded by \(\max_i \sum_{j=1}^{N} |\sigma_{ij}|\). By assuming that \(\sum_{j=1}^{N} |\sigma_{ij}| \leq M\) for all \(i\) and \(N\) as in (b), the assumptions of an approximate factor model in the sense of Chamberlain and Rothschild (1983) are satisfied. As Heaton and Solo (2006) noted, the maximum eigenvalue bound of Chamberlain and Rothschild can be obtained even if the number of correlated series
increases with $N$. Our assumption, also used by Stock and Watson (2002a), indeed allows the number of strongly cross-correlated errors to grow at a rate slower than $\sqrt{N}$, and in this sense, we permit a stronger degree of cross-section correlation in the errors than the approximate factor model of Chamberlain and Rothschild. Part (d) permits weak dependence between the factors and the idiosyncratic errors and falls short of requiring $F_t$ to be independent of $e_{it}$.

Within group dependence in Assumption LFE means that $F_t$ can be serially correlated, $\lambda_i$ can be correlated over $i$, and $e_{it}$ can have serial and cross-sectional correlations. None of these correlations can be too strong if Assumption E is to hold. However, we assume no dependence between the factor loadings and the factors, or between the factors and the idiosyncratic errors, which is the meaning of mutual independence among the three groups.

Assumption IE strengthens E. When $e_{it}$ is independent over time, the first part of (IE) is equivalent to requiring $E(e_{it}^2) \leq M$ for all $t$. Similarly, under cross-section independence, the second part of (IE) is equivalent to $E(e_{it}^2) \leq M$ for all $i$. Thus, under time series and cross-section independence, both parts of (IE) are implied by (E).

Let $\hat{F}^k = \tilde{F}^k(\tilde{V})^{1/2} = \tilde{F}^k\tilde{V}^k$. By the relationship given in Section 3,

$$\hat{F}^k = \tilde{F}^k(\tilde{V})^{1/2} = \tilde{F}^k\tilde{V}^k.$$ 

And for $k = r$, dropping the superscript, we have $\hat{F} = \tilde{F}\tilde{V}$. Equivalently, for every $t$,

$$\hat{F}_t = \tilde{V}\tilde{F}_t.$$ 

Let $V$ be an $r \times r$ diagonal matrix with the eigenvalues of $\Sigma_1^{1/2}\Sigma_F\Sigma_1^{1/2}$ as its elements. The matrix $\tilde{V}$ has been shown to have an invertible limit, given by $V$, see Bai (2003). Since $\tilde{V}$ is invertible, $\tilde{F}$ and $\tilde{F}$ are equivalent in the sense that knowing one will lead to the other.

Our main results concerning the factor estimates can be summarized as follows. The first result is for an arbitrary $k$, and the remaining results are for $k = r$. Throughout, we let $C_{NT} = \min[\sqrt{N}, \sqrt{T}]$. 


Result A: Factor Space

A.1. For any fixed \( k > 1 \) and under Assumptions \( F(0), L, \) and \( E, \)

\[
C_{NT}^2 \left( \frac{1}{T} \sum_{t=1}^{T} \| \hat{F}_t^k - \hat{H}^{k} F_0^t \|^2 \right) = O_p(1),
\]

where \( \hat{H}^k = (\hat{F}^{k'} F^0 / T)(\Lambda^0 \Lambda^0 / N) \) with \( \text{rank} (\hat{H}^k) = \min (k,r) \). If, in addition, \( \sum_{s=1}^{T} \tau_{s,t} \leq M \) for all \( t \) and \( T \), then

\[
C_{NT}^2 \| \hat{F}_t^k - \hat{H}^{k} F_0^t \|^2 = O_p(1)
\]

for each \( t \). Note that matrix \( H^k \) depends on both \( N \) and \( T \), but not on \( t \).

A.2. Let \( H = \hat{V}^{-1}(\hat{F}' F^0 / T)(\Lambda^0 \Lambda^0 / N) \), \( Q = V^{1/2} \Psi \Sigma^{-1/2} \), where \( \Psi \) are the eigenvectors corresponding to the eigenvalues \( V \) of the matrix \( \Sigma^{-1/2} \Sigma_F \Sigma^{-1/2} \). Under Assumptions \( F(0), L, E, \) and \( LFE, \)

(a) if \( \sqrt{N} / T \to 0 \), then for each \( t, \)

\[
\sqrt{N}(\hat{F}_t^k - H' F_0^t) \overset{d}{\to} N(0, V^{-1} \Gamma_t \Psi' \Psi V^{-1}).
\]

(b) if \( \sqrt{T} / N \to 0 \), then for each \( i, \)

\[
\sqrt{T}(\hat{\lambda}_i^k - H^{-1} \lambda_i^0) \overset{d}{\to} N(0, (Q')^{-1} \Phi_i Q^{-1}).
\]

A.3. Let \( A_{it} = \lambda_i^0 \Sigma^{-1}_0 \Gamma_t \Sigma^{-1}_t \lambda_i^0 \) and \( B_{it} = F_i^0 \Sigma_{F}^{-1} \Phi_i \Sigma_{F}^{-1} F_t^0 \), where \( \Phi_i \) is the variance of \( T^{-1/2} \sum_{t=1}^{T} F_t^0 e_{it}; \)

(a) Under Assumptions \( F(0), L, E, LFE, \) and \( IE \)

\[
(N^{-1} A_{it} + T^{-1} B_{it})^{-1/2} (\hat{C}_{it} - C_{it}) \overset{d}{\to} N(0, 1)
\]

(b) if \( N / T \to 0 \), then \( \sqrt{N}(\hat{C}_{it} - C_{it}) \overset{d}{\to} N(0, A_{it}); \)

(c) if \( T / N \to 0 \), then \( \sqrt{T}(\hat{C}_{it} - C_{it}) \overset{d}{\to} N(0, B_{it}). \)

A.4. Suppose Assumptions \( F(0), L, E, \) and \( LFE \) hold. Then

\[
\max_{1 \leq t \leq T} \| \hat{F}_t - H' F_0^t \| = O_p(T^{-1/2}) + O_p((T/N)^{1/2}).
\]
As we can only estimate the space spanned by the factors, all results concerning the factor estimates are stated in terms of the difference between the estimated and the true factor space, $H^k F^0_t$. Part 1, derived by Bai and Ng (2002), says that if we estimate $k$ (not necessarily the same as $r$) factors, the average squared deviation between the $k$ estimated factors and the space spanned by $k$ of the true factors will vanish at rate $\min[N,T]$, or in other words, the smaller of the sample size in the two dimensions. Result A.1 is useful for determining the number of factors, which will be considered later on. The second part of A.1 imposes stronger condition on the error correlations.

Result A.1 has been extended by Anderson and Vahid (2007) to allow for jumps, an issue that is relevant in volatility analysis. The authors argue that jumps can distort the principal components estimates. Treating jumps as measurement error, an IV approach is used to correct for the bias. Their IV estimate of the factors are the eigenvectors of the covariance matrix of $\hat{X}$, where $\hat{X}$ is the orthogonal projection of $X$ on lags of $X$.

Whereas A.1 provides the starting point for estimation, the asymptotic distribution is required to assess the sampling variability of the factor estimates. These results are derived by Bai (2003) and are summarized in A.2 and A.3. Essentially, for each $t$, $\tilde{F}_t$ is $\sqrt{N}$ consistent for the true factor space while for each $i$, $\tilde{\lambda}_i$ is $\sqrt{T}$ consistent for the space spanned by the true factor loadings. Although these results put restrictions between $N$ and $T$, note that $N$ and $T$ pass to infinity jointly, not sequentially. It is clear that $\sqrt{N}/T \to 0$ and $\sqrt{T}/N \to 0$ are not strong conditions. The approximation should work well even for panels with just 50 units in each dimension. A.4 provides an upper bound on the maximum deviation of the estimated factors from the space spanned by the true ones. The upper bound goes to zero if $T/N \to 0$. A sharper bound is obtained in Bai and Ng (2008). If there exists $\ell \geq 4$ such that $E\|F_t\|^{\ell} \leq M$ and $E\|N^{-1/2}\sum_{i=1}^{N} \lambda^0_i e_{it}\|^{\ell} \leq M$ for all $t$, then

$$\max_{1 \leq t \leq T} \|\tilde{F}_t - H' F^0_t\| = O_p(T^{1+1/\ell}) + O_p(T^{1/\ell}/N^{1/2}). \quad (4.1)$$

For $\ell = 4$ as is assumed, the maximum deviation goes to zero provided that $T/N^2 \to 0$. 

The sampling uncertainty of $\tilde{F}_t$ is captured by $\text{Avar}(\tilde{F}_t) = V^{-1} Q \Gamma_t Q' V^{-1}$, which can be rewritten as $V^{-1} Q H (H^{-1} \Gamma_t H^{-1'}) H' Q' V^{-1}$. Matrix $V^{-1}$ is a consistent estimate for $V^{-1}$. Matrix $Q$ is the limit of $\tilde{F} F^0 / T$. An approximation for $Q H$ is obtained by substituting $\tilde{F}$ for $F^0 H$ since $\tilde{F}$ provides an approximation for $F^0 H$. Note that $\tilde{Q} = \tilde{F} \tilde{F} / T$ is just an identity matrix by construction (the limit of $Q H$ is indeed an identity). It remains to obtain a consistent estimate for $H^{-1} \Gamma_t H^{-1'}$. Denote its consistent estimate by $\tilde{\Gamma}_t$. This leads to

$$\text{Avar}(\tilde{F}_t) = V^{-1} \tilde{\Gamma}_t V^{-1}.$$ 

Several estimators of the $r \times r$ matrix $H^{-1} \Gamma_t H^{-1'}$ are possible. There is no need to estimate $H$. When estimating $\Gamma_t$, we have to use $\tilde{\lambda}_i$ in place of $\lambda_i$ because $\lambda_i$ cannot be observed. Since $\tilde{\lambda}_i = H^{-1} \lambda_i + o_p(1)$, $H^{-1}$ is implicitly estimated.

Result B: Estimation of $\Gamma_t$. Let $\tilde{e}_{it} = x_{it} - \tilde{\lambda}'_i \tilde{F}_i$:

B.1. Cross-sectionally independent but heterogeneous panels:

let

$$\tilde{\Gamma}_t = \frac{1}{N} \sum_{i=1}^{N} \tilde{e}_{it}^2 \tilde{\lambda}_i \tilde{\lambda}'_i.$$ 

Under Assumption F(0), L, E, and LFE, we have $\| \tilde{\Gamma}_t - H^{-1} \Gamma_t H^{-1'} \| \xrightarrow{p} 0$.

B.2. Cross-sectionally independent and homogeneous panels:

let

$$\tilde{\Gamma}_t = \tilde{\sigma}_e^2 \frac{1}{N} \sum_{i=1}^{N} \tilde{\lambda}_i \tilde{\lambda}'_i,$$

where $\tilde{\sigma}_e^2 = \frac{1}{NT-r(T+N-r)} \sum_{i=1}^{N} \sum_{t=1}^{T} \tilde{e}_{it}^2$. The result of B.1 still holds.

B.3. Cross-sectionally correlated but stationary panels: let

$$\tilde{\Gamma} = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{\lambda}_i \tilde{\lambda}'_j \frac{1}{T} \sum_{t=1}^{T} \tilde{e}_{it} \tilde{e}_{jt}.$$
Suppose Assumption F(0), IE, E, and LFE hold, and $E(e_{it}e_{jt}) = \sigma_{ij}$ for all $t$ so that $\Gamma_t = \Gamma$ not depending on $t$.

If $n$ is such that with $n/\min[N,T] \to 0$, then

$$\|\hat{\Gamma} - H^{-1}\Gamma H^{-1}\| \overset{p}{\to} 0.$$ 

The estimator defined in B.3 is referred to the CS-HAC estimator in Bai and Ng (2006a). It is robust to cross-section correlation and cross-section heteroskedasticity but requires the assumption of covariance stationarity. If there is cross-section correlation of an unknown form and the data have no natural ordering, estimation of $\Gamma_t$ runs into problems analogous to the estimation of the spectrum in time series. Specifically, some truncation is necessary to obtain a consistent estimate of $\Gamma_t$. Here in a panel setting, we use the fact that if covariance stationarity holds, the time series observations will allow us to consistently estimate the cross-section correlations provided $T$ is large. Furthermore, the covariance matrix of interest is of dimension $(r \times r)$ and can be estimated with $n < N$ observations. The assumption of covariance stationarity is not necessary for B.1 and B.2 since these assume cross-sectionally uncorrelated idiosyncratic errors.

### 4.1 Estimating the Number of Factors

Some economic models have a natural role for factors and thus determining the number of factors is of interest in its own right. For example, underlying the APT theory of Ross (1976) is the assumption that there are common risk factors across assets, while underlying consumer demand analysis is the notion that there are individual factors common across goods. From a statistical standpoint, being able to consistently estimate the number of factors enables the researcher to treat $r$ as known, so that we can simply deal with the $r \times 1$ vector of factor estimates $\tilde{F}_t$, instead of a sequence of factor estimates $\tilde{F}_k^t$. The result concerning estimation of $r$ can be summarized as follows:

Let $S(k) = (NT)^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} (x_{it} - \hat{\lambda}_i^k \tilde{F}_t^k)^2$ be the sum of squared residuals (divided by $NT$) when $k$ factors are estimated. Let $g(N,T)$
be a penalty function. Define the information criteria

\[ PCP(k) = S(k) + k \hat{\sigma}^2 g(N,T) \]
\[ IC(k) = \ln(S(k)) + kg(N,T), \]

where \( \hat{\sigma}^2 \) is equal to \( S(k_{\text{max}}) \) for a pre-specified value \( k_{\text{max}} \). The second criterion does not depend on \( \hat{\sigma}^2 \). The estimator for the number of factors is defined as

\[ \hat{k}_{PCP} = \arg\min_{0 \leq k \leq k_{\text{max}}} PCP(k) \]
\[ \hat{k}_{IC} = \arg\min_{0 \leq k \leq k_{\text{max}}} IC(k). \]

**Result C: Number of Static Factors**

C.1. Suppose Assumptions F(0), L, E, and LFE hold. If (i) \( g(N,T) \to 0 \) and (ii) \( C_{NT}^2 g(N,T) \to 0 \) as \( N,T \to \infty \), then
\[
\text{prob}(\hat{k}_{PCP} = r) \to 1, \quad \text{and} \quad \text{prob}(\hat{k}_{IC} = r) \to 1.
\]

Result C is based on Bai and Ng (2002). The first condition in C.1 prevents under-fitting while the second condition prevents over-fitting. Similar looking conditions underlie well-known model selection procedures such as the AIC and the BIC that are often used in time series and cross section analysis. The notable and important difference is that our penalty factor \( g(N,T) \) depends on the sample size in both dimensions, not just \( N \) or \( T \). In problems for which \( \sqrt{T} \) consistent estimates are available, condition (ii) would require divergence of \( Tg(T) \). But in our particular panel setting, the convergence rate of the estimated factor space dictates that \( \min[N,T]g(N,T) \) must diverge. Examples of \( g(N,T) \) that satisfy the required conditions are

\[ g_1(N,T) = \frac{N + T}{NT} \ln \left( \frac{NT}{N + T} \right) \]
\[ g_2(N,T) = \frac{N + T}{NT} \ln C_{NT}^2 \]
\[ g_3(N,T) = \frac{\ln C_{NT}^2}{C_{NT}^2}. \]
We have frequently used \( g_2(N,T) \) in empirical work because it tends to be more stable. In our experience, \( g_4(N,T) = (N + T - k) \ln(NT)/NT \) has good properties especially when the errors are cross correlated. Strictly speaking, \( g_4(N,T) \) fails condition (i) when \( T = \exp(N) \) or \( N = \exp(T) \). But these configurations of \( N \) and \( T \) do not seem empirically relevant. Thus, \( g_4(N,T) \) should not be ruled out in practice.

