1. Suppose that you want to perform a fair coin toss, but you only have a tack. How can you use the tack to simulate a fair coin toss?

The tack can either land on its side, with the point touching the ground, or on its head, with the point up. Call these two outcomes “side” (S) and “up” (U). However, these outcomes cannot be expected to occur with probability 1/2. To get independent events with probability exactly 1/2, toss the tack twice: Call the outcome SU a coin head, and the outcome US a coin tail. If outcome SS or UU occurs, repeat the experiment until either SU or US occurs.

2. A laboratory blood test is 90% effective in detecting the dreaded disease Sleepthruklazia when it is in fact present. However, the blood test also yields a “false positive” result for 20% of healthy persons tested. (That is, if a healthy person is tested, then with probability 0.20 the test will conclude that the person has the dreaded disease Sleepthruklazia.) Suppose that 1.0% of the population actually has the dreaded disease Sleepthruklazia.

What is the probability that a person has the dreaded disease Sleepthruklazia given that his test result is positive?

Let Pos be the event that the test is positive. Let D be the event that a random person has the disease and let $D^c$ be the complement of that event. Let $A \cap B$ denote the intersection of the events $A$ and $B$. Then, from the definition of conditional probability,

$$P(D|Pos) = \frac{P(D \cap Pos)}{P(Pos)} = \frac{P(Pos|D)P(D)}{P(Pos)} = \frac{P(Pos|D)P(D)}{P(Pos \cap D) + P(Pos \cap D^c)} = \frac{P(Pos|D)P(D)}{P(Pos|D)P(D) + P(Pos|D^c)P(D^c)} = \frac{(0.90)(0.01)}{(0.90)(0.01) + (0.2)(0.99)} = \frac{0.009}{0.009 + 0.198} = \frac{0.009}{0.207} = 0.043$$

3. Suppose you are on a game show, and you are given the choice of three doors. You win what is behind the door you choose. Behind one door is a new car; behind the other two doors
are goats. You pick a door, say door number 1. Afterwards, the game show host, who knows what is behind all the doors, opens another door, say door number 3, and shows you a goat. He says to you, "Do you want to change your pick to door number 2?"

Is it to your advantage to switch your choice of doors? Why?

This is the famous Monte Hall Quiz-Show Problem. For lengthy discussion, see the webpage: http://math.rice.edu/~ddonovan/montyurl.html

It is important to note that the problem is not well formulated. There are important non-probabilistic aspects. We are not told whether the game host will always open one of the other doors. He might only do it under selected circumstances. Then much depends on the motivation of the game host.

Under the assumption that the game host would always open one of the other doors, you should be able to persuade yourself that indeed it is to your advantage to switch. You would win with probability 2/3 instead of 1/3. With probability 1/3, you originally pick the door with the car, if you switch, then you lose. With probability 2/3 you pick a door with one of the goats. If you switch, then you win in either case.

4. Suppose that you flip a coin many times, with the outcome of each flip being a “head” or “tail”. Suppose that each outcome is equally likely.

(a) After many independent coin tosses, approximately what will be the proportion of heads? Why?

The long-run average value is the expected value in one experiment, which is (0+1)/2 = 1/2. The sample average in n experiments converges to the expected value by virtue of the law of large numbers.

(b) In 1,000,000 coin tosses, what is the approximate probability of getting more than 500,000 heads?

1/2 — This can be obtained from the normal approximation, which follows from the central limit theorem. It is obvious by symmetry.

(c) In 1,000,000 coin tosses, what is the approximate probability of getting more than 505,000 heads?

0 — This also can be obtained from the normal approximation, which follows from the central limit theorem. Note that 505,000 is 5000 above the mean. However, the variance of the total number of heads is 1,000,000/4, so that the standard deviation is 1000/2 = 500. Thus 5000 is 10 standard deviations above the mean.
(d) In $1,000,000$ coin tosses, what is the approximate probability of getting more than 500,500 heads?

