Problem 4.1 Let $D_n$ be the random demand of time period $n$. Clearly $D_n$ is i.i.d. and independent of all $X_k$ for $k < n$. Then we can represent $X_{n+1}$ by

$$X_{n+1} = \max\{0, X_n \cdot 1_{[s, \infty)}(X_n) + S \cdot 1_{[0, s)}(X_n) - D_{n+1}\}$$

which depends only on $X_n$ since $D_{n+1}$ is independent of all history. Hence $\{X_n, n \geq 1\}$ is a Markov chain. It is easy to see assuming $\alpha_k = 0$ for $k < 0$, $P_{ij} = \begin{cases} \alpha_{s-j} & \text{if } i < s, j > 0 \\ \sum_{k=S}^{\infty} \alpha_k & \text{if } i < s, j = 0 \\ \alpha_{i-j} & \text{if } i \geq s, j > 0 \\ \sum_{k=i}^{\infty} \alpha_k & \text{if } i \geq s, j = 0 \end{cases}$

The following three problems (4.2, 4.4, 4.5) needs a fact:

$$\mathbb{P}(A \cap B|C) = \mathbb{P}(A|B \cap C)\mathbb{P}(B|C)$$

which requires a proof to use. Try to prove it by yourself.

Problem 4.2 Let $S$ be the state space. First we show that

$$\mathbb{P}(X_{n_k+1} = j|X_{n_1} = i_1, \cdots, X_{n_k} = i_k) = \mathbb{P}(X_{n_k+1} = j|X_{n_k} = i_k)$$

by the following: Let $A = \{X_{n_k+1} = j\}$, $B = \{X_{n_1} = i_1, \cdots, X_{n_k} = i_k\}$ and $B_b, b \in \mathcal{I}$ are elements of $\{(X_l, l \leq n_k, l \neq n_1, \cdots, l \neq n_k) : X_l \in S\}$.

$$\mathbb{P}(A|B) = \sum_{b \in \mathcal{I}} \mathbb{P}(A \cap B_b|B)$$

$$= \sum_{b \in \mathcal{I}} \mathbb{P}(A|B_b \cap B)\mathbb{P}(B_b|B)$$

$$= \sum_{b \in \mathcal{I}} \mathbb{P}(A|X_{n_k} = i_k)\mathbb{P}(B_b|B)$$

$$= \mathbb{P}(A|X_{n_k} = i_k) \sum_{b \in \mathcal{I}} \mathbb{P}(B_b|B)$$

$$= \mathbb{P}(A|X_{n_k} = i_k)\mathbb{P}(\Omega|B)$$

$$= \mathbb{P}(X_{n_k+1} = j|X_{n_k} = i_k)$$
We consider the mathematical induction on \( l = n - m \). For \( l = 1 \), we just showed. Now assume that the statement is true for all \( l \leq l^* \) and consider \( l = l^* + 1 \):

\[
P(X_n = j | X_{n_1} = i_1, \cdots, X_{n_k} = i_k) = \sum_{i \in S} P(X_n = j | X_{n-1} = i | X_{n_1} = i_1, \cdots, X_{n_k} = i_k)
\]

which completes the proof for \( l = l^* + 1 \) case.

**Problem 4.3** Simply by *Pigeon hole principle* which saying that if \( n \) pigeons return to their \( m(< n) \) home (through hole), then at least one home contains more than one pigeon.

Consider any path of states \( i_0 = i, i_1, \cdots, i_n = j \) such that \( P_{i_k, i_{k+1}} > 0 \). Call this a path from \( i \) to \( j \). If \( j \) can be reached from \( i \), then there must be a path from \( i \) to \( j \). Let \( i_0, \cdots, i_n \) be such a path. If all of values \( i_0, \cdots, i_n \) are not distinct, then there must be a subpath from \( i \) to \( j \) having fewer elements (for instance, if \( i, 1, 2, 4, 1, 3, j \) is a path, then so is \( i, 1, 3, j \)). Hence, if a path exists, there must be one with all distinct states.

**Problem 4.4** Let \( Y \) be the first passage time to the state \( j \) starting the state \( i \) at time 0.

\[
P^n_{ij} = P(X_n = j | X_0 = i)
\]

\[
= \sum_{k=0}^{n} P(X_n = j, Y = k | X_0 = i)
\]

\[
= \sum_{k=0}^{n} P(X_n = j | Y = k, X_0 = i) P(Y = k | X_0 = i)
\]

\[
= \sum_{k=0}^{n} P(X_n = j | X_k = j) P(Y = k | X_0 = i)
\]

\[
= \sum_{k=0}^{n} \sum_{f_{ij}} P^n_{ij}^{k} f_{ij}^k
\]

**Problem 4.5** *(a)* The probability that the chain, starting in state \( i \), will be in state \( j \) at time \( n \) without ever having made a transition into state \( k \).
(b) Let $Y$ be the last time leaving the state $i$ before first reaching to the state $j$ starting the state $i$ at time 0.