Random matrix theory has recently been used to determine the number of factors. The idea is to exploit the largest and the smallest eigenvalue of large matrices whose properties are known for iid normal data. See Onatski (2005) and Kapetanios (2007). These methods use different and often stronger assumptions than the ones stated in this survey. In simulations, the tests seem to have good properties when the factor structure is weak, or the idiosyncratic errors are highly serially or cross-sectionally correlated. Whether a factor model with these characteristics is a better characterization of the data remains debatable. It should be noted that a positive limit for \( \Lambda'\Lambda/N \) still permits zero factor loadings for many series, implying weak factors for the corresponding series.

Onatski (2006b) developed a formal test for the number of factors in data with correlated Gaussian idiosyncratic errors. The idea is to test the slope of the scree plot to identify changes in curvature.\(^1\) His analysis provides a formal justification for the use of the scree plot in determining the number of factors, and appears to be the first to obtain the asymptotic distribution of tests that determine the number of factors. Under the assumptions in Onatski (2006b) and the large sample properties of random matrices, his proposed test is characterized by the Tracy-Widom distribution.

There are specific instances when knowledge of the number of dynamic factors, \( q \), is useful. If the static factors \( F_t \) are driven by \( q \leq r \) primitive innovations (say, \( \epsilon_t \)), then the innovations of \( F_t \) and \( \epsilon_t \) are related by \( u_t = R\epsilon_t \). It then follows that \( \Sigma_u = R\Sigma\epsilon R' \) has rank \( q \leq r \).

\(^1\)A scree diagram is a plot of the ordered eigenvalues against the corresponding order number.
4.1 Estimating the Number of Factors

and \( F_t \) can be represented as

\[
A(L)F_t = R\epsilon_t.
\]

These features lead to two ways of determining \( q \) using time domain methods.

The first, from Stock and Watson (2005) and Amengual and Watson (2007), takes the dynamic factor model

\[
x_{it} = \lambda_i' A(L) F_{t-1} + \rho_i(L) x_{it-1} + e_{it}.
\]

Rewriting \( A(L) = I - A^+ (L) L \) together with \( A(L) F_t = R\epsilon_t \) yields

\[
x_{it} = \lambda_i' A^+(L) F_{t-1} + \rho_i(L) x_{it-1} + \lambda_i' R\epsilon_t + e_{it}.
\]

Then \( w_t = \lambda_i' R\epsilon_t + e_{it} \) has a factor structure. This factor structure has

\( q = \text{dim}(\epsilon_t) \) dynamic factors. The PCP and IC criteria can be used to
determine \( q \), upon replacing \( w_{it} \) by a consistent estimate.

The second, developed in Bai and Ng (2007), starts with the premise

that if the \( r \times r \) matrix \( \Sigma_u \) has rank \( q \), then the \( r - q \) smallest eigenvalues are zero. Let \( c_1 > c_2 \cdots c_N \) be the ordered eigenvalues of \( \Sigma_u \). Define

\[
D_{1,k} = \left( \frac{c_{k+1}^2}{\sum_{j=1}^k c_j^2} \right)^{1/2} \quad \text{and} \quad D_{2,k} = \left( \frac{\sum_{j=k+1}^r c_j^2}{\sum_{j=1}^k c_j^2} \right).
\]

The test is based on the idea that when the true eigenvalues \( c_{q+1}, \ldots, c_r \) are zero, \( D_{1,k} \) and \( D_{2,k} \) should be zero for \( k \geq q \). To estimate \( \Sigma_u \),
a VAR model is fitted to the estimated factors \( \hat{F}_t \) (but not \( \tilde{F}_t \)). Let \( \tilde{u}_t \) be the residuals from estimation of a VAR(p) in \( \tilde{F}_t \), and let

\[
\hat{\Sigma}_u = \frac{1}{T-p} \sum_{t=1}^T \tilde{u}_t \tilde{u}_t'.
\]

Let \( \tilde{c}_1 > \tilde{c}_2 \cdots \tilde{c}_r \) be the eigenvalues of \( \hat{\Sigma}_u \). Replace \( c_j \) in \( D_{1,k} \) and \( D_{2,k} \) by \( \tilde{c}_j \) to obtain \( \hat{D}_{1,k} \) and \( \hat{D}_{2,k} \). For some \( 0 < m < \infty \) and \( 0 < \delta < 1/2 \), let

\[
M_{NT}(\delta) = \frac{m}{\min[N^{1/2-\delta}, T^{1/2-\delta}]}.
\]

As there are sampling errors from estimation of \( \Sigma_u \), \( \hat{D}_{1,k} \) and \( \hat{D}_{2,k} \) will not be zero exactly. The cut-off point of \( M_{NT}(\delta) \) is used to account for estimation error. We now summarize the properties of these two tests.
Result D: Number of Dynamic Factors

D.1. Stock and Watson (2005). Let $\hat{w}_{it}$ be the residuals from a restricted regression of $x_{it}$ on the lags of $\tilde{F}_t$ and $x_{it}$. Let $S(k)$ be the sum of squared residuals when $k$ factors are estimated from the data matrix $\{\hat{w}_{it}\}$. Let $\hat{q} = \arg\min_k PCP(k)$ or $\hat{q} = \arg\min_k IC(k)$. If (i) $g(N,T) \to 0$, and (ii) $C_{NT}^2 g(N,T) \to \infty$ as $N,T \to \infty$, then
$$\text{prob}(\hat{q} = q) \to 1.$$

D.2. Bai and Ng (2007). Let $\hat{q}_1$ be the smallest $k$ such that $\hat{D}_{1,k} < M_{NT}(\delta)$ and $\hat{q}_2$ be the smallest $k$ such that $\hat{D}_{2,k} < M_{NT}(\delta)$. For $0 < m < \infty$ and $0 < \delta < 1/2$, as $N,T \to \infty$
$$P(\hat{q} = q) \to 1.$$

Both approaches enable $q$ to be determined without having to estimate the spectrum of $F$. Note, however, that the Bai–Ng method assumes $\tilde{F}_t$ is estimated by $\bar{F}_t$ (i.e., with normalization $A'A/N = I_k$) and is not valid when $\tilde{F}_t$ is used. The Stock–Watson method is flexible on which normalization is used to construct the principle component estimates.

Hallin and Liska (2007) proposed a method for determining the number of factors in a generalized dynamic factor model. The test is based on the eigenvalues $c_i(\theta)$ of the spectral density matrix of the data at frequency $\theta$, denoted $\Sigma(\theta)$. Specifically, for $m = -M_T, \ldots, M_T$, and $\theta_m = \frac{\pi m}{M_T+1/2}$ with $M_T \to \infty$ and $M_T T^{-1} \to 0$, define
$$PCP(k) = \frac{1}{N} \sum_{i=k+1}^{N} \frac{1}{2M_T+1} \sum_{m=-M_T}^{M_T} c_i(\theta_m) + k\bar{g}(N,T), \quad 0 \leq k \leq q_{\max}$$
and
$$IC(k) = \log \left[ \frac{1}{N} \sum_{i=k+1}^{N} \frac{1}{2M_T+1} \sum_{m=-M_T}^{M_T} c_i(\theta_m) \right] + k\bar{g}(N,T).$$

Under somewhat different assumptions from the ones considered here, the authors showed that if (i) $\bar{g}(N,T) \to 0$, and (ii) $\min[N,M_T^{-2}$,
4.1 Estimating the Number of Factors

$M_T^{1/2}T^{1/2}]\bar{g}(N,T) \rightarrow \infty$, then $P(\hat{q} = q) \rightarrow 1$. The authors suggest using $M_T = 0.5\sqrt{T}$ and penalties $\bar{g}_1(N,T) = (M_T^{-2} + M_T^{1/2}T^{-1/2} + N^{-1})\log A_T$, $\bar{g}_2(N,T) = A_T$, and $\bar{g}_3(N,T) = A_T^{-1}\log(A_T)$ with $A_T = \min[N, M_T^2, M_T^{-1/2}T^{1/2}]$. Not surprisingly, their method outperforms $\hat{D}_{1,k}$ and $\hat{D}_{2,k}$ when the loadings have an autoregressive structure, which does not satisfy the assumption of Bai and Ng (2007). Implementation of their procedure requires careful selection of the auxiliary parameters. The authors suggest cross-validation as a possibility.

Work is on-going to find improved ways to estimate $r$. The issue is important as estimation and inference often hinge on precise estimation of $r$ and $q$. One can expect new and improved tests to appear.
The motivation for considering factor analysis is to deal with large number of variables. Use of a small number of factors as conditioning variables is a parsimonious way to capture information in a data rich environment, and there should be efficient gains over (carefully) selecting a handful of predictors. There are three strands of research that evolve around this theme. The first concerns using the estimated factors as predictors; the second uses the estimated factors as improved instruments over observed variables. The third concerns testing the validity of observed proxies. These are discussed in the next three subsections.

5.1 Factor-Augmented Regressions

The distinctive feature of a so-called “factor-augmented” regression is to add factors estimated by the method of principal components to an otherwise standard regression:

\[ y_{t+h} = \alpha' \tilde{F}_t + \beta' W_t + \epsilon_{t+h} \]

\[ = \tilde{z}'_{t+h} \delta + \epsilon_{t+h}, \quad (5.1) \]
where $W_t$ are predetermined variables (such as lags) that the researcher would include whether or not $\tilde{F}_t$ is available. The following assumptions will be used.

**Assumption FAR:**

(a) Let $z_t = (F_t' W_t')'$, $E\|z_t\|^4 \leq M$; $E(\epsilon_{t+h}|y_t, z_t, y_{t-1}, z_{t-1}, \ldots) = 0$ for any $h > 0$; $z_t$ and $\epsilon_t$ are independent of the idiosyncratic errors $\epsilon_{ts}$ for all $i$ and $s$. Furthermore, 
\[ \frac{1}{T} \sum_{t=1}^{T} z_t z_t' \xrightarrow{d} N(0, \Sigma_{zz, \epsilon}), \]
where $\Sigma_{zz, \epsilon} = \text{plim} \frac{1}{T} \sum_{t=1}^{T} (\epsilon_t^2 + h z_t z_t') > 0$.

The regression model given by (5.1) encompasses many applications of interest. If $h = 0$, (5.1) is simply a regression with generated regressors $\tilde{F}_t$, and $\tilde{z}_t' \hat{\delta}$ is the estimated conditional mean of $y_t$. For example, if $y_t$ is stock returns, then $\tilde{z}_t' \hat{\delta}$ is the estimated conditional mean of stock returns, and if $y_t$ is the volatility of stock returns, then $\tilde{z}_t' \hat{\delta}$ is the estimated conditional volatility of stock returns with conditioning information $\tilde{z}_t$. The ratio of the two estimates is the conditional Sharp ratio, which is useful in studying the risk-return trade-off of stock returns.

When $h > 0$, (5.1) is a forecasting equation and forms the basis of the so-called Diffusion Index (DI) forecasting methodology of Stock and Watson (2002a). The advantage of DI is that $\tilde{F}_t$ drastically reduces the dimension of the information spanned by the predictors from $N$ to a much smaller number as determined by the number of factors included. As such, it is capable of exploiting the information in a large panel of data while keeping the size of the forecasting model small. The DI framework is now used by various government agencies in a number of countries, as well as by independent and academic researchers alike.

In addition to forecasting, Equation (5.1) also provides a framework for assessing predictability. For example, under the expectations hypothesis, excess bond returns should be unpredictable given the information available. Predictability tests are often sensitive to the choice...
of the predictors. A finding of non-predictability in a small information set may not be robust to changes in the set of predictors, or the extent to which the variable of interest is predictable. The factor augmented framework is well suited for these analysis because the inclusion of $\tilde{F}_t$ brings the empirical problem closer to the conceptual problem of testing predictability with respect to “all information available.” Not only can we assess if the “usual suspects” $W_t$ contain all relevant information (and that all other predictors are irrelevant), we can do so without much efficiency loss because $\tilde{F}_t$ is of small dimension.

As written, Equation (5.1) is a single equation. But multivariate models can also accommodate $\tilde{F}_t$. More specifically, if $y_t$ is a vector of $m$ series, and $F_t$ is a vector of $r$ factors, a Factor Augmented Vector Autoregression of order $p$, or simply FAVAR($p$), can be obtained.

$$y_{t+1} = \sum_{k=0}^{p} a_{11}(k)y_{t-k} + \sum_{k=0}^{p} a_{12}(k)\tilde{F}_{t-k} + \epsilon_{1t+1}$$

$$\tilde{F}_{t+1} = \sum_{k=0}^{p} a_{21}(k)y_{t-k} + \sum_{k=0}^{p} a_{22}(k)\tilde{F}_{t-k} + \epsilon_{2t+1},$$

where $a_{11}(k)$ and $a_{21}(k)$ are coefficients on $y_{t-k}$, while $a_{12}(k)$ and $a_{22}(k)$ are coefficients on $\tilde{F}_{t-k}$.

The following results validate estimation and inference based upon (5.1).