0.16 — This also can be obtained from the normal approximation, which follows from the central limit theorem. Note that 500,500 is 500 above the mean, which is 1 standard deviation. The probability of being at least one standard deviation above the mean is $1 - 0.8413 \approx 0.16$. Here are more details:

$$P(S_{1,000,000} > 500,500)$$
$$= P((S_{1,000,000} - E[S_{1,000,000}]) / SD(S_{1,000,000}) > (500,500 - E[S_{1,000,000}]) / SD(S_{1,000,000}))$$
$$= P((S_{1,000,000} - 500,000) / 500 > (500,500 - 500,000) / 500)$$
$$\approx P(N(0,1) > 1) = 1 - 0.8413.$$

5. What is the difference between the weak law of large numbers and the strong law of large numbers?

The classical law of large numbers (LLN) concerns the average or sample mean of $n$ independent and identically distributed (IID) random variables. Given a sequence \( \{X_n : n \geq 1\} \) of IID random variables, the sample mean is \( \bar{X}_n \equiv n^{-1}S_n \), where \( S_n \equiv X_1 + \cdots + X_n \) is the $n^{th}$ partial sum. The LLN states that \( \bar{X}_n \) converges to the mean \( EX \). The weak LLN (WLLN) is convergence in probability; the strong LLN (SLLN) is convergence with probability one (w.p.1). Generalizations of the classical LLN draw the same conclusion with the IID conditions weakened. A version of the SLLN is proved in the Appendix to Chapter 1 in Ross. It requires that \( E[X_1^4] < \infty \). The SLLN actually holds under the weaker condition that \( E[|X_1|] < \infty \).

6. Consider the following limits (as $n \to \infty$):

(i) \( X_n \to X \) (convergence in distribution)
(ii) \( EX_n \to EX \)
(iii) \( E[|X_n - X|] \to 0 \)
(iv) \( P(X_n \to X) = 1 \)
(v) \( X_n \to X \) in probability

What are the implications among these limits? For example, does limit (i) imply limit (ii)?

See Chapter 4 of Chung, *A Course in Probability Theory* for a discussion of convergence concepts. The following implications hold:

\((iv) \to (v) \to (i), \ (iii) \to (ii) \) and \( (iii) \to (v) \)

To see that (iv) does not imply (ii) or (iii), let the underlying probability space be the unit interval \([0,1]\) with the uniform distribution (which coincides with Lebesgue measure). Let \( X = 0 \) w.p.1 and let \( X_n = 2^n \) on the interval \((a_n, a_n + 2^{-n})\) where \( a_n = 2^{-1} + 2^{-2} + \cdots + 2^{-(n-1)} \) with \( a_1 = 0 \), and let \( X_n = 0 \) otherwise. Then \( P(X_n \to X \equiv 0) = 1 \), but \( E[|X_n - X|] = EX_n = 1 \).
for all \( n \), but \( EX = 0 \). (To see that indeed \( P(X_n \to X = 0) = 1 \), note that the interval on which \( X_k \) is positive for any \( k > n \) has probability going to 0 as \( n \to \infty \).

From the example above, it follows that (v) does not imply (ii) and that (i) does not imply (ii). However, (i) does imply (ii) under regularity conditions, namely, under uniform integrability. See p. 32 of Billingsley, *Convergence of Probability Measures*, 1968, for more discussion.

To see that (iii) does not imply (iv), again let the underlying probability space be the unit interval \([0, 1]\) with the uniform distribution (which coincides with Lebesgue measure). Let \( X = 0 \) w.p.1. Somewhat like before, let \( X_n = 1 \) on the interval \((a_n, a_n + n^{-1})\) where \( a_n = a_{n-1} + (n - 1)^{-1} \mod 1 \), with \( a_1 = 0 \), and let \( X_n = 0 \) otherwise. (The mod1 means that there is “wrap around” from 1 back to 0.) (To see that indeed \( P(X_n \to X = 0) = 0 \), note that the \( X_k = 1 \) infinitely often for each sample point. On the other hand, \( E[|X_n - X|] = EX_n = 1/n \to 0 \) as \( n \to \infty \).