Problem 5.3 (a) Let $N(t)$ denote the number of transitions be $t$. It is easy to show in this case that

$$P(N(t) \geq n) \leq \sum_{j=n}^{\infty} e^{-Mt} \frac{(Mt)^j}{j!}$$

and thus $P(N(t) < \infty) = 1$.

(b) Let $X_{n+1}$ denote the time between the $n$-th and $(n+1)$-st transition and let $J_n$ denote the $n$-th state visited. Also if we let

$$N(t) \triangleq \sup\{n : X_1 + \cdots + X_n \leq t\}$$

then $N(t)$ denotes the number of transitions by $t$. Now let $j$ be the first recurrent state that is reached and suppose it was reached at the $n_0$-th transition ($n_0$ must be finite by assumption). Let $n_1, n_2, \cdots$ be the successive integers $n$ at which $J_n = j$. (Such integers exist since $j$ is recurrent.) Set $T_0 = X_1 + \cdots + X_{n_0}$, and

$$T_k \triangleq X_{n_{k-1}+1} + \cdots + X_{n_k}.$$

In other words, $T_k$ denote the amount of time between the $k$-th and $(k+1)$-th visit to $j$. Therefore it follows that $\{T_k, k \geq 1\}$ forms a renewal process, and so $\sum_{k=1}^{\infty} T_k = \infty$ with probability 1. Since

$$\sum_{n=1}^{\infty} X_n = \sum_{k=0}^{\infty} T_k$$

it follows that $\sum_{n=1}^{\infty} X_n = \infty$.

Problem 5.4 Let $T_i$ denote the time to go from $i$ to $i+1$, $i \geq 0$. Then $\sum_{i=0}^{N-1} T_i$ is the time to go from 0 to $N$. Now $T_i$ is exponential with rate $\lambda_i$ and the $T_i$ are independent. Hence

$$E\left[e^{s \sum_{i=0}^{N-1} T_i}\right] = \prod_{i=0}^{N-1} \frac{\lambda_i}{\lambda_i - s}.$$

We may use it to compute the mean and variance or we can do directly and mean $= 1/\lambda_0 + \cdots + 1/\lambda_{N-1}$, variance $= 1/\lambda_0^2 + \cdots + 1/\lambda_{N-1}^2$. 

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Problem 5.9

\[ P_{ij}(t+s) = \sum_k P(X(t+s) = j \mid X_0 = i, X(t) = k) P(X(t) = k \mid X_0 = i) \]

\[ = \sum_k P_{kj}(s) P_{ik}(t) . \]

Problem 5.10 (a)

\[ \lim_{t \to 0} \frac{1 - P(t)}{t} = v_0 . \]

(b) The first inequality follows from exercise 5.9 and

\[ P(t + s) = P(X(t + s) = 0 \mid X(0) = 0, X(s) = 0) P(s) + P(X(t + s) = 0 \mid X(0) = 0, X(s) \neq 0)(1 - P(s)) \]

\[ \leq P(t) P(s) + 1 - P(s) . \]

(c) From (b)

\[ P(s)P(t - s) \leq P(t) \leq P(s)P(t - s) + 1 - P(t - s) \]

or

\[ P(s) + P(t - s) - 1 \leq P(t) \leq P(s) + 1 - P(t - s) \]

where the left hand inequality follows from

\[ P(s)(1 - P(t - s)) \leq 1 - P(t - s) . \]

\[ \lim_{s \to t} P(s - t) = 1 \text{ implies the continuity of } P . \]

Problem 5.13

\[ \prod_{j=i}^{i+k-1} \frac{\lambda_j}{\lambda_j + \mu_j} . \]

Problem 5.15 (a) Birth and death process.

(b) \( \lambda_n = n \lambda + \theta, \mu_n = n \mu. \)

(c) Set \( M(t) = \mathbb{E}[X(t) \mid X(0) = i] . \) Then

\[ \mathbb{E}[X(t+h) \mid X(t)] = X(t) + (\lambda X(t) + \theta)h - \mu X(t)h + o(h) \]

and so \( M(t+h) = M(t) + (\lambda - \mu)M(t)h + \theta h + o(h) . \) Therefore, \( M'(t) = (\lambda - \mu)M(t) + \theta \)

or \( e^{-(\lambda - \mu) t}[M'(t) - (\lambda - \mu)M(t)] = \theta e^{-(\lambda - \mu) t} . \) Integrating both sides gives

\[ e^{-(\lambda - \mu) t} M(t) = -\frac{\theta}{\lambda - \mu} e^{-(\lambda - \mu) t} + C . \]
or

\[ M(t) = C e^{-(\lambda-\mu)t} - \frac{\theta}{\lambda - \mu} . \]

As \( M(0) = i \) we obtain

\[ M(t) = \frac{\theta}{\lambda - \mu} \left( e^{-(\lambda-\mu)t} - 1 \right) + i e^{-(\lambda-\mu)t} . \]

**Problem 5.21** With the number of customers in the shop as the state, we get a birth and death process with \( \lambda_0 = \lambda_1 = 3, \mu_1 = \mu_2 = 4 \). Therefore \( P_1 = \frac{3}{4} P_0, P_2 = \frac{3}{4} P_1 = \left( \frac{3}{4} \right)^2 P_0 \). And since \( P_0 + P_1 + P_2 = 1 \), we get \( P_0 = 16/37 \).

(a)

\[ P_1 + 2P_2 = \left[ \frac{3}{4} + 2 \left( \frac{3}{4} \right)^2 \right] P_0 = \frac{30}{37} \]

(b) The proportion of customers that enter the shop is

\[ \frac{\lambda(1 - P_2)}{\lambda} = 1 - P_2 = 1 - \frac{9}{37} = \frac{28}{37} . \]

(c) \( \mu = 8 \), and so \( P_0 = \frac{64}{97} \). So the proportion of customers who now enter the shop is

\[ 1 - P_2 = 1 - \left( \frac{3}{8} \right)^2 \frac{64}{97} = \frac{88}{97} . \]

The rate of added customers is therefore

\[ \lambda \frac{88}{97} - \lambda \frac{28}{37} \approx 0.45 . \]

The business he does would improve by 0.45 customers per hour.