**Result E: Linear Factor Augmented Regressions.** Bai and Ng (2006a). Suppose Assumptions F(0), L, E, and FAR hold.

E.1. If $\sqrt{T}/N \rightarrow 0$, then

$$\sqrt{\tilde{T}}(\hat{\delta} - \delta) \xrightarrow{d} N(0, \Sigma_\delta),$$

where $\Sigma_\delta = \Phi_0^{-1}\Sigma_{zz}^{-1}\Sigma_{zz,e}\Sigma_{zz}^{-1}\Phi_0^{-1}$, $\Phi_0 = \text{diag}(V^{-1}Q\Sigma_A, I)$ is block diagonal, $V = \text{plim} \tilde{V}$, $Q = \text{plim} \tilde{F}'F/T$, and $\Sigma_A$ defined in Assumption E. A consistent estimator for $\text{Avar}(\hat{\delta}) = \Sigma_\delta$ is

$$\hat{\text{Avar}}(\hat{\delta}) = \left( \frac{1}{T} \sum_{t=1}^{T-h} \tilde{z}_t \tilde{z}'_t \right)^{-1} \left( \frac{1}{T} \sum_{t=1}^{T-h} \tilde{z}_t \tilde{z}'_{t+h} \tilde{z}_t \right) \left( \frac{1}{T} \sum_{t=1}^{T-h} \tilde{z}_t \tilde{z}'_t \right)^{-1}.$$
5.1 Factor-Augmented Regressions

E.2. Consider a \( p \)th order vector autoregression in \( m \) observable variables \( y_t \) and \( r \) factors, \( \tilde{F}_t \), estimated by the method of principal components. Let \( \tilde{z}_t = (y'_t, \ldots, y'_{t-p}, \tilde{F}'_t, \ldots, \tilde{F}'_{t-p})' \). Define \( \tilde{Y}_t = (y'_t, \tilde{F}'_t) \) and let \( \tilde{Y}_{jt} \) be the \( j \)th element of \( \tilde{Y}_t \).

For \( j = 1, \ldots, m + r \), let \( \hat{\delta}_j \) be obtained by least squares from regressing \( \tilde{Y}_{jt+1} \) on \( \tilde{z}_t \). Let \( \tilde{\epsilon}_{jt+1} = \tilde{Y}_{jt+1} - \hat{\delta}'_j \tilde{z}_t \). If \( \sqrt{T}/N \to 0 \) as \( N, T \to \infty \),

\[
\sqrt{T}(\hat{\delta}_j - \delta_j) \xrightarrow{d} N\left(0, \lim_{T,N \to \infty} \left(\frac{1}{T} \sum_{t=1}^{T} \tilde{z}_t \tilde{z}_t' \right)^{-1} \left(\frac{1}{T} \sum_{t=1}^{T} (\tilde{\epsilon}_{jt})^2 \tilde{z}_t \tilde{z}_t' \right) \times \left(\frac{1}{T} \sum_{t=1}^{T} \tilde{z}_t \tilde{z}_t' \right)^{-1} \right).
\]

E.3. Let \( \tilde{y}_{T+h|T} = \tilde{\delta}' \tilde{z}_T \). If \( \sqrt{N}/T \to 0 \) and Assumptions F(0), L, and E hold, then for any \( h \geq 0 \),

\[
\frac{(\tilde{y}_{T+h|T} - y_{T+h|T})}{\sqrt{\text{var}(\tilde{y}_{T+h|T})}} \xrightarrow{d} N(0, 1),
\]

where \( \text{var}(\tilde{y}_{T+h|T}) = \frac{1}{T} \tilde{z}_T' \tilde{A} \text{var}(\hat{\delta}) \tilde{z}_T + \frac{1}{N} \tilde{z}_T' \tilde{A} \text{var}(\tilde{F}_T) \tilde{z}_T \).

Result E states that parameter estimates of equations involving \( \tilde{F}_{t+1} \), whether as regressand or regressors, are \( \sqrt{T} \) consistent. Result E also shows how standard errors can be computed and provides a complete inferential theory for factor augmented regressions.

Conventionally generated regressors are obtained as the fitted values from a first step regression of some observed variable related to the latent variable of interest on a finite set of other observed regressors. As shown in Pagan (1984), sampling variability from the first step estimation is \( O_p(1) \) in the second stage. Consequently, the standard errors of the second step parameter estimates must account for the estimation error from the first step. As indicated by Result E, no such adjustment is necessary when the generated regressors are \( \tilde{F}_t \) if \( \sqrt{T}/N \to 0 \). This is because the term that is \( O_p(1) \) in a conventional setting is \( O_p(\frac{\sqrt{T}}{\min(N,T)}) \) in the factor augmented regression setting, and this term vanishes if
\[ \sqrt{T}/N \to 0. \] Note, however, that although the condition \( \sqrt{T}/N \to 0 \) is not stringent, it does put discipline on when estimated factors can be used in regression analysis.

Result E.3 concerns the prediction interval for the conditional mean. There are two terms in \( \text{var}(\hat{y}_{T+h|T}) \), and the overall convergence rate for \( \hat{y}_{T+h|T} \) is \( \min[\sqrt{T}, \sqrt{N}] \). In a standard setting, \( \text{var}(\hat{y}_{T+h|T}) \) falls at rate \( T \), and for a given \( T \), it increases with the number of observed predictors through a loss in degrees of freedom. With factor forecasts, the error variance decreases at rate \( \min[N,T] \), and for a given \( T \), efficiency improves with the number of predictors used to estimate \( F_t \). A large \( N \) enables better estimation of the common factors and thus directly affects the efficiency of subsequent estimations involving \( \hat{F}_t \). Note that if the estimation of \( F_t \) was based upon a fixed \( N \), consistent estimation of the factor space would not be possible however large \( T \) becomes. That is to say, Result E will not apply if we simply use \( \hat{F}_t \) to reduce the dimension of an already small set of predictors.

5.2 Extremum Estimation

The estimated factors can be used in estimation not just in linear models, but also when the models are nonlinear in the parameters. Suppose we observe \( z_t = (y_t, w_t, F_t) \), where \( y_t \) is typically the dependent variable and \( (w_t, F_t) \) is a set of explanatory variables \( (t = 1, \ldots, T) \). Consider the problem of estimating \( \theta \) as

\[
\hat{\theta} = \arg\max_{\theta} Q_T(\theta), \quad Q_T = \frac{1}{T} \sum_{t=1}^{T} m(z_t, \theta).
\]

If \( z_t \) are iid with density function \( f(z_t|\theta) \), the MLE can be written as \( m(z_t, \theta) = \log f(z_t|\theta) \). For nonlinear regressions, \( y_t = g(w_t, F_t, \theta) + \varepsilon_t \), the nonlinear squares method is equivalent to \( m(z_t, \theta) = -[y_t - g(w_t, F_t, \theta)]^2 \). Under some regularity conditions, \( \hat{\theta} \) can be shown to be \( \sqrt{T} \) consistent and asymptotically normal. However, if \( F_t \) is not observed but we instead observe a large panel of data \( x_{it} \) with a factor structure, \( x_{it} = \lambda_i^t F_t + \varepsilon_{it} \), we can replace \( F_t \) by \( \hat{F}_t \) in estimation.
Replacing $z_t$ by $\tilde{z}_t = (y_t, w_t, \tilde{F}_t)$ gives the feasible objective function

$$\tilde{Q}_T(\theta) = \frac{1}{T} \sum_{t=1}^{T} m(\tilde{z}_t, \theta)$$

and the estimator is defined as

$$\tilde{\theta} = \arg\max_{\theta} \tilde{Q}_T(\theta).$$

Let $h(z_t, \theta) = \frac{\partial m(z_t, \theta)}{\partial \theta}$ and $K(z_t, \theta) = \frac{\partial^2 m(z_t, \theta)}{\partial \theta \partial \theta^T}$. The following assumptions are made:

**Assumption EE:**

(i) $E h(z_t; \theta_0) = 0$, $E h(z_t; \theta) \neq 0$ for all $\theta \neq \theta_0$ and $\frac{1}{\sqrt{T}} \sum_{t=1}^{T} h(z_t, \theta_0) \to^d N(0, \Sigma)$, where $\Sigma$ is positive definite.

(ii) $E \sup_{\theta} \|h(z_t; \theta)\| < \infty$.

(iii) $h(z; \theta)$ is twice continuously differentiable with respect to $F_t$ and $\theta$, and $K_0 = E K(z_t, \theta_0)$ is invertible.

(iv) For some $b_{NT} \to 0$

$$\sup_{\{F_t^* : \|F_t^* - F_t\| \leq b_{NT}, \forall t\}} \sup_{\theta \in \Theta} \frac{1}{T} \sum_{t=1}^{T} \|h(y_t, x_t, F_t^*, \theta)\|^2 = O_p(1).$$

(v) $\xi_t = (\partial/\partial F_t) h(z_t, \theta_0)$ is uncorrelated with $e_{it}$ and $E \|\xi_t\|^2 \leq M$ for all $t$.

(vi) For $j = 1, 2, \ldots, p = \dim(\theta)$, and for some $b_{NT} \to 0$,

$$\sup_{\{F_t^* : \|F_t^* - F_t\| \leq b_{NT}, \forall t\}} \sup_{\{\theta^* : \|\theta^* - \theta_0\| \leq b_{NT}\}} \frac{1}{T} \sum_{t=1}^{T} \left\| \frac{\partial^2 h_j(y_t, x_t, F_t^*, \theta^*)}{\partial \eta_1 \partial \eta_2} \right\|^2 = O_p(1)$$

for $\eta_1, \eta_2 \in \{F_t, \theta\}$.

Assumptions EE(i)–(iii) are sufficient conditions for consistency and asymptotic normality when $z_t$ is observable. Assumption EE(iv) guarantees consistency of $\tilde{\theta}$ when $F_t$ is replaced by $\tilde{F}_t$. The remaining conditions are for asymptotic normality when using $\tilde{F}_t$ in place of $F_t$. 

5.2 Extremum Estimation 119

Under Assumptions of EE, F(0), L, E, and $T^{5/8}/N \rightarrow 0$, we have

$$\sqrt{T}(\hat{\theta} - \theta_0) \xrightarrow{d} N(0, K_0^{-1}\Sigma K_0^{-1}).$$

The limiting distribution is the same as if $F_t$ are observable. For linear models, only $T^{1/2}/N \rightarrow 0$ is required. A larger $N$ is required for nonlinear models. Bai and Ng (2008) also consider a GMM estimator and obtain similar results.

5.3 Instrumental Variable Estimation

A different use of $\tilde{F}_t$ is as instrumental variables. Consider the regression model

$$y_t = x_{1t}'\beta_1 + x_{2t}'\beta_2 + \epsilon_t$$

$$= x_t'\beta + \epsilon_t,$$

where $x_t = (x_{1t}', x_{2t}')'$ is $K \times 1$. The $K_1 \times 1$ regressors $x_{1t}$ are exogenous or predetermined and may include lags of $y_t$. The $K_2 \times 1$ regressors $x_{2t}$ are endogenous; $K = K_1 + k_2$. Suppose that

$$x_{2t} = \Psi'F_t + u_t$$

and $E(u_t\epsilon_t) \neq 0$ and $E(F_t\epsilon_t) = 0$. This implies that $E(x_{2t}\epsilon_t) \neq 0$, and thus $x_{2t}$ is endogenous. The least squares estimator of $\beta$ will be inconsistent. The conventional treatment of endogeneity bias is to use lags of $y, x_1,$ and $x_2$ as instruments for $x_2$. Our point of departure is to note that in this setting, $F_t$ are the ideal instruments in the sense of satisfying both instrument exogeneity and instrument relevance. The reason why the moments $g_t = F_t\epsilon_t$ are not used to estimate $\beta$ is that $F_t$ are not observed. We assume that there is a “large” panel of valid instruments, $z_{1t}, \ldots, z_{Nt}$ that are weakly exogenous for $\beta$ and generated as follows:

$$z_{it} = \lambda_i'F_t + \epsilon_{it}. \quad (5.3)$$

The idea is to estimate $F$ from the above factor model and then use the estimated $F$ as instruments for $x_{2t}$. For a related study, see Kapetanios and Marcellino (2006a). The following analysis assumes no $x_{1t}$ for simplicity. Otherwise, use $F_t^+ = (x_{1t}', F_t')'$ in place of $F_t$ as instrument.
Assumption IV:

(a) $E(\epsilon_t) = 0$, $E(|\epsilon_t|^{4+\delta} < \infty$ for some $\delta > 0$. The $r \times 1$ vector process $g_t(\beta^0) = F_t \epsilon_t(\beta^0)$ satisfies $E[g_t(\beta)] = E(g_t^0) = 0$ with $E(g_t(\beta)) \neq 0$ when $\beta \neq \beta^0$. Furthermore, $\sqrt{T}g_0 \overset{d}{\rightarrow} N(0, S^0)$, where $S^0$ is the asymptotic variance of $\sqrt{T}g_0$, and $g_0 = \frac{1}{T} \sum_{t=1}^{T} F_t \epsilon_t(\beta^0)$.

(b) $x_{1t}$ is predetermined such that $E(x_{1t} \epsilon_t) = 0$.

(c) $x_{2t} = \Psi' F_t + u_t$ with $\Psi' \Psi > 0$, $E(F_t u_t) = 0$, and $E(u_t \epsilon_t) \neq 0$.

(d) For all $i = 1, \ldots, N$, $E(e_{it} u_t) = 0$, and $E(e_{it} \epsilon_t) = 0$.

Assumption IV(c) restricts consideration to the case of strong instruments, while IV(d) assumes that $z_{it}$ are valid instruments. Let $g_t = \tilde{F}_t \epsilon_t$, $\bar{g} = \frac{1}{T} \sum_{t=1}^{T} g_t$, and let $\bar{S} = \frac{1}{T} \sum_{t=1}^{T} (\tilde{g}_t \tilde{g}_t')$, with $\tilde{g}_t = \tilde{F}_t \tilde{\epsilon}_t$, and $\tilde{\epsilon}_t = y_t - x_t' \tilde{\beta}_{FIV}$, where $\tilde{\beta}_{FIV}$ is an initial consistent estimator for $\beta$. The two-step efficient estimator with $\tilde{F}$ as instruments is defined as

$$\hat{\beta}_{FIV} = \arg\min_{\beta} \tilde{g}(\beta)' \tilde{S}^{-1} \tilde{g}(\beta).$$

Result G: Factor IV

G.1. Under Assumptions F, L, E, LFE, and FIV,

$$\sqrt{T}(\tilde{\beta}_{FIV} - \beta^0) \overset{d}{\rightarrow} N(0, Avar(\tilde{\beta}_{FIV})),$$

where $Avar(\tilde{\beta}_{FIV})$ is the probability limit of $(S_{\tilde{F} x}(\bar{S})^{-1} S_{\tilde{F} x}'^{-1})$, which is therefore a consistent estimator for the asymptotic variance, and $S_{\tilde{F} x} = \frac{1}{T} \sum_{t=1}^{T} \tilde{F}_t x_t$.