$$P_{ij}^n = \mathbb{P}(X_n = j | X_0 = i)$$

$$= \sum_{k=0}^{n} \mathbb{P}(X_n = j, Y = k | X_0 = i)$$

$$= \sum_{k=0}^{n} \mathbb{P}(X_n = j, Y = k, X_k = i | X_0 = i) \mathbb{P}(X_k = i | X_0 = i)$$

$$= \sum_{k=0}^{n} \mathbb{P}(X_n = j, X_l \neq i, l = k + 1, \ldots, n - 1 | X_k = i) P_{ii}^k$$

$$= \sum_{k=0}^{n} P_{ij}^{n-k} \cdot P_{ii}^k$$

Problem 4.7

(a) $\infty$

Here is an argument: Let $x$ be the expected number of steps required to return to the initial state (the origin). Let $y$ be the expected number of steps to move to the left 2 steps, which is the same as the expected number of steps required to move to the right 2 steps. Note that the expected number of steps required to go to the left 4 steps is clearly $2y$, because you first need to go to the left 2 steps, and from there you need to go to the left 2 steps again. Then, consider what happens in successive pairs of steps: Using symmetry, we get

$$x = 2 + (0 \times (1/2) + y \times (1/2)) = 2 + y/2$$

and

$$y = 2 + (0 \times (1/4) + y \times (1/2) + (2 \times y) \times (1/4))$$

If we subtract $y$ from both sides, this last equation yields

$$2 = 0.$$

Hence there is no finite solution. The quantity $y$ must be infinite; a finite value cannot solve the equation.

(b) Note that the expected number of returns in $2n$ steps is the sum of the probabilities of returning in $2k$ steps for $k$ from 1 to $n$, each term of which is binomial. Thus, we have

$$E[N_{2n}] = \sum_{k=1}^{n} \frac{(2k)!}{k!k!} (1/2)^{2k},$$

3
which can be shown to be equal to the given expression by mathematical induction.

(c) We say that \( f(n) \sim g(n) \) as \( n \to \infty \) if
\[
f(n)/g(n) \to 1 \quad \text{as} \quad n \to \infty.
\]

By Stirling’s approximation,
\[
(2n + 1)(2n)!/(n!)^2 (1/2)^n \sim 2 \sqrt{n/\pi},
\]
so that
\[
E[N_n] \sim \sqrt{2n/\pi} \quad \text{as} \quad n \to \infty.
\]

Problem 4.8 (a)
\[
P_{ij} = \frac{\alpha_j}{\sum_{k=1}^{\infty} \alpha_k}, \quad j > i
\]

(b) \( \{T_i, i \geq 1\} \) is not a Markov chain - the distribution of \( T_i \) does depend on \( R_i \). \( \{(R_{i+1}, T_i), i \geq 1\} \) is a Markov chain.

\[
P(R_{i+1} = j, T_i = n|R_i = l, T_{i-1} = m) = \frac{\alpha_j}{\sum_{k=l+1}^{\infty} \alpha_k} \left( \sum_{k=0}^{l} \alpha_k \right)^{n-1} \sum_{k=l+1}^{\infty} \alpha_k
\]
\[
= \alpha_j \left( \sum_{k=0}^{l} \alpha_k \right)^{n-1}, \quad j > l
\]

(c) If \( S_n = j \) then the \((n+1)st\) record occurred at time \( j \). However, knowledge of when these \( n + 1 \) records occurred does not yield any information about the set of values \( \{X_1, \cdots, X_j\} \). Hence, the probability that the next record occurs at time \( k, k > j \), is the probability that both \( \max\{X_1, \cdots, X_j\} = \max\{X_1, \cdots, X_{k-1}\} \) and that \( X_k = \max\{X_1, \cdots, X_k\} \). Therefore, we see that \( \{S_n\} \) is a Markov chain with
\[
P_{jk} = \frac{j}{k} \frac{1}{1 - \frac{1}{k}}, \quad k > j.
\]

Problem 4.11 (a)
\[
\sum_{n=1}^{\infty} P_{ij}^n = E[\text{number of visits to } j|X_0 = i]
\]
\[
= E[\text{number of visits to } j| \text{ever visit } j, X_0 = i]f_{ij}
\]
\[
= (1 + E[\text{number of visits to } j|X_0 = j])f_{ij}
\]
\[
= \frac{f_{ij}}{1 - f_{jj}} < \infty.
\]

since \( 1 + \text{number of visits to } j|X_0 = j \) is geometric with mean \( \frac{1}{1 - f_{jj}} \).
(b) Follows from above since

\[
\frac{1}{1 - f_{jj}} = 1 + \mathbb{E}[\text{number of visits to } j|X_0 = j] = 1 + \sum_{n=1}^{\infty} P_{jj}^n.
\]

**Problem 4.12** If we add the irreducibility of \( P \), it is easy to see that \( \pi = \frac{1}{n} \mathbf{1} \) is a (and the unique) limiting probability.