Problem 4.40 Consider a segment of a sample path beginning and ending in state \( i \), with no visit to \( i \) in between, i.e., the vector \((i, j_1, j_2, j_3, \ldots, j_{n-1}, j_n = i)\), where \( j_k \neq i \) for the non-end states \( j_k \). Going forward in time, the probability of this segment is 
\[
\pi_i P_{i,j_1} P_{j_1,j_2} P_{j_2,j_3} \cdots P_{j_{n-1},i}.
\]
The probability, say \( p \), of the reversed sequence \((i, j_{n-1}, j_{n-2}, j_{n-3}, \ldots, j_1, j_0 = i)\) under the reverse DTMC with transition matrix 
\[
\overrightarrow{P}_{i,j} \equiv \frac{\pi_j P_{j,i}}{\pi_i}
\]
is 
\[
p = \pi_i \overrightarrow{P}_{i,j_{n-1}} \overrightarrow{P}_{j_{n-1},j_{n-2}} \overrightarrow{P}_{j_{n-2},j_{n-3}} \cdots \overrightarrow{P}_{j_1,i}.
\]
However, successively substituting in the reverse-chain transition probabilities, we get 
\[
p = \frac{\pi_i \pi_{j_{n-1}} P_{j_{n-1},i}}{\pi_i} \overrightarrow{P}_{j_{n-1},j_{n-2}} \overrightarrow{P}_{j_{n-2},j_{n-3}} \cdots \overrightarrow{P}_{j_1,i}
\]
\[
= \pi_i \pi_{j_{n-1}} \overrightarrow{P}_{j_{n-1},j_{n-2}} \overrightarrow{P}_{j_{n-2},j_{n-3}} \cdots \overrightarrow{P}_{j_1,i}
\]
\[
= \pi_i \pi_{j_{n-1}} \pi_{j_{n-2}} P_{j_{n-2},j_{n-1}} \overrightarrow{P}_{j_{n-1},j_{n-2}} \overrightarrow{P}_{j_{n-2},j_{n-3}} \cdots \overrightarrow{P}_{j_1,i}
\]
\[
= \pi_i \pi_{j_{n-1}} \pi_{j_{n-2}} \pi_{j_{n-3}} P_{j_{n-3},j_{n-2}} \cdots \overrightarrow{P}_{j_1,i}
\]
\[
= \pi_i \pi_{j_{n-1}} \pi_{j_{n-2}} \pi_{j_{n-3}} \cdots \overrightarrow{P}_{j_1,i}
\]

Problem 4.41 (a) The reverse time chain has transition matrix 
\[
\overrightarrow{P}_{i,j} \equiv \frac{\pi_j P_{j,i}}{\pi_i}
\]
To find it, we need to first find the stationary vector \( \pi \). By symmetry (or by noting that the chain is doubly stochastic), \( \pi_j = 1/n, \ j = 1, \cdots, n \). Hence, 
\[
P_{ij}^* = \frac{\pi_j P_{j,i}}{\pi_i} = \begin{cases} p & \text{if } j = i - 1 \\ 1 - p & \text{if } j = i + 1 \end{cases}
\]

(b) In general, the DTMC is not time reversible. It is in the special case \( p = 1/2 \). Otherwise, the probabilities of clockwise and counterclockwise motion are reversed.
Problem 4.42 Imagine that there are edges between each of the pair of nodes \( i \) and \( i + 1 \), \( i = 0, \ldots, n - 1 \), and let the weight on edge \((i, i + 1)\) be \( w_i \), where
\[
w_0 = 1 \quad w_i = \prod_{j=1}^{i} \frac{p_j}{q_j}, \quad i \geq 1
\]
where \( q_j = 1 - p_j \). As a check, note that with these weights
\[
P_{i,i+1} = \frac{w_i}{w_{i-1} + w_i} = \frac{p_i/q_i}{1 + p_i/q_i} = p_i, \quad 0 < i < n.
\]
Since the sum of the weights on edges out of node \( i \) is \( w_{i-1} + w_i \), \( i = 1, \ldots, n - 1 \), it follows that
\[
\begin{align*}
\pi_i &= c \left[ \prod_{j=1}^{i-1} \frac{p_j}{q_j} + \prod_{j=1}^{i} \frac{p_j}{q_j} \right] = c \frac{\pi_0}{\prod_{j=1}^{i} \frac{p_j}{q_j}}, \quad 0 < i < n \\
\pi_n &= c \prod_{j=1}^{n-1} \frac{p_j}{q_j}
\end{align*}
\]
where \( c \) is chosen to make \( \sum_{j=0}^{n} \pi_j = 1 \).

Problem 4.46 (a) Yes, it is a Markov chain. It suffices to construct the transition matrix and verify that the process has the Markov property. Let \( P^* \) be the new transition matrix. Then we have, for \( 0 \leq i \leq N \) and \( 0 \leq j \leq N \),
\[
P^*_{i,j} = P_{i,j} + \sum_{k=N+1}^{\infty} P_{i,k} B^{(N)}_{k,j},
\]
where \( B^{(N)}_{k,j} \) is the probability of absorption into the absorbing state \( j \) in the absorbing Markov chain, where the states \( N + 1, N + 2, \ldots \) are the transient states, while the state \( 1, 2, \ldots N \) are the \( N \) absorbing states. In other words, \( B^{(N)}_{k,j} \) is the probability that the next state with index in the set \( \{1, 2, \ldots, N\} \) visited by the Markov chain, starting with \( k > N \) is in fact \( j \). It is easy to see that the markov property is still present.

(b) The proportion of time in \( j \) is \( \pi_j / \sum_{i=1}^{N} \pi_i \).