G.2. Let $z_2$ be a subset of $r$ of the $N$ observed instruments ($z_{1t}, \ldots, z_{Nt}$). Let $m_t = z_{2t}(y_t - x_t' \beta)$ with $\sqrt{T}m \overset{d}{\rightarrow} N(0, Q)$. Let $\tilde{\beta}_{IV}$ be the minimizer of $m'(Q)^{-1} m$ with the property that $\sqrt{T}(\tilde{\beta}_{IV} - \beta^0) \overset{d}{\rightarrow} N(0, Avar(\tilde{\beta}_{IV}))$. Under the assumption that $E(e_{2i}^2) > 0$ for all $i$ in the subset of $z_2$,

$$Avar(\tilde{\beta}_{IV}) - Avar(\tilde{\beta}_{FIV}) \geq 0.$$
Result G is based on Bai and Ng (2006c). G.1 and G.2 say that not only do the instruments $\tilde{F}_t$ yield $\sqrt{T}$ consistent and asymptotically normal estimates, but that the estimates so obtained are more efficient than using an equal number of observed instruments. The intuition is straightforward. The observed instruments are the ideal instruments contaminated with errors while $\tilde{F}$ is consistent for the ideal instrument space. More efficient instruments thus lead to more efficient estimates. Pooling information across the observed variables washes out the noise to generate more efficient instruments for $x_2$. Result F.1 is stated assuming that all instruments are valid, but this can be relaxed. Result F.1 still holds if $\sum_{i=1}^N |E(\epsilon_i\epsilon_{it})| \leq M < \infty$ for all $N$ with $M$ not depending on $N$, and $\sqrt{T}/N \to 0$. Thus in a data rich environment, use of invalid instruments does not preclude consistent estimation. However, use of invalid instruments will not yield consistent estimates if $N$ is fixed; this highlights how a large $N$ and $T$ can open up new horizons for estimation and inference.

The factor estimates can also be used as instruments in a panel context.

$$y_{it} = x_{it}'\beta + \epsilon_{it},$$

where $x_{it}$ is $K \times 1$. We continue to maintain the assumption that

$$x_{it} = \Lambda_i'F_t + u_{it} = C_{it} + u_{it}.$$ 

**Assumption E-Panel:** Same as Assumption L and E with three changes. Part (i) holds with $\lambda_i$ replaced by $\Lambda_i$; part (ii) holds with $e_{it}$ replaced by each component of $u_{it}$ (note that $u_{it}$ is a vector).

**Assumption IV-Panel:**

(a) $E(\epsilon_{it}) = 0$, $E|\epsilon_{it}|^{4+\delta} < M < \infty$ for all $i,t$, and for some $\delta > 0$; $\epsilon_{it}$ is independent over $i$ and $\epsilon_{it}$ is independent of $F_t$ and $\Lambda_i$.

(b) $x_{it} = \Lambda_i'F_t + u_{it}$; $E(u_{it}\epsilon_{it}) \neq 0$; $u_{it}$ is independent over $i$. 


5.3 Instrumental Variable Estimation

(c) \((NT)^{-1/2} \sum_{i=1}^{N} \sum_{t=1}^{T} C_{it} \varepsilon_{it} \xrightarrow{d} N(0, S)\), where

\[
S = \lim_{N,T \to \infty} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t,s=1}^{T} E(C_{it} C'_{is} \varepsilon_{it} \varepsilon_{is}),
\]

which is the long run variance of \(\xi_t = N^{-1/2} \sum_{i=1}^{N} C_{it} \varepsilon_{it}\).

The pooled two-stage least-squares estimator with \(\tilde{C}_{it}\) as instruments is

\[
\hat{\beta}_{PFIIV} = \left( \sum_{i=1}^{N} \sum_{t=1}^{T} \tilde{C}_{it} x_{it}' \right)^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} \tilde{C}_{it} y_{it}.
\]

Result G: Panel FIV. Under Assumptions E-Panel and IV-Panel,

G.3. (i) \(\hat{\beta}_{PFIIV} - \beta^0 = O_p(T^{-1}) + O_p(N^{-1})\) and thus \(\hat{\beta}_{PFIIV} \xrightarrow{p} \beta^0\).

(ii) If \(T/N \to \tau > 0\), then

\[
\sqrt{NT}(\hat{\beta}_{PFIIV} - \beta^0) \xrightarrow{d} N(\tau^{1/2} \Delta_1^0 + \tau^{-1/2} \Delta_2^0, \Omega),
\]

where \(\Omega = \text{plim}[S_{xx}]^{-1} S[S_{xx}']^{-1}\) with \(S_{xx} = (NT)^{-1} \sum_{i=1}^{N} \tilde{C}_{it} x_{it}'\), and \(\Delta_1^0\) and \(\Delta_2^0\) are bias terms.

(iii) Suppose \(\varepsilon_{it}\) are serially uncorrelated. Let \(\hat{\Delta} = \frac{1}{N} \hat{\Delta}_1 + \frac{1}{T} \hat{\Delta}_2\), where \(\hat{\Delta}_k\) is a consistent estimate of \(\Delta_k\) (\(k = 1, 2\)), see Bai and Ng (2006c). Suppose Assumptions F(0), L, E-Panel, LFE, and FAR-Panel hold. If \(T/N^2 \to 0\), and \(N/T^2 \to 0\), then

\[
\sqrt{NT}(\hat{\beta}_{PFIIV} - \hat{\Delta} - \beta^0) \xrightarrow{d} N(0, \Omega).
\]

Result G.3 indicates that there can be no valid instrument in the conventional sense, yet, we can still consistently estimate the large simultaneous equations system. In a data-rich environment, the information in the data collectively permits consistent instrumental variable estimation under much weaker conditions on individual instruments. However, \(C_{it}\) is not observed, and biases arise from the estimation of \(C_{it}\). More precisely, \(\tilde{C}_{it}\) contains elements of \(u_{it}\), which are correlated.
Applications with $\varepsilon_{it}$. This motivates the bias-corrected estimator stated in (iii). Although the biased estimator is reasonably precise, the bias correction terms, defined in Bai and Ng (2006c), are necessary for the $t$ statistic to have the appropriate size.

5.4 Testing the Validity of Observed Proxies

Finance theory postulates that systematic risks should be priced. Many observed variables including inflation, term and premia, as well as variables constructed by Fama and French have been used as proxies for the latent risk factors. But are these factors guided by economic theory close to the statistical factors constructed in a data rich environment?

Let $G_t = (G_1t, \ldots, G_{mt})$ be a $m \times 1$ vector of observed variables that are thought to be useful proxies of a set of otherwise unobserved factors, $F_t$. The objective is to get a sense of how close $F$ and $G$ are. We consider two cases: (i) $G_{jt}$ is an exact linear combination of $F_t$ such that $G_{jt} = \delta_j' F_t$ for some vector $\delta_j$; (ii) $G_{jt}$ is a linear combination of $F_t$ plus an error term such that $G_{jt} = \delta_j' F_t + \varepsilon_{jt}$. In the second case, we want to measure how big is the error $\varepsilon_{jt}$ relative to $\delta_j' F_t$. We also want to make inference about the correlation between the vector $G_t$ and $F_t$. But $F_t$ is not observable, so we use $\tilde{F}_t$ instead.

Let $\hat{\gamma}_j$ be obtained from the least squares estimation of

$$G_{jt} = \gamma_j' \tilde{F}_t + \text{error}.$$ 

Let $\hat{\tau}_t(j)$ be the corresponding $t$ statistic and let $\Phi_\alpha$ be the $\alpha$ percentage point of the limiting distribution of $\hat{\tau}_t(j)$.

Result H: Testing Validity of Observed Factors

H.1. Let $A(j) = \frac{1}{T} \sum_{t=1}^{T} I(|\hat{\tau}_t(j)| > \Phi_\alpha)$ and $M(j) = \max_{1 \leq t \leq T} |\hat{\tau}_t(j)|$. Under the null hypothesis of exact linear combination $G_{jt} = \delta_j' F_t$ and $\sqrt{N/T} \rightarrow 0$ as $N, T \rightarrow \infty$, then $A(j) \rightarrow 2\alpha$. If, in addition, $e_{it}$ is serially uncorrelated, $P(M(j) \leq x) \approx [2\Phi(x) - 1]^T$, where $\Phi(x)$ is the cdf of a standard normal random variable.
H.2. Consider the null hypothesis $G_{jt} = \delta_j' F_t + \epsilon_{jt}$. Let $\bar{\epsilon}_{jt} = G_{jt} - \hat{G}_{jt}$. As $N,T \to \infty$ and with $s^2_{jt} = T^{-1} \tilde{F}_t (T^{-1} \sum_{s=1}^T \tilde{F}_s \tilde{F}_s' (\epsilon^2_{js})^{-1} \tilde{F}_t + N^{-1} \text{Avar}(\hat{G}_{jt}))$, then for each $t$

$$\frac{\bar{\epsilon}_{jt} - \epsilon_{jt}}{s_{jt}} \xrightarrow{d} N(0,1).$$

An estimate of $\frac{1}{N} \text{Avar}(\hat{G}_{jt})$ is $N^{-1} \tilde{\gamma}_j' \tilde{V}^{-1} \tilde{\Gamma}_T \tilde{V}^{-1} \hat{\gamma}_j$, where $\tilde{V}$ and $\tilde{\Gamma}_T$ are defined in previous sections. Also define two overall statistics (not depending on $t$)

$$NS(j) = \frac{\text{var}(\bar{\epsilon}(j))}{\text{var}(\hat{G}(j))} \quad \text{and} \quad R^2(j) = \frac{\text{var}(\hat{G}(j))}{\text{var}(G(j))},$$

where $\text{var}(\cdot)$ is simply the sample variance. Then $NS(j)$ should be close to zero and $R^2(j)$ should be close to one under the null hypothesis of an exact linear combination.

H.3. Let $\tilde{\rho}^2_1, \ldots, \tilde{\rho}^2_p$ be the largest $p = \min[m,r]$ sample squared canonical correlations between $\tilde{F}$ and $G$. Suppose that $(F'_t, G'_t)'$ are iid normally distributed and the true canonical correlation coefficient between $F_t$ and $G_t$ are given by $\rho^2_1, \ldots, \rho^2_p$. As $N,T \to \infty$ with $\sqrt{T}/N \to 0$,

$$\tilde{z}_k = \frac{\sqrt{T} (\rho^2_k - \tilde{\rho}^2_k)}{2 \tilde{\rho}_k (1 - \tilde{\rho}^2_k)} \xrightarrow{d} N(0,1), \quad k = 1, \ldots, \min[m,r].$$

These results are based on Bai and Ng (2006b). Both $A(j)$ and $M(j)$ are tests for exact factors. The $A(j)$ statistic allows $G_{jt}$ to deviate from $\hat{G}_{jt}$ for a specified number of time periods as specified by $\alpha$. The $M(j)$ test is stronger and requires $G_{jt}$ not to deviate from $\hat{G}_{jt}$ by more than the sampling error at every $t$. As measurement error and time aggregation can be responsible for deviations between the observed and latent factors, an approximate test is given in G.2. Instead of asking how large the measurement errors in the proxy variables are, the two overall statistics, $NS(j)$ and $R^2(j)$, provide a guide to the size of the measurement error. Under normality, how close is the set $G_t$ to the set of latent factors can also be assessed by testing the canonical correlation coefficients. The normality assumption can be relaxed to iid elliptically distributed errors with a suitable rescaling of $z_k$. 
Consider the following model

\[ Y_{it} = X_{it}' \beta + u_{it} \]

and

\[ u_{it} = \lambda_i' F_t + \varepsilon_{it}, \]

where \( X_{it} \) is a \( p \times 1 \) vector of observable regressors, \( \beta \) is a \( p \times 1 \) vector of unknown coefficients; \( u_{it} \) has a factor structure, but \( \lambda_i, F_t, \) and \( \varepsilon_{it} \) are all unobserved. We are interested in the estimation of \( \beta \), the common slope coefficients.

We assume that \( \varepsilon_{it} \) are independent of the regressors. If both \( \lambda_i \) and \( F_t \) are also independent of the regressor \( X_{it} \), the aggregate error \( u_{it} \) will also be independent of the regressor and the model can be estimated with pooled least squares. More efficient estimation could be obtained by GLS based on the factor error structure. However, we allow the regressors to be correlated with either the factors \( F_t \) or the factor loadings or both. In this case GLS will be inconsistent.

The above model with a factor error structure encompasses the fixed effect model as a special case. To see this, let \( r = 2 \), and for all \( i \) and
all $t$, define

$$F_t = \begin{bmatrix} 1 \\ \xi_t \end{bmatrix} \quad \text{and} \quad \lambda_i = \begin{bmatrix} \alpha_i \\ 1 \end{bmatrix}.$$ 

Then

$$\lambda'_i F_t = \alpha_i + \xi_t.$$ 

Hence, the model reduces to

$$Y_{it} = X_{it}' \beta + \alpha_i + \xi_t + \varepsilon_{it},$$

which is the fixed effect model where the individual effects $\alpha_i$ and the time effects $\xi_t$ enter the model additively instead of interactively.

