Problem 3.19

\[ P\{S_{N(t)} \leq s\} = \sum_{n=0}^{\infty} P\{S_n \leq s, S_{n+1} > t\} \]

\[ = \bar{G}(t) + \sum_{n=1}^{\infty} P\{S_n \leq s, S_{n+1} > t\} \]

\[ = \bar{G}(t) + \sum_{n=1}^{\infty} \int_0^s \bar{F}(t-y)d(A \ast F_{n-1})(s) \]

Problem 3.20

(a) Say that a renewal occurs when that pattern appears. By Blackwell’s theorem for renewal processes we obtain

\[ E[\text{time}] = \frac{1}{(1/2)^7} = 2^7 \]

(b) By Blackwell’s theorem

\[ E[\text{time between HHTT renewals}] = E[\text{time between HTHT renewals}] = 16 \]

But the HHTT renewal process is an ordinary one and so the mean time until HHTT occurs is 16 whereas the HTHT process is a delayed renewal process and so the mean time until HTHT occurs is greater than 16.

Problem 3.29

Let \( L \) denote the lifetime of a car with distribution function \( F(\cdot) \).

(a) Under the policy of replacements at \( A \),

\[
\text{Cost of cycle} = \begin{cases} 
C_1 + C_2 & \text{if } L \leq A \\
C_1 - R(A) & \text{if } L > A 
\end{cases}
\]

and

\[
\text{Length of cycle} = \begin{cases} 
L & \text{if } L \leq A \\
A & \text{if } L > A 
\end{cases}
\]

Then

\[
\frac{\mathbb{E}[\text{Cost}]}{\mathbb{E}[\text{Time}]} = \frac{C_1 + C_2 F(A) - R(A) \bar{F}(A)}{\int_0^A xF(x) + AF(A)}
\]

(Validate the final formula by yourself. If you are confusing, utilize the \textit{indicator} to combine the \textit{if}-clauses into one function as I said in the first recitation.)
(b) Condition on the life of the initial car.

\[
E[\text{Length of cycle}] = \int_0^\infty E[\text{Length}|L = x]dF(x)
\]

\[
= \int_0^A x dF(x) + \int_A^\infty (A + E[\text{Length}])dF(x)
\]

\[
= \int_0^A x dF(x) + (A + E[\text{Length}])\bar{F}(A)
\]

\[
= \int_0^A x dF(x) + A\bar{F}(A)
\]

and similarly

\[
E[\text{Cost of cycle}] = \int_0^\infty E[\text{Cost}|L = x]dF(x)
\]

\[
= \int_0^A (C_1 + C_2)dF(x) + (C_1 - R(A) + E[\text{Cost}])\bar{F}(A)
\]

\[
= \frac{C_1 + C_2F(A) - R(A)\bar{F}(A)}{F(A)}
\]

Then

\[
\frac{E[\text{Cost}]}{E[\text{Time}]} = \text{same as in (a)}.
\]

**Problem 3.31**

Let \(\mu_i\) and \(\nu_i\) denote the means of \(F_i\) and \(G_i\), respectively for \(i = 1, 2, 3, 4\). Then,

\[
l\lim P\{\text{i is working at } t\} = \mu_i/(\mu_i + \nu_i), i = 1, 2, 3, 4
\]

Now, if \(p_i\) is the probability that component \(i\) is working, then

\[
P\{\text{system works}\} = (p_1 + p_2 - p_1p_2)(p_3 + p_4 - p_3p_4)
\]

Hence \(\lim P\{\text{system works at } t\}\) is equal to the preceding expression with \(p_i = \mu_i/(\mu_i + \nu_i), i = 1, 2, 3, 4\)

**Problem 3.32**

(a) \(1 - P_0 = \text{average number in service} = \lambda\mu\)

(b) By alternating renewal processes

\[
P_0 = \text{proportion of time empty} = \frac{E[I]}{E[I] + E[B]}
\]

where \(I\) is an idle period and \(B\) a busy period. But clearly \(I\) is exponential with rate \(\lambda\) and so
\[ 1 - \lambda \mu = \frac{1/\lambda}{1/\lambda + E[B]} \text{ or } E[B] = \frac{\mu}{1 - \lambda \mu} \]

(c) Let \( C \) denote the number of customers served in a busy period \( B \) and let \( S_i \) denote the service time of the \( i \)-th customer, \( i \geq 1 \). Then

\[ B = \sum_{i=1}^{C} S_i \]

and by Wald’s equation

\[ E[C] = \frac{E[B]}{E[S]} = \frac{1}{1 - \lambda \mu} \]