Problem 3.19

\[ P\{S_{N(t)} \leq s\} = \sum_{n=0}^{\infty} P\{S_n \leq s, S_{n+1} > t\} = \bar{G}(t) + \sum_{n=1}^{\infty} P\{S_n \leq s, S_{n+1} > t\} \]

\[ = \bar{G}(t) + \sum_{n=1}^{\infty} \int_{0}^{s} P\{S_n \leq s, S_{n+1} > t | S_n = y\} d(G*F_{n-1})(s) \]

\[ = \bar{G}(t) + \int_{0}^{s} F(t-y) dm_G(y) \]

Problem 3.20

(a) Say that a renewal occurs when that pattern appears. By Blackwell’s theorem for renewal processes we obtain

\[ E[\text{time}] = \frac{1}{(1/2)^7} = 2^7 \]

(b) By Blackwell’s theorem

\[ E[\text{time between HHTT renewals}] = E[\text{time between HTHT renewals}] = 16 \]

But the HHTT renewal process is an ordinary one and so the mean time until HHTT occurs is 16 whereas the HTHT process is a delayed renewal process and so the mean time until HTHT occurs is greater than 16.

Problem 3.29

Let \( L \) denote the lifetime of a car with distribution function \( F(\cdot) \).

(a) Under the policy of replacements at \( A \),

\[ \text{Cost of cycle} = \begin{cases} C_1 + C_2 & \text{if } L \leq A \\ C_1 - R(A) & \text{if } L > A \end{cases} \]

and

\[ \text{Length of cycle} = \begin{cases} L & \text{if } L \leq A \\ A & \text{if } L > A \end{cases} \]

Then

\[ \frac{E[\text{Cost}]}{E[\text{Time}]} = \frac{C_1 + C_2 F(A) - R(A) \tilde{F}(A)}{\int_{0}^{A} x dF(x) + AF(A)} \]

(Validate the final formula by yourself. If you are confusing, utilize the indicator to combine the \textit{if}-clauses into one function as I said in the first recitation.)
(b) Condition on the life of the initial car.

\[ E[\text{Length of cycle}] = \int_0^\infty E[\text{Length}|L = x]dF(x) \]
\[ = \int_0^A x dF(x) + \int_A^\infty (A + E[\text{Length}])dF(x) \]
\[ = \int_0^A x dF(x) + (A + E[\text{Length}])\bar{F}(A) \]
\[ = \frac{\int_0^A x dF(x) + A\bar{F}(A)}{F(A)} \]

and similarly

\[ E[\text{Cost of cycle}] = \int_0^\infty E[\text{Cost}|L = x]dF(x) \]
\[ = \int_0^A (C_1 + C_2)dF(x) + (C_1 - R(A) + E[\text{Cost}])\bar{F}(A) \]
\[ = \frac{C_1 + C_2F(A) - R(A)\bar{F}(A)}{F(A)} \]

Then

\[ \frac{E[\text{Cost}]}{E[\text{Time}]} = \text{same as in (a)}. \]

**Problem 3.31**

Let \( \mu_i \) and \( \nu_i \) denote the means of \( F_i \) and \( G_i \), respectively for \( i = 1, 2, 3, 4 \). Then,

\[ \lim P\{\text{i is working at t}\} = \frac{\mu_i}{\mu_i + \nu_i}, i = 1, 2, 3, 4 \]

Now, if \( p_i \) is the probability that component \( i \) is working, then

\[ P\{\text{system works}\} = (p_1 + p_2 - p_1p_2)(p_3 + p_4 - p_3p_4) \]

Hence \( \lim P\{\text{system works at t}\} \) is equal to the preceding expression with \( p_i = \mu_i/(\mu_i + \nu_i), i = 1, 2, 3, 4 \)

**Problem 3.32**

(a) \( 1 - P_0 = \text{average number in service} = \lambda\mu \)

(b) By alternating renewal processes

\[ P_0 = \text{proportion of time empty} = \frac{E[I]}{E[I] + E[B]} \]

where \( I \) is an idle period and \( B \) a busy period. But clearly \( I \) is exponential with rate \( \lambda \) and so
\[ 1 - \lambda \mu = \frac{1/\lambda}{1/\lambda + E[B]} \text{ or } E[B] = \frac{\mu}{1 - \lambda \mu} \]

(c) Let \( C \) denote the number of customers served in a busy period \( B \) and let \( S_i \) denote the service time of the \( i \)-th customer, \( i \geq 1 \). Then

\[ B = \sum_{i=1}^{C} S_i \]

and by Wald’s equation

\[ E[C] = \frac{E[B]}{E[S]} = \frac{1}{1 - \lambda \mu} \]