The fixed effect model is usually estimated by the least squares dummy variable approach which treats $\alpha_i$ and $\xi_t$ as parameters to be estimated. This suggests that for models with factor errors, we can also treat $\lambda_i$ and $F_t$ as parameters. The unknown parameters $\beta$, $\lambda_i$, and $F_t$ can be estimated by simply minimizing the least squares objective function $\sum_{i=1}^{N} \sum_{t=1}^{T} (Y_{it} - X_{it}' \beta - \lambda'_i F_t)^2$. The model is, however, overparameterized since $\lambda'_i F_t = \lambda_i AA^{-1} F_t$ for an arbitrary invertible matrix $A$. We need $r^2$ restrictions since an arbitrary invertible $r \times r$ matrix has $r^2$ free parameters. These restrictions are most clearly stated using vector representation of the model

$$Y_i = X_i \beta + F \lambda_i + \varepsilon_i,$$

where for $i = 1, 2, \ldots, N$, $Y_i = (Y_{i1}, \ldots, Y_{iT})'$, $X_i = (X_{i1}, \ldots, X_{iT})'$ ($T \times k$), and $F = (F_{i1}, \ldots, F_{iT})'$ ($T \times r$). Also let $\Lambda = (\lambda_{i1}, \ldots, \lambda_{iN})'$ ($N \times r$). The constraint

$$F'F/T = I$$

implies $r(r + 1)/2$ restrictions since a symmetric matrix has $r(r + 1)/2$ free parameters. The additional constraint that $\Lambda' \Lambda$ is a diagonal matrix (i.e., the off diagonal elements are zero) gives $r(r - 1)/2$ restrictions since the off-diagonal elements of a symmetric matrix has $r(r - 1)/2$ free parameters. These restrictions are similar to those in the pure factor model stated in previous sections. They are restated here for completeness.
Now consider minimizing the least squares objective function

\[ SSR(\beta, F, \Lambda) = \sum_{i=1}^{N} (Y_i - X_i\beta - F\lambda_i)'(Y_i - X_i\beta - F\lambda_i) \]

subject to the constraint \( F'F/T = I_r \) and \( \Lambda'\Lambda \) being diagonal. Define the projection matrix

\[ M_F = I_T - F(F'F)^{-1}F = I_T - FF'/T. \]

The least squares estimator for \( \beta \) for each given \( F \) is simply

\[ \hat{\beta}(F) = \left( \sum_{i=1}^{N} X'_i M_F X_i \right)^{-1} \sum_{i=1}^{N} X'_i M_F Y_i. \]

Given \( \beta \), \( W_i = Y_i - X_i\beta \) has a pure factor structure given by

\[ W_i = F\lambda_i + \varepsilon_i. \]

Define \( W = (W_1, W_2, \ldots, W_N) \), a \( T \times N \) matrix. Thus \( F \) is estimated as the first \( r \) eigenvectors associated with first \( r \) largest eigenvalues of the matrix

\[ WW' = \sum_{i=1}^{N} W_i W'_i = \sum_{i=1}^{N} (Y_i - X_i\beta)(Y_i - X_i\beta)'. \]

Denote this estimate by \( \hat{F}(\beta) \). Then \( \hat{\Lambda}(\beta) = W'\hat{F}(\beta)/T \). Therefore, given \( F \), we can estimate \( \beta \), and given \( \beta \), we can estimate \( F \). The final least squares estimator \((\hat{\beta}, \hat{F})\) is the solution to the following set of nonlinear equations

\[ \hat{\beta} = \left( \sum_{i=1}^{N} X'_i M_{\hat{F}} X_i \right)^{-1} \sum_{i=1}^{N} X'_i M_{\hat{F}} Y_i, \]  
\[ (6.1) \]

and

\[ \left[ \frac{1}{NT} \sum_{i=1}^{N} (Y_i - X_i\hat{\beta})(Y_i - X_i\hat{\beta})' \right] \hat{F} = \hat{F}V_{NT}, \]
\[ (6.2) \]

where \( V_{NT} \) is a diagonal matrix consisting of the \( r \) largest eigenvalues of the above matrix\(^1\) in the brackets, arranged in decreasing order.

\(^1\)We divide this matrix by \( NT \) to make \( V_{NT} \) have a proper limit. The scaling does not affect \( \hat{F} \).
The solution \((\hat{\beta}, \hat{F})\) can be simply obtained by iteration. Given \(\hat{F}\), we have \(\hat{\Lambda} = W'\hat{F}/T\). An alternative estimation procedure is suggested by Pesaran (2006), in which time averages of the dependent variable and independent variables are included as additional regressors. These averages play the role of estimated common factors.

**Assumptions: EF-Panel.** Assume \(F(0), L, E,\) and IE of Section 3 hold, together with one of the following assumptions

(i) \(\varepsilon_{it}\) are iid for all \(i\) and \(t\);  
(ii) \(\varepsilon_{it}\) are correlated and heteroskedastic only in the cross-section dimension, and \(T/N \to 0\);  
(iii) \(\varepsilon_{it}\) are correlated and heteroskedastic only in the time dimension, and \(N/T \to 0\).

Under the above assumption, the estimator \(\hat{\beta}\) has the following asymptotic distribution

\[
\sqrt{NT}(\hat{\beta} - \beta^0) = \left( \frac{1}{NT} \sum_{i=1}^{N} Z_i'Z_i \right)^{-1} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} Z_i'\varepsilon_i + o_p(1),
\]

where

\[
Z_i = MFX_i - \frac{1}{N} \sum_{k=1}^{N} a_{ik}MFX_k
\]

and \(a_{ik} = \lambda_i'(A'A/N)^{-1}\lambda_k\). The right-hand side of the representation does not depend on any estimated quantity.

**Result I: Panel Data Models with Interactive Effects.** Under the assumption

\[
\frac{1}{NT} \sum_{i=1}^{N} Z_i'Z_i \xrightarrow{p} D_0 \quad \text{and} \quad \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} Z_i'\varepsilon_i \xrightarrow{d} N(0, D_Z),
\]

I.I. If (i) of EF-Panel holds, then

\[
\sqrt{NT}(\hat{\beta} - \beta) \xrightarrow{d} N(0, D_0^{-1}D_ZD_0^{-1}).
\]
I.2. If (ii) or (iii) of EF-PANEL holds, and $T/N \to \rho > 0$,

$$\sqrt{NT}(\hat{\beta} - \beta) \overset{d}{\to} N(\rho^{1/2}B_0 + \rho^{-1/2}C_0, D_0^{-1}DZD_0^{-1}).$$

If the $\varepsilon_{it}$ are allowed to be correlated and heteroskedastic in both dimensions and if $T/N \to \rho > 0$, asymptotic bias exists. The expression of $B_0$ and $C_0$ are given in Bai (2005), who also derived the biased-corrected estimator. It is possible to test factor error structure against additive fixed effect. The model can be extended to

$$Y_{it} = X'_{it}\beta + \alpha_i + \xi_t + \tau'_i\delta_t + \varepsilon_{it}.$$  

But in this model, $\tau'_i\delta_t$ must have a genuine factor structure in the sense that $\tau_i$ cannot be one for all $i$ or $F_t$ cannot be one for all $t$. The details are given by Bai (2005).
Consider the following data generating process:

\[ X_{it} = D_{it} + \lambda_i'F_t + e_{it} \]

\[ (1 - L)F_t = C(L)\eta_t \]

\[ e_{it} = \rho_t e_{i,t-1} + \varepsilon_{it}, \]

where \( D_{it} = \sum_{i=0}^{p} \delta_{it} \) is the deterministic component. When \( p = 0 \), \( D_{it} = \delta_i \) is the individual specific fixed effect, and when \( p = 1 \), an individual specific time effect is also present. When there is no deterministic term, \( D_{it} \) is null and we will refer to this as case \( p = -1 \).

**Assumption F(1):**

(i) \( \eta_t \sim \text{iid}(0, \Sigma_\eta) \), \( E\|\eta_t\|^4 \leq M \), (ii) \( \text{var}(\Delta F_t) = \sum_{j=0}^{\infty} C_j \Sigma_\eta C_j' > 0 \), (iii) \( \sum_{j=0}^{\infty} j \|C_j\| < M \), (iv) \( C(1) \) has rank \( r_1 \), \( 0 \leq r_1 \leq r \), and (v) \( E\|F_0\| \leq M \).

**Assumption E(1):** \( E|\varepsilon_{i0}| < M \) for all \( i = 1, \ldots, N \).

Under F(1) (non-stationary factors), the short-run variance of \( \Delta F_t \) is positive definite but the long run variance can be reduced rank. Initial conditions are also imposed on the factors and the errors to permit asymptotic analysis.
The number of common stochastic trends is determined by $r_1$, the rank of $C(1)$. When $r_1 = 0$, $\Delta F_t$ is overdiffereced and the common factors are stationary.

In this model, $X_{it}$ can be non-stationary when $F_t$ has unit roots, or $\rho_i = 1$, or both. Clearly, if the common factors share a stochastic trend, $X_{it}$ will all be non-stationary. But even if a series indeed has a common and trending component, the series may appear stationary if the idiosyncratic component is stationary.

7.1 Estimation of $F_t$ when $e_{it}$ may be I(1)

Suppose now that we observe only $X_{it}$ and we do not know if it is the factors or the idiosyncratic errors are stationary. The possibility that $e_{it}$ is non-stationary poses a serious problem for estimation, as any regression with a non-stationary error (observed or otherwise) is spurious. We now consider how to estimate the factors when we do not know a priori if $e_{it}$ is stationary.

(a) Case $p = 0$: Let $\Delta X_{it} = \lambda'_i \Delta F_t + \Delta e_{it}$. Denote $x_{it} = \Delta X_{it}$, $f_t = \Delta F_t$, and $z_{it} = \Delta e_{it}$. Let $(\lambda_1, \ldots, \lambda_N)$ and $(f_1, \ldots, f_T)$ and $z_{it}$ for all $i$ and $t$ be the principal components estimates of $\lambda_i$, $f_t$, and $z_{it}$.

(b) Case $p = 1$: Let $x_{it} = \Delta X_{it} - \Delta \bar{X}_i$, $f_t = \Delta F_t - \Delta \bar{F}$, and $z_{it} = \Delta e_{it} - \Delta \bar{e}_i$. Let $\hat{f}_t$ and $\hat{\lambda}_i$ be obtained by applying the method of principal components to the differenced and demeaned data. Let $\hat{F}_t = \sum_{s=2}^t \hat{f}_s$ and $\hat{e}_{it} = \sum_{s=2}^t \hat{z}_{is}$.

**Result A1.** Let $\hat{F}_t = \sum_{s=2}^t \hat{f}_s$ and $\hat{e}_{it} = \sum_{s=2}^t \hat{z}_{is}$. Suppose $f_t$ and $\lambda_i$ satisfy Assumption F(0), and $z_{it}$ satisfies Assumption L and E. Then

$$\max_{1 \leq t \leq T} \| \hat{F}_t - HF_t + HF_1 \| = O_p(T^{1/2}N^{-1/2}) + O_p(T^{-1/4}).$$

The basis of Result A1, developed in Bai and Ng (2004), is that $x_{it} = \lambda'_i f_t + z_{it}$ is a pure factor model satisfying Assumption F(0). By Result A0, $\hat{f}_t$, $\hat{\lambda}_i$, and $\hat{z}_{it}$ are consistent for $f_t$, $\lambda_i$, and $z_{it}$. Result A1 says that if $T/N \rightarrow 0$ as $N,T \rightarrow \infty$, then $\hat{F}_t$ is uniformly consistent for $HF_t$. Pointwise convergence (for each given $t$) does not require $T/N \rightarrow 0$;
the deviation of $\hat{F}_t$ from $HF_t$ is of order $\min[N^{-1/2}, T^{-1/2}]$. This result is quite remarkable, as it is obtained without knowledge of whether $F_t$ or $e_{it}$ is I(1) or I(0). This means that even if each cross-section equation is a spurious regression, the common stochastic trends are well defined and can be consistently estimated, if they exist. This is certainly not possible within the framework of traditional time series analysis in which $N$ is fixed.

To see the method actually work, we simulate an I(1) common factor process $F_t = F_{t-1} + \eta_t$, and independent I(1) idiosyncratic errors $e_{it} = e_{i,t-1} + \varepsilon_{it}$ for $t = 1, 2, \ldots, T; i = 1, 2, \ldots, N$ ($N = 40, T = 100$) to form $X_{it} = \lambda_i F_t + e_{it}$ with $\lambda_i$ iid $N(0,1)$. The observable processes $X_{it}$ are not cointegrated because $e_{it}$ are independent I(1) processes. Once the data are generated, we treat $F_t$ and $e_{it}$ as unobservable. We then estimate $F_t$ by the difference-recumulating method discussed earlier. To demonstrate that the estimated common trend $\hat{F}_t$ is close to the actual $F_t$, we rotate $\hat{F}_t$ toward $F_t$ by running the regression $F_t = b\hat{F}_t + \text{error}$. Figure 7.1 displays both $F_t$ and $\hat{F}_t$. The estimated $\hat{F}_t$ tracks $F_t$ quite well. If the data are not differenced, one cannot expect consistent estimation of $F_t$, unless the data are I(0). To see this, we present, in Figure 7.2, the estimate without differencing the data. As can be seen, the estimate is unsatisfactory.

Result A1 pertains to the case when we do not know if $F_t$ and $e_{it}$ are stationary or not. The case in which $F_t$ is a vector of integrated processes and $e_{it}$ is stationary is considered by Bai (2004). He shows that the estimated common factors have a faster rate of convergence than the case in which $F_t$ being I(0). He also derives the limiting distribution for the estimated common factors.

### 7.2 Unit Root Tests

The ability to estimate $F_t$ when the data are non-stationary opens up new possibilities of re-examining hypotheses that have been difficult to test. For example, when $x_{it}$ has a factor structure, non-stationarity can arise because $F_t$ is I(1), or $e_{it}$ is I(1), or both. When one of the components is stationary but the other is not, testing $x_{it}$ for non-stationarity can be misleading. A more informative approach is to test
Fig. 7.1 Estimated common trend from a large spurious system: Differenced data approach.

\( F_t \) and \( e_{it} \) separately. Result A1 shows that this is possible. Consider the regression

\[
\Delta \hat{e}_{it} = d_0 \hat{e}_{it-1} + d_{i1} \Delta \hat{e}_{it-1} + \cdots + d_{ik} \Delta \hat{e}_{it-k} + \text{error.} \tag{7.2}
\]

**Result J: PANIC (Panel Analysis of Nonstationarity in Idiosyncratic and Common components).** Suppose \( F(1) \), and \( E(1) \) hold. Let \( ADF^c_i \) be the \( t \) test for \( d_0 \) in (7.2) with \( k \) chosen such that \( k^3 \min \{N, T\} \to 0 \). Let \( W_{\epsilon i} \) be a standard Brownian and \( V_{\epsilon i}(s) \) be a Brownian bridge.

J.1. Case \( p = 0 \):

\[
ADF^0_i \Rightarrow \frac{\int_0^1 W_{\epsilon i}(s) dW_{\epsilon i}(s)}{(\int_0^1 W_{\epsilon i}(s)^2 ds)^{1/2}}.
\]
7.2 Unit Root Tests

Fig. 7.2 Estimated common trend from a large spurious system: Level data approach.

J.2. Case \( p = 1 \):

\[
ADF_i^1 \Rightarrow -\frac{1}{2} \left( \int_0^1 V_{ei}(s)^2 ds \right)^{-1/2}.
\]

Result J says that the said Dickey–Fuller test can be used to test if \( \hat{e}_{it} \) is non-stationary, one series at a time. When \( p = 0 \), the test has the same limiting distribution as the usual Dickey–Fuller test without an intercept. When \( p = 1 \), the limiting distribution is not a DF type but is proportional to the reciprocal of a Brownian bridge. Result I suggests that other unit root and stationary tests, not just the ADF, can be used to test \( \hat{e}_{it} \). Our conjecture is that the limiting distribution when \( p = 0 \) will be the same as the one when an observed series is tested but without the intercept. When \( p = 1 \), the limiting distribution will likely be different.
Perhaps the most exploited aspect of the factor structure is that it enables construction of panel unit root tests in the presence of cross-section dependence. By assuming that the cross-section correlations are strong and are induced by the common factors, they can be directly removed from $x_{it}$ to yield $\hat{e}_{it}$. Thus, while pooling $x_{it}$ will yield tests with size distortions, pooling $\hat{e}_{it}$ is valid. Result K below considers three types of pooled tests.

