Problem 4.40 Consider a segment of a sample path beginning and ending in state $i$, with no visit to $i$ in between, i.e., the vector $(i, j_1, j_2, j_3, \ldots, j_{n-1}, j_n = i)$, where $j_k \neq i$ for the non-end states $j_k$. Going forward in time, the probability of this segment is

$$\pi_i P_{i,j_1} P_{j_1,j_2} P_{j_2,j_3} \cdots P_{j_{n-1},i}.$$

The probability, say $p$, of the reversed sequence $(i, j_{n-1}, j_{n-2}, j_{n-3}, \ldots, j_1, j_0 = i)$ under the reverse DTMC with transition matrix $\overrightarrow{P} = \pi_j P_{j,i}$ is

$$p = \pi_j \overrightarrow{P}_{i,j_{n-1}} \overrightarrow{P}_{j_{n-1},j_{n-2}} \overrightarrow{P}_{j_{n-2},j_{n-3}} \cdots \overrightarrow{P}_{j_1,i}.$$

However, successively substituting in the reverse-chain transition probabilities, we get

$$p = \pi_j \overrightarrow{P}_{i,j_{n-1}} \overrightarrow{P}_{j_{n-1},j_{n-2}} \overrightarrow{P}_{j_{n-2},j_{n-3}} \cdots \overrightarrow{P}_{j_1,i}.$$

Problem 4.41 (a) The reverse time chain has transition matrix $\overrightarrow{P} = \pi_j P_{j,i}$

To find it, we need to first find the stationary vector $\pi$. By symmetry (or by noting that the chain is doubly stochastic), $\pi_j = 1/n$, $j = 1, \cdots, n$. Hence,

$$P^*_j = \pi_j P_{j,i}/\pi_i = P_{j,i} = \begin{cases} p & \text{if } j = i - 1 \\ 1 - p & \text{if } j = i + 1 \end{cases}$$

(b) In general, the DTMC is not time reversible. It is in the special case $p = 1/2$. Otherwise, the probabilities of clockwise and counterclockwise motion are reversed.
Problem 4.42 Imagine that there are edges between each of the pair of nodes $i$ and $i + 1$, $i = 0, \cdots, n - 1$, and let the weight on edge $(i, i + 1)$ be $w_i$, where

\[
\begin{align*}
    w_0 &= 1 \\
    w_i &= \prod_{j=1}^{i} \frac{p_j}{q_j}, \quad i \geq 1
\end{align*}
\]

where $q_j = 1 - p_j$. As a check, note that with these weights

\[
P_{i,i+1} = \frac{w_i}{w_{i-1} + w_i} = \frac{p_i/q_i}{1 + p_i/q_i} = p_i, \quad 0 < i < n.
\]

Since the sum of the weights on edges out of node $i$ is $w_{i-1} + w_i$, $i = 1, \cdots, n - 1$, it follows that

\[
\begin{align*}
    \pi_i &= c \left[ \prod_{j=1}^{i-1} \frac{p_j}{q_j} + \prod_{j=1}^{i} \frac{p_j}{q_j} \right] = c \frac{i-1}{q_i} \prod_{j=1}^{i-1} \frac{p_j}{q_j}, \quad 0 < i < n \\
    \pi_n &= c \prod_{j=1}^{n-1} \frac{p_j}{q_j}
\end{align*}
\]

where $c$ is chosen to make $\sum_{j=0}^{n} \pi_j = 1$.

Problem 4.46 (a) Yes, it is a Markov chain. It suffices to construct the transition matrix and verify that the process has the Markov property. Let $P^*$ be the new transition matrix. Then we have, for $0 \leq i \leq N$ and $0 \leq j \leq N$, \[
P_{i,j}^* = P_{i,j} + \sum_{k=N+1}^{\infty} P_{i,k} B_{k,j}^{(N)},
\]

where $B_{k,j}^{(N)}$ is the probability of absorption into the absorbing state $j$ in the absorbing Markov chain, where the states $N + 1, N + 2, \ldots$ are the transient states, while the state $1, 2, \ldots N$ are the $N$ absorbing states. In other words, $B_{k,j}^{(N)}$ is the probability that the next state with index in the set $\{1, 2, \ldots, N\}$ visited by the Markov chain, starting with $k > N$ is in fact $j$. It is easy to see that the markov property is still present.

(b) The proportion of time in $j$ is $\pi_j / \sum_{i=1}^{N} \pi_i$.

(c) Let $\pi_i(N)$ be the steady-state probabilities for the chain, only counting to visits among the states in the subset $\{1, 2, \ldots, N\}$. (This chain is necessarily positive recurrent.) By renewal theory, \[
\pi_i(N) = (E[\text{Number of } Y \text{ - transitions between } Y \text{ - visits to } i])^{-1}
\]
and

\[ \pi_j(N) = \frac{E[\text{No. } Y\text{-transitions to } j \text{ between } Y \text{ visits to } i]}{E[\text{No. } Y\text{-transitions to } i \text{ between } Y \text{ visits to } i]} = \frac{E[\text{No. } X\text{-transitions to } j \text{ between } X \text{ visits to } i]}{1/\pi_i(N)} \]

(d) For the symmetric random walk, the new MC is doubly stochastic, so \( \pi_i(N) = 1/(N+1) \) for all \( i \). By part (c), we have the conclusion.