(c) Let \( \pi_i(N) \) be the steady-state probabilities for the chain, only counting to visits among the states in the subset \( \{1, 2, \ldots, N\} \). (This chain is necessarily positive recurrent.) By renewal theory,
\[
\pi_i(N) = (E[\text{Number of } Y - \text{ transitions between } Y - \text{ visits to } i])^{-1}
\]
and
\[ \pi_j(N) = \frac{E[\text{No. Y-transitions to } j \text{ between } Y \text{ visits to } i]}{E[\text{No. Y-transitions to } i \text{ between } Y \text{ visits to } i]} = \frac{E[\text{No. X-transitions to } j \text{ between } X \text{ visits to } i]}{1/\pi_i(N)} \]

(d) For the symmetric random walk, the new MC is doubly stochastic, so \( \pi_i(N) = 1/(N+1) \) for all \( i \). By part (c), we have the conclusion.

(e) It suffices to show that
\[ \pi_i(N)P^*_i,j = \pi_j(N)P^*_j,i \]
for all \( i \) and \( j \) with \( i \leq N \) and \( j \leq N \). However, by above,
\[ \pi_i(N)P^*_i,j = \pi_i(N)P_{i,j} + \pi_i(N) \sum_{k=N+1}^{\infty} P_{i,k}B_{k,j}^{(N)} , \]
and
\[ \pi_j(N)P^*_j,i = \pi_j(N)P_{j,i} + \pi_j(N) \sum_{k=N+1}^{\infty} P_{j,k}B_{k,i}^{(N)} , \]
The two terms on the right are equal in these two displays. First, by the original reversibility, we have
\[ \pi_i(N)P_{i,j} = \pi_j(N)P_{j,i} . \]
Second, by Theorem 4.7.2, we have
\[ \pi_j(N) \sum_{k=N+1}^{\infty} P_{j,k}B_{k,i}^{(N)} = \pi_i(N) \sum_{k=N+1}^{\infty} P_{i,k}B_{k,j}^{(N)} . \]
We see that by expanding into the individual paths, and seeing that there is a reverse path.

**Problem 4.47** Intuitively, in steady state each ball is equally likely to be in any of the urns and the positions of the balls are independent. Hence it seems intuitive that
\[ \pi(n) = \frac{M!}{n_1! \cdots n_m!} \left( \frac{1}{m} \right)^M . \]
To check the above and simultaneously establish time reversibility let
\[ n' = (n_1, \cdots, n_{i-1}, n_i - 1, n_{i+1}, \cdots, n_{j-1}, n_j + 1, n_{j+1}, \cdots, n_m) \]
and note that

\[
\pi(n) P(n, n') = \frac{M!}{n_1! \cdots n_m!} \left( \frac{1}{m} \right)^M \frac{n_i}{M} \frac{1}{m - 1} \\
= \frac{M!}{n_1! \cdots (n_i - 1)! \cdots (n_j + 1)! \cdots n_m!} \left( \frac{1}{m} \right)^M \frac{n_j + 1}{M} \frac{1}{m - 1} \\
= \pi(n') P(n', n).
\]

**Problem 4.48** (a) Each transition into \(i\) begins a new cycle. A reward of 1 is earned if state visited from \(i\) is \(j\). Hence average reward per unit time is \(P_{ij}/\mu_{ii}\).

(b) Follows from (a) since \(1/\mu_{jj}\) is the rate at which transitions into \(j\) occur.

(c) Suppose a reward rate of 1 per unit time when in \(i\) and heading for \(j\). New cycle whenever enter \(i\). Hence, average reward per unit time is \(P_{ij}\eta_{ij}/\mu_{ii}\).

(d) Consider (c) but now only give a reward at rate 1 per unit time when the transition time from \(i\) to \(j\) is within \(x\) time units. Average reward is

\[
\frac{E[\text{Reward per cycle}]}{E[\text{Time of cycle}]} = \frac{P_{ij}E[\min(X_{ij}, x)]}{\mu_{ii}} \\
= \frac{P_{ij}\int_0^x \bar{F}_{ij}(y)dy}{\mu_{ii}} \\
= \frac{P_{ij}\eta_{ij}F_e(x)}{\mu_{ii}}
\]

where \(X_{ij} \sim F_{ij}\).

**Problem 4.49**

\[
\lim_{t \to \infty} P(S(t) = j | X(t) = i) = \frac{\lim_{t \to \infty} P(S(t) = j, X(t) = i)}{P(X(t) = i)} \\
= \frac{P_{ij} \int_0^\infty \bar{F}_{ij}(y)dy/\mu_{ii}}{P_i} \quad \text{by Theorem 4.8.4} \\
= \frac{P_{ij} \eta_{ij}}{\mu_i}
\]

**Problem 4.50** \(\pi = (6, 3, 5)/14, \mu_1 = 25, \mu_2 = 80/3, \text{ and } \mu_3 = 30\).

(a)

\[
\begin{align*}
P_1 &= \frac{6 \times 25}{6 \times 25 + 3 \times \frac{80}{3} + 5 \times 30} = \frac{15}{38} \\
P_2 &= \frac{3 \times \frac{80}{3}}{6 \times 25 + 3 \times \frac{80}{3} + 5 \times 30} = \frac{8}{38} \\
P_3 &= \frac{5 \times 30}{6 \times 25 + 3 \times \frac{80}{3} + 5 \times 30} = \frac{15}{38}
\end{align*}
\]
(b) \[ P(\text{heading for 2}) = P_1 \frac{P_{12} t_{12}}{\mu_1} = \frac{15}{38} \times \frac{10}{25} = \frac{3}{19} \]

(e) \[ \text{fraction of time from 2 to 3} = P_2 \frac{P_{23} t_{23}}{\mu_2} = \frac{8}{38} \times \frac{60}{80} = \frac{3}{19} \]