Result K: Pooled Unit Root Tests

K.1. Suppose the $e_{it}$ are iid across $i$. Let $p_i$ be the $p$-values associated with $ADF^c_{\hat{e}}$ or $ADF^\tau_{\hat{e}}$. Then

$$P_i = \frac{-2\sum_{i=1}^{N} \log p_i - 2N}{\sqrt{4N}} \xrightarrow{d} N(0,1).$$

K.2. Given $\hat{e}_{it}$ from PANIC, define the PMSB (panel MSB) as

$$PMSB = \frac{\sqrt{N}(\text{tr}(\frac{1}{N}\hat{e}'\hat{e}) - \hat{\psi})}{\sqrt{\hat{\phi}^4_i/K}},$$

where $\hat{\psi} = \hat{\omega}^2/2$ and $K = 3$ when $p = 0$, and $\hat{\psi} = \hat{\omega}^2/6$ and $K = 45$ when $p = 1$. Under the null hypothesis that all $e_{it}$ are non-stationary and iid across $i$, then as $N,T \to \infty$ with $N/T \to 0$,

$$PMSB \xrightarrow{d} N(0,1).$$

Result K.1 pools the $p$ values of the test and is especially appropriate when there is substantial heterogeneity in the data. Result K.2 exploits an important feature that distinguishes stationary from non-stationary processes, namely, that their sample moments require different rates of normalization in order to be bounded asymptotically. Assuming cross-section independence, Phillips and Ploberger (2002) proposed a point optimal test for panel unit root in the presence of incidental trends that has some resemblance to the Sargan–Bhargava test, first proposed in Sargan and Bhargava (1983). Stock (1990) modified the test to allow
for serial correlation in the errors. The properties of the test are analyzed in Perron and Ng (1998). Our PMSB is explicitly motivated by the Sargan–Bhargava test and permits \( x_{it} \) to be cross-sectionally correlated. Standard methods that do not take into account common stochastic trends perform poorly, as documented by Banerjee et al. (2005). Other procedures that allow factor structures can be found in Moon and Perron (2004), Phillips and Sul (2003), Breitung and Das (2007), Pesaran (2007), among others. A recent survey on panel unit root and cointegration is provided by Breitung and Pesaran (2007).

Moon and Perron (2004) considered a panel unit root test that estimates the pooled autoregressive coefficient of the defactored data. The test, like \( P_\epsilon \), is also a test of the idiosyncratic errors. As such, a PANIC test of the pooled autoregressive coefficient in \( \varepsilon_{it} \) can also be developed. Consider pooled OLS estimation of the model \( \hat{\varepsilon}_{it} = \rho \hat{\varepsilon}_{it-1} + \varepsilon_{it} \), where \( \hat{\varepsilon}_{-1} \) and \( \hat{\varepsilon} \) are \((T - 2) \times N\) matrices. When \( p = 1 \), we add an intercept and a linear trend in the above regression. Define the bias corrected estimator

\[
\hat{\rho}^+ = \frac{\text{tr}(\hat{\varepsilon}'_{-1}M_z\hat{\varepsilon} - NT\hat{\psi})}{\text{tr}(\hat{\varepsilon}'_{-1}M_z\hat{\varepsilon}_{-1})},
\]

where \( \hat{\psi} \) is the bias correction estimated from \( \hat{\varepsilon} = M_z\hat{\varepsilon} - \hat{\rho}M_z\hat{\varepsilon}_{-1} \), where \( M_1 = I_{T-2} - z(z'z)z' \) with \( z = (z_1, z_2, \ldots, z_{T-2})' \). Let \( \hat{\omega}^2 = \frac{1}{N} \sum_{i=1}^{N} \hat{\omega}_i^2 \), \( \hat{\omega}_i^2 \) being the estimate for the long run variance of \( \varepsilon_{it} \), computed from the residuals \( \hat{\varepsilon}_{it} \). Also let \( \hat{\sigma}^2 = \frac{1}{N} \sum_{i=1}^{N} \hat{\sigma}_i^2 \) with \( \hat{\sigma}_i^2 \) being the variance estimate for \( \varepsilon_{it} \), and \( \hat{\phi}_4^4 = \frac{1}{N} \sum_{i=1}^{N} \hat{\omega}_i^4 \).

K.3 Consider the test statistics

\[
P_a = \frac{\sqrt{NT}(\hat{\rho}^+ - 1)}{\sqrt{K_a \hat{\phi}_4^4 / \hat{\omega}^4}}
\]

\[
P_b = \sqrt{NT}(\hat{\rho}^+ - 1) \sqrt{\frac{1}{NT^2} \text{tr}(\hat{\varepsilon}_{-1}M\hat{\varepsilon}_{-1})K_b \hat{\omega}^2 / \hat{\phi}_4^4}.
\]

Then \( P_a, P_b \xrightarrow{d} N(0,1) \), where the parameter values are given in the following table:

Result K.3 is a test of whether the pooled autoregressive coefficient is unity, and was recently developed in Bai and Ng (2006d).
The statistics $P_{a,b}$ are the analog of $t_{a,b}$ of Moon and Perron (2004), except that (i) the tests are based on PANIC residuals and (ii) the method of "defactoring" of the data is different from MP. Improved versions of the tests can also be obtained by iteration. As discussed in Bai and Ng (2006d), iteration ensures that the estimate of $\rho$ used to construct the long-run variance estimates coincides with the estimate of $\rho$ that is being test for deviation from unity.

When $p = 0$, whether one tests the pooled autoregressive coefficient using $P_a, P_b$, or the divergence of the sample moments using PMSB, the results in terms of both size and power are quite similar. Both tests have better power than testing the pooled $p$ values. However, we find in Bai and Ng (2006d) that $P_e$ and PMSB are superior to tests of a common unit root against incidental trends. Moon et al. (2007) and Phillips and Ploberger (2002) also documented that panel unit root tests that rely on a pooled estimate of autoregressive parameter do not have good power properties against deterministic trends.

In a recent paper, Westerlund and Larsson (2007) point out that the asymptotic distribution of a test that pools the individual unit root tests is not centered around zero. While this suggests that the $P_e$ (which pools $p$ values instead of the tests) may have size distortion, simulations suggest that this is a problem in finite samples only when $N$ is very small (less than 10). On the other hand, the $P_e$ is still the test with the best properties when there are incidental trends.

### 7.3 Common Trends

If there are common factors in the data $x_{it}$, and one or more of the factors are non-stationary, any two series $x_{it}$ and $x_{jt}$ are cointegrated in the sense of Engle and Granger (1987). Result A1 shows that $\hat{F}_t$ can be consistently estimated without knowing if the data are stationary or not. Testing if $\hat{F}_t$ is non-stationary, however, is more involved because $\hat{F}_t$ is a vector process and there can be multiple unit roots. Define $\hat{F}_t^c =$ [Table]

<table>
<thead>
<tr>
<th>$p$</th>
<th>$M_c$</th>
<th>$\psi$</th>
<th>$K_a$</th>
<th>$K_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$T^{-1}$</td>
<td>$\lambda_c$</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$M_1$</td>
<td>$-\tilde{\sigma}_c^2/2$</td>
<td>$15/4$</td>
<td>4</td>
</tr>
</tbody>
</table>
\( \hat{F}_t - \bar{F}, \overline{F} = (T - 1)^{-1} \sum_{t=2}^{T} \hat{F}_t \). For some \( m \leq r \), let \( \hat{Y}_t^c = \tilde{\beta}_1 \hat{F}_t^c \), where \( \tilde{\beta}_1 \) are the \( m \) eigenvectors associated with the \( m \) largest eigenvalues of \( T^{-2} \sum_{t=2}^{T} \hat{F}_t^c \hat{F}_t^{c'} \). Let \( \tilde{\gamma}_t^c = \hat{\Pi}(L) \hat{Y}_t^c \) and \( \tilde{\gamma}_t^{c'} \) be the smallest eigenvalue of \( \hat{\Phi}_t^c(m) = \frac{1}{2} \sum_{t=2}^{T} (\tilde{\gamma}_t^c \tilde{\gamma}_{t-1}^c + \tilde{\gamma}_{t-1}^c \tilde{\gamma}_t^c) \left( \sum_{t=2}^{T} \tilde{\gamma}_t^c \tilde{\gamma}_{t-1}^c \right)^{-1} \). Similarly define \( \tilde{\gamma}_t^c \) and \( \hat{\gamma}_t^{c'} \) when there is a time trend.

**Result L: Testing for Common Trends.** Consider the null hypothesis that \( F_t \) has \( m \) stochastic trends with finite VAR representation.

- **L.1 (intercept model).** Let \( W_m^c \) be a vector of \( m \) dimensional demeaned Brownian motions. Then \( MQ_f^c(m) = T(\tilde{\gamma}_t^c(m) - 1) \xrightarrow{d} \nu_c^c(m) \), where \( \nu_c^c(m) \) is the smallest eigenvalue of
  \[
  \Phi_c^c(m) = \frac{1}{2} [W_m^c(1)W_m^c(1)' - I_m] \left[ \int_0^1 W_m^c(s)W_m^c(s)'ds \right]^{-1}.
  \]
- **L.2 (linear trend case).** Let \( W_m^r \) be a vector of \( m \) dimensional detrended Brownian motions. Then \( MQ_f^r(m) = T(\tilde{\gamma}_t^r(m) - 1) \xrightarrow{d} \nu_r^r(m) \), where \( \nu_r^r(m) \) is the smallest eigenvalue of
  \[
  \Phi_r^r(m) = \frac{1}{2} [W_m^r(1)W_m^r(1)' - I_m] \left[ \int_0^1 W_m^r(s)W_m^r(s)'ds \right]^{-1}.
  \]

Result L states that the number of common trends in \( F_t \) can be tested using \( \hat{F}_t \). The proposed statistic is a modified variation of Stock and Watson (1988). Their test is based on the idea that the real part of the smallest eigenvalue of an autoregressive coefficient matrix should be unity. Our modification ensures that the eigenvalues are always real and enables simplifications to the proofs.

### 7.4 Panel Unit Roots with Structural Breaks

The model has the same form as (7.1) except that the deterministic components have structural breaks. Bai and Carrion-i-Silvestre (2004) consider two specifications, referred to as Model 1 and Model 2,
respectively,

Model 1:  \[ D_{i,t} = \mu_i + \sum_{j=1}^{l_i} \theta_{i,j} DU_{i,j,t} \]  \( (7.4) \)

Model 2:  \[ D_{i,t} = \mu_i + \beta_i t + \sum_{j=1}^{l_i} \theta_{i,j} DU_{i,j,t} + \sum_{k=1}^{m_i} \gamma_{i,k} DT_{i,k,t}^*, \]  \( (7.5) \)

where \( DU_{i,j,t} = 1 \) for \( t > T_{i,a,j}^* \) and 0 elsewhere, and \( DT_{i,k,t}^* = (t - T_{i,b,k}^*) \) for \( t > T_{i,b,k}^* \) and 0 elsewhere, where \( T_{i,a,j}^* \) and \( T_{i,b,k}^* \) denote the \( j \)th and \( k \)th dates of the break for the \( i \)th individual, \( j = 1, \ldots, l_i \), \( k = 1, \ldots, m_i \). There are \( l_i \) structural breaks affecting the mean and \( m_i \) structural breaks affecting the trend of the time series, where \( l_i \) is not necessarily equal to \( m_i \).

The structural breaks are heterogeneous across individuals because (i) the break dates \( T_{i,a,j}^* \) and \( T_{i,b,k}^* \) are individual specific, (ii) the magnitude of shifts are also individual specific, (iii) each individual may have a different number of structural breaks, and (iv) the break points for the level and the slope can be at different times. While Model 1 is a special case of Model 2, the limiting distribution of the test statistic for Model 1 is not a special case for Model 2. Thus we will consider the two models separately.

To estimate the model, we follow Bai and Ng (2004). For Model 1, differencing leads to

\[ \Delta X_{i,t} = \Delta F_{i} \pi_{i} + \Delta e_{i,t}^*, \]

where

\[ \Delta e_{i,t}^* = \Delta e_{i,t} + \sum_{j=1}^{l_i} \theta_{i,j} D(T_{i,a,j}^*)_{t}, \]  \( (7.6) \)

with \( D(T_{i,a,j}^*)_{t} = 1 \) if \( t = T_{i,a,j}^* \) and \( D(T_{i,a,j}^*)_{t} = 0 \) otherwise. In matrix notation, we can rewrite the above as

\[ \Delta X_i = \Delta F \pi_i + \Delta e_i^*, \]

where \( \Delta X_i = (\Delta X_{i,2}, \Delta X_{i,3}, \ldots, \Delta X_{i,T})' \) and \( \Delta e_i^* = (\Delta e_{i,2}^*, \Delta e_{i,3}^*, \ldots, \Delta e_{i,T}^*)' \) are two \((T - 1) \times 1\) vectors for the \( i \)th individual, \( \Delta F = \)
[\Delta F_1, \Delta F_2, \ldots, \Delta F_T]' is a \((T - 1) \times r\) matrix and \(\pi_i = (\pi_{i,1}, \ldots, \pi_{i,r})'\) is the \((r \times 1)\)-vector of loading parameters for the \(i\)th individual, \(i = 1, \ldots, N\).

We can rewrite the model more compactly as

\[ x_i = f\pi_i + z_i, \] (7.7)

where \(x_i = \Delta X_i\), \(f = \Delta F\), and \(z_i = \Delta e_i^*\). Now \(f\) and \(z\) can be estimated by the principal components method. The unit root test statistic for each individual time series can be based on

\[ \hat{e}_{i,t} = \sum_{s=2}^{T} \hat{z}_{i,s}. \]

A consistent unit root test is the modified Sargan–Bhargava (MSB) statistic, defined as

\[ MSB_i = \frac{T^{-2}\sum_{t=1}^{T} \hat{e}_{i,t}^2}{\hat{\sigma}_i^2}, \] (7.8)

where \(\hat{\sigma}_i^2\) is a consistent estimator of the long-run variance of \(e_{it} - \rho_i e_{i,t-1}\).