(e) It suffices to show that

\[ \pi_i(N)P_{i,j}^* = \pi_j(N)P_{j,i}^* \]

for all \( i \) and \( j \) with \( i \leq N \) and \( j \leq N \). However, by above,

\[ \pi_i(N)P_{i,j}^* = \pi_i(N)P_{i,j} + \pi_i(N) \sum_{k=N+1}^{\infty} P_{i,k} B_{k,j}^{(N)} \]

and

\[ \pi_j(N)P_{j,i}^* = \pi_j(N)P_{j,i} + \pi_j(N) \sum_{k=N+1}^{\infty} P_{j,k} B_{k,i}^{(N)} \]

The two terms on the right are equal in these two displays. First, by the original reversibility, we have

\[ \pi_i(N)P_{i,j} = \pi_j(N)P_{j,i}. \]

Second, by Theorem 4.7.2, we have

\[ \pi_j(N) \sum_{k=N+1}^{\infty} P_{j,k} B_{k,i}^{(N)} = \pi_i(N) \sum_{k=N+1}^{\infty} P_{i,k} B_{k,j}^{(N)}. \]

We see that by expanding into the individual paths, and seeing that there is a reverse path.

**Problem 4.47** Intuitively, in steady state each ball is equally likely to be in any of the urns and the positions of the balls are independent. Hence it seems intuitive that

\[ \pi(n) = \frac{M!}{n_1! \cdots n_m!} \left( \frac{1}{m} \right)^M. \]

To check the above and simultaneously establish time reversibility let

\[ n' = (n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_{j-1}, n_j + 1, n_{j+1}, \ldots, n_m) \]
and note that
\[
\pi(n)P(n, n') = M! \cdot \frac{1}{n_1! \cdots n_m!} \cdot \frac{1}{M} \cdot \frac{1}{m} \cdot \frac{1}{M} \cdot \frac{1}{m - 1}
= \pi(n')P(n', n).
\]

**Problem 4.48 (a)** Each transition into \(i\) begins a new cycle. A reward of 1 is earned if state visited from \(i\) is \(j\). Hence average reward per unit time is \(P_{ij}/\mu_{ii}\).

(b) Follows from (a) since \(1/\mu_{jj}\) is the rate at which transitions into \(j\) occur.

(c) Suppose a reward rate of 1 per unit time when in \(i\) and heading for \(j\). New cycle whenever enter \(i\). Hence, average reward per unit time is \(P_{ij}\eta_{ij}/\mu_{ii}\).

(d) Consider (c) but now only give a reward at rate 1 per unit time when the transition time from \(i\) to \(j\) is within \(x\) time units. Average reward is
\[
\frac{E[\text{Reward per cycle}]}{E[\text{Time of cycle}]} = \frac{P_{ij}E[\min(X_{ij}, x)]}{\mu_{ii}} = \frac{P_{ij} \int_0^x \bar{F}_{ij}(y)dy}{\mu_{ii}} = \frac{P_{ij}\eta_{ij}F_{ij}(x)}{\mu_{ii}}
\]
where \(X_{ij} \sim F_{ij}\).

**Problem 4.49**
\[
\lim_{t \to \infty} P(S(t) = j|X(t) = i) = \frac{\lim_{t \to \infty} P(S(t) = j, X(t) = i)}{P(X(t) = i)}
= \frac{P_{ij} \int_0^\infty \bar{F}_{ij}(y)dy/\mu_{ii}}{P_i} \text{ by Theorem 4.8.4}
= \frac{P_{ij}\eta_{ij}}{\mu_{ii}}
\]

**Problem 4.50** \(\pi = (6, 3, 5)/14, \mu_1 = 25, \mu_2 = 80/3,\) and \(\mu_3 = 30\).

(a)
\[
P_1 = \frac{6 \times 25}{6 \times 25 + 3 \times \frac{80}{3} + 5 \times 30} = \frac{15}{38}

P_2 = \frac{3 \times \frac{80}{3}}{6 \times 25 + 3 \times \frac{80}{3} + 5 \times 30} = \frac{8}{38}

P_3 = \frac{5 \times 30}{6 \times 25 + 3 \times \frac{80}{3} + 5 \times 30} = \frac{15}{38}
\]
(b) \[ P(\text{heading for 2}) = P_1 \frac{P_{12} t_{12}}{\mu_1} = \frac{15}{38} \times \frac{10}{25} = \frac{3}{19} \]

(e) \[ \text{fraction of time from 2 to 3} = P_2 \frac{P_{23} t_{23}}{\mu_2} = \frac{8}{38} \times \frac{60}{80} = \frac{3}{19} \]