For Model 2, differencing leads to

\[ \Delta X_{i,t} = \Delta F_t'\pi_i + \beta_i + \sum_{k=1}^{m_i} \gamma_{i,k} DU_{i,k,t} + \Delta e_{i,t}^*, \]

where \(DU_{i,k,t} = 1\) when \(t > T_{b,k}^i\) and 0 otherwise. The dummy variables result from differencing the broken trends, see Equation (7.5). There is no loss of generality by assuming \(\Delta F_t\) to have zero mean, otherwise, define \(f_t = \Delta F_t - \tau\) with \(\tau = E(\Delta F_t)\) and redefine the intercept as \(\beta_i + \tau'\pi_i\). More compactly, we also have

\[ x_i = f\pi_i + a_i\delta_i + z_i, \] (7.9)

where \(x_i\), \(f\), and \(z_i\) are defined earlier; \(\delta_i = (\beta_i, \gamma_{i,1}, \ldots, \gamma_{i,m_i})'\) and \(a_i = (a_{i,2}, \ldots, a_{i,T})'\) with \(a_{i,t} = (1, DU_{i,1,t}, \ldots, DU_{i,m_i,t})'\). Thus \(a_i\) is the matrix of regressors for the \(i\)th cross section, and \(\delta_i\) is the corresponding vector of coefficients.
The matrix $a_i$ is not completely specified, as it depends on unknown break points in the slope. But a slight modification of the iterative approach considered in Section 4 can be used. Suppose the break points are known. Then we can start the iteration by first estimating $\delta_i$ with the least squares, i.e., $\tilde{\delta}_i = (a_i' a_i)^{-1} a_i' x_i$. Because $f$ has zero mean and may therefore be treated as a part of regression errors, $\tilde{\delta}_i$ is in fact root-$T$ consistent in the first step. Given $\tilde{\delta}_i$, we estimate the factors and factor loadings based on $\tilde{w}_i = x_i - a_i \tilde{\delta}_i$ ($i = 1, 2, \ldots, N$). Due to the good starting value for $\tilde{\delta}_i$, convergence is rapid, where convergence is achieved when the successive change in sum of squared residuals is small than a prespecified small number. The iteration procedure is equivalent to simultaneously estimating $f$, $\pi_i$, and $\delta_i$.

In general, we need to estimate the break points. Because $\Delta F_t$ has zero mean, we can regard $f \pi_i + z \epsilon_i$ as the overall disturbance. Then Equation (7.9) is a simple model with mean breaks. We can apply the Bai–Perron dynamic programming algorithm to estimate the number and the location of the breaks. This is done equation by equation, unless one assumes common break dates across equations. Owing to the consistency and fast convergence rate for the estimated break points, the regression matrix $a_i$ indeed can be treated as known. A formal argument is presented in Bai and Carrion-i-Silvestre (2004).

With breaks in slope, the estimation procedure consists of two steps. The first step estimates $m_i$ and the break points $T_{b,k}^i$, $k = 1, \ldots, m_i$, to obtain $(\hat{m}_i, \hat{T}_{b,k}^i; k = 1, \ldots, m_i)$, from which we define the regressor matrix $\hat{a}_i$. The second step estimates $f$, $\pi_i$, and $\delta_i$ ($i = 1, 2, \ldots, N$) from the equation $x_i = f \pi_i + \hat{a}_i \delta_i + z_i$ with an iteration procedure described earlier. Let $\hat{f}$, $\hat{\pi}_i$, and $\hat{\delta}_i$ be the final estimates. The residual vector is given by

$$\hat{z}_i = x_i - \hat{f} \hat{\pi}_i - \hat{a}_i \hat{\delta}_i.$$ 

The cumulative sum of $\hat{z}_{i,t}$ gives $\hat{e}_{i,t} = \sum_{s=2}^{t} \hat{z}_{i,s}$. The MSB test based on the sequence $\hat{e}_{i,t}$ is computed as in Equation (7.8) for each $i$ to test the null hypothesis that $\rho_i = 1$ in Equation (7.1), against the alternative hypothesis that $|\rho_i| < 1$. The limiting distribution of the MSB statistics when there are common factors and multiple structural breaks is given in the following result.
Result M: Unit Root Test with Breaks. Assume that the trend break points satisfy $T^i_{b,k} = [T\lambda_{i,k}]$, where $[x]$ denotes the largest integer less than or equal to $x$, and $\lambda_{i,k} \in (0,1)$, $k = 1, 2, \ldots, m_i$, $i = 1, 2, \ldots, N$. Under the null hypothesis that $\rho_i = 1$, the MSB test in (7.8) converges in distribution to

1. Model 1: $MSB_i \Rightarrow \int_0^1 W_i^2 (r) \, dr$

2. Model 2: $MSB_i (\lambda_i) \Rightarrow \sum_{k=1}^{m_i+1} (\lambda_{i,k} - \lambda_{i,k-1})^2 \int_0^1 V_{i,k}^2 (b) \, db$,

where $W_i (r)$ is the standard Brownian motion independent across $i$, $V_{i,k} (b) = W_{i,k} (b) - bW_{i,k} (1)$ is a Brownian bridge independent across $i$ and $k$, $k = 1, \ldots, m_i + 1$, and $\lambda_{i,0} = 0$ and $\lambda_{i,m_i+1} = 1$, $i = 1, \ldots, N$.

Since the common factors $F$ do not appear in the limiting distribution, the test statistics $MSB_i (\lambda)$ are asymptotically independent provided that $e_{it}$ are independent over $i$. This implies that one can construct a valid pooled test statistic. Let $\xi_i$ denote the limiting distribution. It is very easy to derive the mean and variance of $\xi_i$ which are given in Bai and Carrion-i-Silvestre (2005). One way of pooling is through standardization

$$P_1 = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{MSB_i (\lambda_i) - E(\xi_i)}{\text{var}(\xi_i)}.$$ 

The other way of pooling is the combination of individual $p$-values. Let $p_i$ denote the $p$-value for $MSB_i (\lambda_i)$. Define

$$P_2 = -\frac{2 \sum_{i=1}^{N} \log(p_i)}{2N} - 2N.$$ 

Given a value of test statistic $MSB_i (\lambda_i)$, its $p$-value can be easily obtained via the response surfaces computed by Bai and Carrion-i-Silvestre (2004). The above discussion focuses on breaks in the mean and slopes. Breaks in factor loadings are studied by Kao et al. (2007).

7.5 Panel Cointegration with Global Trends

Consider now the model

$$Y_{it} = X'_{it} \beta + u_{it},$$
where for $i = 1, \ldots, N$, $t = 1, \ldots, T$, $Y_{it}$ is a scalar, $X_{it} = X_{it-1} + \nu_{it}$. If $u_{it}$ is stationary and iid across $i$, we can then say the panel shares a common cointegrating vector, $(1, -\beta')$. It is straightforward to show that $\hat{\beta}$ is superconsistent, even though its limiting distribution is, in general, non-standard. The assumption that $u_{it}$ is cross-sectionally independent is strong. To remove this assumption, we assume

$$u_{it} = \lambda_i' F_t + e_{it},$$

where $F_t$ is a $r \times 1$ vector of latent common factors, $\lambda_i$ is a $r \times 1$ vector of factor loadings and $e_{it}$ is the idiosyncratic error. If $F_t$ is stationary, $Y_{it}$ and $X_{it}$ are still cointegrated in a panel sense. But if $F_t$ is non-stationary, panel cointegration effectively occurs between $Y_{it}, X_{it}$ and $F_t$. For obvious reasons, the model can be termed panel cointegration with global stochastic trends. Define

$$w_i = Y_i - X_i\beta = F\lambda_i + e_i.$$

Our proposed continuous updated estimator (Cup) for $(\beta, F)$ is defined as

$$(\hat{\beta}_{\text{Cup}}, \hat{F}_{\text{Cup}}) = \arg\min_{\beta, F} S_{NT}(\beta, F).$$

More precisely, $(\hat{\beta}_{\text{Cup}}, \hat{F}_{\text{Cup}})$ is the solution to the following two non-linear equations

$$\hat{\beta} = \left( \sum_{i=1}^{N} X_i' M_{\hat{F}} X_i \right)^{-1} \sum_{i=1}^{N} X_i' M_{\hat{F}} Y_i$$

$$\hat{F} V_{NT} = \left[ \frac{1}{NT^2} \sum_{i=1}^{N} \left( Y_i - X_i\hat{\beta} \right) \left( Y_i - X_i\hat{\beta} \right)' \right] \hat{F}$$

where $M_{\hat{F}} = I_T - T^{-2} \hat{F}' \hat{F}$ since $\hat{F}' \hat{F} / T^2 = I_r$, and $V_{NT}$ is a diagonal matrix consisting of the $r$ largest eigenvalues of the matrix inside the brackets, arranged in decreasing order. The estimator of $\hat{\beta}$ has the same form as in the stationary case considered earlier. But the limiting distribution is different.
Result N: Panel Cointegration and Global Trends. Define the bias-corrected Cup estimator as 
\[ \hat{\beta}_{\text{Cup}}^{\text{BC}} = \hat{\beta}_{\text{Cup}} - \frac{1}{T} \hat{\phi}_{NT}, \]
where \( \hat{\phi}_{NT} \) is a bias correction term defined in Bai et al. (2006). Then
\[ \sqrt{NT}(\hat{\beta}_{\text{Cup}}^{\text{BC}} - \beta^0) \xrightarrow{d} N(0, \Sigma) \]
for some positive definite matrix \( \Sigma \).

The result is based on Bai et al. (2006). Because of asymptotic normality, the usual chi-square test for hypothesis on the coefficient \( \beta \) can be performed. The Cup estimator is obtained by iteratively solving for \( \hat{\beta} \) and \( \hat{F} \) using (7.10) and (7.11). It is a nonlinear estimator even though linear least squares estimation is involved at each iteration. The CupBC estimator is consistent with a limiting distribution that is centered at zero as long as \( (N,T) \to \infty \) and \( \frac{N}{T} \to 0 \). A fully modified estimator can also be obtained along the lines of Phillips and Hansen (1990). The resulting bias-corrected CupFM estimator is also \( \sqrt{NT} \) consistent and asymptotically normally. An extension to include incidental trends such as
\[ Y_{it} = \mu_i + \gamma_i t + X_{it}' \beta + \lambda_i' F_t + e_{it} \]
is also possible. The presence of incidental trends will affect the limiting covariance matrix \( \Sigma \), but asymptotic normality and and thus chi-squared distribution for hypothesis testing on \( \beta \) remain valid.

7.6 Testing Panel Cointegration with Global Trends

Panel cointegration imposes the restriction that \( e_{it} \) in (7.12) are all I(0). This restriction can be tested. Bai and Carrion-i-Silvestre (2005) consider testing the null hypothesis of no cointegration versus cointegration. More specifically, consider testing \( H_0 : e_{it} \sim I(1) \) versus \( H_1 : e_{it} \sim I(0) \) in the following heterogeneous slope panel cointegration model:
\[ Y_{it} = \mu_i + \gamma_i t + X_{it}' \beta_i + \lambda_i' F_t + e_{it}. \]  
(7.12)
We assume both \( X_{it} \) and \( F_t \) are I(1). That is, \( X_{it} = X_{it-1} + v_{it} \) and \( F_t = F_{t-1} + \eta_t \). As shown in Bai and Carrion-i-Silvestre (2005), there is no loss of generality for testing purpose by assuming \( (\Delta X_{it}', \Delta F_t') = (v_{it}', \eta_t') \) is uncorrelated with \( \Delta e_{it} \). This implies that the product \( \Delta X_{it} \Delta e_{it} \) will be a sequence of zero mean random variables. As long as \( \Delta e_{it} \) and \( \Delta X_{it} \) have weak serial correlations, the product
$\Delta X_{it}\Delta e_{it}$ is also serially correlated but only weakly. So we can assume $T^{-1/2} \sum_{t=1}^{T} \Delta X_{it}\Delta e_{it}$ to be $O_p(1)$. Similarly, $T^{-1/2} \sum_{t=1}^{T} \Delta F_t \Delta e_{it}$ is also $O_p(1)$.

Again as in Bai and Ng (2004), differencing in the intercept only case leads to

$$\Delta Y_{it} = \Delta X_{it}' \beta_i + \lambda_i \Delta F_t + \Delta e_{it}.$$ 

In the absence of correlation between $\Delta X_{it}$ and $\Delta F_t$, we can first estimate $\beta_i$ by least squares, treating $\lambda_i F_t + \Delta e_{it}$ as the regression error. The lack of correlation between $\Delta X_{it}$ and the error $\lambda_i \Delta F_t + \Delta e_{it}$ makes it possible to obtain $\sqrt{T}$ consistent estimation of $\beta_i$. We can then estimate $\Lambda$ and $F$ by the method of principal components treating $\Delta X_{it} \beta_i + \Delta e_{it}$ as the idiosyncratic errors. But when $\Delta X_{it}$ and $\Delta F_t$ are correlated, the two step procedure will be inconsistent; we must estimate the unknown quantities simultaneously. This can be achieved by the iteration procedure outline in Result N, with $F_t$ replaced by $f_t = \Delta F_t$.

Let $\tilde{z}_i = y_i - x_i \tilde{\beta}_i - \tilde{f} \lambda_i$, and define $\tilde{e}_{it} = \sum_{s=2}^{t} \tilde{z}_is$. The MSB statistic is again computed as

$$MSB_i = \frac{T^{-2} \sum_{t=2}^{T} \tilde{e}_{it}^2}{\tilde{\sigma}_i^2},$$

where $\tilde{\sigma}_i^2$ is an estimator for the long-run variance of $e_{it} - \rho e_{it-1}$. The estimator is obtained from the residuals $\tilde{e}_{it} = \tilde{e}_{it} - \hat{\rho}_i \tilde{e}_{it-1}$ by the Newey–West (1987) procedure.

For the linear trend case, we difference and demean the data as in PANIC, and $f_t = \Delta F_t - \Delta F$.

**Result O: Testing Panel Cointegration.** Under the null hypothesis that $\rho_i = 1$:

O.1. For the intercept only case,

$$MSB_i \xrightarrow{d} \int_0^1 W_i(r)^2 dr,$$

where $W_i$ is a Brownian motion.
O.2. For the linear trend case

\[ MSB_i \xrightarrow{d} \int_0^1 B_i(r)^2 dr, \]

where \( B_i \) is Brownian bridge.

It is important to highlight that the limiting distributions do not depend on \( X_{it} \) and \( F_t \). Thus the limiting distributions are independent over \( i \) provided that \( e_{it} \) are cross-sectionally independent. As a result, a valid pooled test statistic for panel cointegration can be constructed. One way of pooling is based on standardization. Let \( U_i = \int_0^1 W_i(r)^2 \) and \( V_i = \int_0^1 B_i(r)^2 dr \). Using \( EU_i = 1/2 \), \( \text{var}(U_i) = 1/6 \), and \( \text{var}(V_i) = 1/45 \), we have, for the intercept only case, \( \sqrt{3N^{-1/2}} \sum_{i=1}^{N} (MSB_i - 1/2) \xrightarrow{d} N(0,1) \), and for the linear trend case, \( \sqrt{45N^{-1/2}} \sum_{i=1}^{N} (MSB_i - 1/6) \xrightarrow{d} N(0,1) \). Pooling based on \( p \)-values can also be easily constructed.

An LM type test for panel cointegration that also permits structural breaks is considered by Westerlund and Edgerton (2007).
How Precise are the Factor Estimates?

The key to being able to use the principal component estimates as though they were the observed factors is that the factor space can be estimated precisely as $N$ and $T$ tend to infinity. In essence, if the principal components can estimate the space spanned by the true factors as $N$ and $T$ increase, and the sample principal components precisely estimate the population principal components as $N$ gets large, the sample principal components will consistently estimate the space spanned by the factors. In simulations, we found that when the errors are iid over $i$ and $t$, $\min[N,T]$ can be as small as 30 and the number of factors can be estimated with almost perfect certainty, suggesting that the factor space can indeed be estimated with high precision even with rather small samples. Figure 8.1 illustrates this for stationary and non-stationary factors.

Can it be possible that increasing $N$ does not improve the precision of the factor estimates? Boivin and Ng (2006) argued that this is possible if the additional data are uninformative about the factor structure. This can arise if the idiosyncratic error variances are large, or if the factor loadings of the additional data are small. More generally, our theory for approximate factor models permits heteroskedasticity,
some cross-section and some serial correlation in the errors. Errors in which such effects are strong would be incompatible with Assumption E. When the importance of the idiosyncratic error is magnified, it will become more difficult to separate out the common from the idiosyncratic component in the data, and data with these characteristics cannot be ruled out in practice.

Efficiency issues surrounding the principal components estimator are best understood in terms of OLS versus GLS. It is well known that when the errors are spherical and other conditions of the Guass–Markov theorem are met, OLS is the most efficient amongst the class of linear unbiased estimators. However, when the errors are non-spherical, a GLS estimator that exploits information in the structure of the error variance is more efficient. In the present context, the principal components estimator is based on an unweighted objective function that minimizes the sum of squared residuals. No consideration is given as to whether a particular observation has errors that may be correlated over
t or i. Inefficient estimates can also be thought of as a consequence of not exploiting the moments implied by the idiosyncratic errors. However, unlike in simple regressions when feasible FGLS is usually possible, a weighted principal components estimator that weights the errors inversely by $\Omega$ is not possible. This is because the estimate of $\Omega$, or the sample covariance of $\tilde{e}$, cannot be full rank if $\tilde{e}$ are the errors associated with data generated by $r \geq 1$ factors. Indeed, $\tilde{\Omega}$ has a rank of $\min[T,N]-r$ making $\tilde{\Omega}$ non-invertible. To remedy this problem, several suggestions have been made to estimate weighted principal components by solving the problem

$$\min_{F_1, \ldots, F_T, \Lambda} \sum_{t=1}^{T} (X_t - \Lambda F_t)' \tilde{\Omega} (X_t - \Lambda F_t),$$

giving $\tilde{\Lambda}$ as the eigenvectors of $\tilde{\Omega}^{-1/2} \tilde{\Sigma} \tilde{\Omega}^{-1/2}$. The methods differ in the choice of $\tilde{\Omega}$. Jones (2001) and Boivin and Ng (2006) use the diagonal of $\tilde{\Omega}$ obtained from the unweighted estimation, while Forni et al. (2005) set $\hat{\Omega} = \hat{\Sigma} - \hat{\Sigma}_C$, where $\hat{\Sigma}_C$ is the covariance matrix of the unweighted estimated common component $\tilde{C}_{it}$. Boivin and Ng (2006) also considered a priori rules that set the off-diagonal elements of $\tilde{\Omega}$ to be either zeros or ones depending on whether the error of a series is deemed correlated with other errors. However, none of these methods are optimal in any formal sense.

Let $\text{eig}_{\text{p}}^{\text{r}}$ be the $r$th largest eigenvalue of the population covariance matrix of the $N$ variables, $z$. Recall that in theory, $\text{eig}_{\text{p}}^{\text{r}}$ should diverge as $N$ increases while $\text{eig}_{\text{p}+1}^{\text{r}}$ should be bounded. Theory also says that $\text{eig}_{\text{p}}^{\text{r}}/\text{eig}_{\text{p}+1}^{\text{r}}$ should diverge since the numerator increases with $N$ and the denominator is bounded. With weak loadings, correlated or large errors, $\text{eig}_{\text{p}}^{\text{r}}$ will not be “too different” from $\text{eig}_{\text{p}+1}^{\text{r}}$. Heaton and Solo (2006) interpret $\text{eig}_{\text{p}}^{\text{r}}/\text{eig}_{\text{p}+1}^{\text{r}}$ as an indicator of signal to noise. The eigenvalue conditions, while conceptually simple, are not easily translated into practical use because with one set of observational data, we cannot assess how the eigenvalues increase with $N$.

To better understand the precision of the factor estimates under more general conditions, we consider a monte carlo experiment. The data are generated according to the dynamic factor model. For $i=
$x_{it} = \lambda_i(L)f_t + \sigma_i e_{it},$

where $\sigma_i^2$ is set so that for a pre-specified $R^2_i \sim U[R^2_L, R^2_U]$, $\frac{\text{var}(\lambda_i(L)f_t)}{\text{var}(e_{it})} = \frac{R^2_i}{1-R^2_i}$ on average. In the simulations, we fix $R^2_U$ to 0.8. The factor loadings $\lambda_i(L) = \lambda_{i0} + \lambda_{i1}L + \cdots + \lambda_iL^s$ are generated with $\lambda_{ij} \sim N(0, 1)$ for $j = 0, \ldots, s$. When $s = 0$, we have $r = q = 1$ static factor. When $s > 0$, we have $r = q(s + 1)$ static factors but $q = 1$ dynamic factor. The single common factor $f_t$ and the idiosyncratic errors evolve according to

$$(1 - \rho_f L)f_t = u_t, \quad u_t \sim N(0, 1)$$

$$(1 - \rho_e L)e_{it} = \epsilon_{it}, \quad E(\epsilon_t \epsilon_t') = \Omega.$$

The error variance matrix $\Omega$ is an identity matrix of order $N$ when the errors are cross-sectionally uncorrelated. Otherwise, it is a positive definite correlation matrix such that a fraction $Nc$ of the $N^2$ elements of $\Omega$ are nonzero. The parameters of the simulations are

- $(N,T) = (20,50), (50,100), (100,50), (100,100), (50,200), (100,200)$;
- $s = 0, 1$;
- $\rho_f = 0, 0.4, 0.8$;
- $\rho_e = 0, U(0, 0.5)$, or $U(0.4, 0.8)$;
- $R^2_L = 0.1, 0.35, 0.6$;
- $Nc = 0, 0.15, 0.3$.

For a given $s$, there are 81 configurations for each sample size, giving a total of 486 configurations. We consider 1000 replications for each configuration. In each run, we keep track of the eigenvalues of $\Sigma_{xx} = x'x/NT$ and of $\Omega = \epsilon'\epsilon/(NT)$. Let $\text{eig}_r$ be average of the $r$th largest eigenvalue of the matrix $\Sigma_{xx}$ over 1000 replications, and let $\text{eig}_1$ be the corresponding largest eigenvalue of $\Omega$. We will use $\text{EIG}_{A,B}(a,b)$ to denote the ratio of the $a$th largest eigenvalue of the covariance matrix of $A$ to the $b$th largest eigenvalue of the covariance matrix of $B$. We also keep track of FIT, which is the $R^2$ from a regression of $\tilde{F}_t$ on $F_t$ and a
Table 8.1 Dependent variable: FIT.

<table>
<thead>
<tr>
<th>Regressor</th>
<th>( t_{\hat{\beta}} )</th>
<th>( t_{\beta} )</th>
<th>( t_{\hat{\beta}} )</th>
<th>( t_{\beta} )</th>
<th>( t_{\hat{\beta}} )</th>
<th>( t_{\beta} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>0.974</td>
<td>21.244</td>
<td>1.000</td>
<td>66.855</td>
<td>0.958</td>
<td>15.048</td>
</tr>
<tr>
<td>( C_{NT}^{-1} )</td>
<td>-4.086</td>
<td>-1.819</td>
<td>-3.030</td>
<td>-4.250</td>
<td>-3.196</td>
<td>-1.184</td>
</tr>
<tr>
<td>( EIG_{x,x}(k+1,k) )</td>
<td>-0.116</td>
<td>-1.700</td>
<td>0.286</td>
<td>7.681</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( EIG_{e,x}(1,1) )</td>
<td>0.025</td>
<td>7.906</td>
<td>-0.019</td>
<td>-5.231</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( EIG_{e,x}(k+1,k)^2 )</td>
<td>-0.952</td>
<td>-6.564</td>
<td>1.007</td>
<td>-19.892</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( EIG_{e,x}(1,1)^2 )</td>
<td>-0.003</td>
<td>-10.694</td>
<td>0.000</td>
<td>-0.214</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \bar{R}^2 )</td>
<td>0.246</td>
<td>0.927</td>
<td>0.121</td>
<td>0.8454</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

constant. When \( \hat{F}_t \) is two-dimensional (as is the case when \( s = 1 \)), we regress each of the \( \tilde{F}_t \) on \( F_t \) and then average the two \( R^2 \).

To evaluate the precision of the factor estimates and to compactly summarize the results, we perform a response surface analysis by regressing FIT on the sample size, and other potential determinants of FIT. Let \( C_{NT} = \min[\sqrt{N}, \sqrt{T}] \) and \( \bar{R}^2 \) be importance of the common component. The estimates and the robust \( t \) statistics are reported in Table 8.1.

Several features should be noted. First, the precision falls with the number of factors. Second, while FIT improves with \( \min[\sqrt{N}, \sqrt{T}] \), the effect is not statistically significant. The variation of FIT is primarily explained by \( EIG_{x,x}(r + 1, r) \), the ratio of the \( r + 1 \)th to the \( r \)th eigenvalue of \( \Sigma_x \). Under the assumption that the factors are strong, this ratio should tend to zero in the population as \( N \) increases. However, if the factors are weak, such as when the idiosyncratic errors are large or correlated, or when the loadings are weak, this ratio may be non-negligible. As suggested by Onatski (2006a) and Heaton and Solo (2006), the larger this ratio, the less precise will be the factor estimates. This ratio, along with the corresponding quadratic term, explain close to 80% of the variation in FIT. Additional explanatory power is provided by \( EIG_{e,x}(1,1) \), which also measures the size of the idiosyncratic component. Thus the relative importance of the idiosyncratic component is indeed an important determinant of the precision of FIT.

The above results suggest that the principal component estimator can indeed be sensitive to the possibility of weak loadings, large error
variances, and cross-section correlation all culminating in a large $\text{eig}_{r+1}$ that is no longer small relative to $\text{eig}_r$ as theory assumes. Two questions arise. Is this a situation of empirical relevance, and are there alternatives?

As for the first question, the availability of a lot of data should never be taken to mean the availability of a lot of data informative about the factor structure. Economic reasoning can usually help screen out data that are overly similar to other series already included. For example, it should not come as a surprise that the idiosyncratic error of two series, one being a finer disaggregation of another, are correlated. Boivin and Ng (2006) found that the errors IPC and IPCD (industrial production of consumer goods and of durable goods) are strongly correlated, and a case can be made to include only one series. Pre-screening of the importance of the common component in each series can also go a long way in removing the uninformative variables from the panel of data used in subsequent estimation. Admittedly, pretesting has its consequences. But improving the precision of the factor estimates may well justify the cost.

What are the alternatives to the principal components estimator? For static factor models, the maximum likelihood estimator via EM (expectation and maximization) algorithm has been considered by many authors, for example, Rubin and Thayer (1982), Lehmann and Modest (1988), and Ghahramani and Hinton (1996). Dynamic factor models fit into the state space framework. Under large $N$, a dynamic factor model is that of a state space model with a small number of state variables but a large number of measurement equations. Naturally, the state space method (especially the EM algorithm combined with the Kalman smoother) can be used to estimate the model, see Watson and Engle (1983), Quah and Sargent (1992), and Shumway and Stoffer (2000). A subspace algorithm is considered by Kapetanios and Marcellino (2006b). Other estimators that have been considered involve re-weighting the data matrix prior to extracting the principal components estimator as discussed earlier. These methods are ad-hoc. Recently, Doz et al. (2007) considered the quasi-maximum likelihood estimator as an alternative. The estimator allows for heterogeneous errors and serially correlated factors. Although similar in spirit to the estimator
considered in Anderson (1984), the theory that Doz et al. (2007) developed assumes that $N$ and $T$ are both large. Their QMLE estimator is based on the Kalman filter and reduces to an iteratively reweighted principal components estimator when the data are assumed to be iid. The weighting serves to downweigh the noisy series and goes some way in resolving the weak loading/large error variance problem. The advantage of the estimator is that it allows restrictions to be imposed on the factor structure, whereas the principal components estimator is best thought of as an unrestricted reduced form type estimator. The limitation, however, is that QMLE is based on an approximation to the “approximate factor model,” where the approximate model is in fact a strict factor model that assumes the errors are cross-sectionally uncorrelated. Therefore, information in the correlated errors is never taken into account, just like the unweighted estimator. Although their estimator appears to work well for a low degree of cross-section correlation, it is unclear if the properties hold when the data with a high noise to signal ratio (in the sense of $\frac{\text{eig}_r}{\text{eig}_{r+1}}$ not tending to zero) are being analyzed. It therefore appears that no frequentist method has yet provided a satisfactory solution to the problem of correlated errors.

Can Bayesian methods solve the problem? At the moment, Bayesian analysis of dynamic factor models still do not allow for cross-sectionally correlated errors. In preparing the monte carlo study, we also obtained Bayesian factor estimates using the method discussed in Otrok and Whiteman (1998), as well as the one described in Kim and Nelson (2000). The Bayesian methods give posterior means very similar results to the principal component estimates, but are tremendously more time consuming to compute with little to no gain in precision. Thus, while the Bayesian methods can be alternatives to the principal components approach, they cannot (so far) be justified on precision grounds. Coming up with alternatives to the principal components estimator remains very much an important area for research.
This survey has surveyed some of the theoretical results relating to the use of principal components as estimated factors in empirical work. The estimator is simple to compute and can estimate well the factor space when conditions required for consistent estimation are satisfied. In practice, the data may be at odds with some of these conditions. New simulation results provided in this survey show that the factor estimates can be severely compromised when the data have a weak factor structure in the sense that $\frac{\text{eig}_r\text{eig}_{r+1}}{\text{eig}_r\text{eig}_{r+1}}$ does not tend to zero. This problem may not arise frequently in practice, but it also cannot be ruled out. A user can alleviate the problem to some extent by using only data that are truly informative about the factor structure. However, how to deal with cross-sectionally correlated errors, which is a genuine feature of an approximate factor model, remains an unresolved issue. Thus, although much work has been accomplished in this research, much more remains to be done.


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