Problem 1.7 Imagine that there are $k$ buckets and we choose $k$ balls and fill each bucket at the same time. Set

$$I_i = \begin{cases} 1 & \text{if } i\text{-th bucket contains a white ball} \\ 0 & \text{if } i\text{-th bucket contains a black ball.} \end{cases}$$

Then $X = \sum_{i=1}^k I_i$. Observe that for $i \neq j$,

$$E[I_i] = E[I_i^2] = P(I_i = 1) = \frac{n}{n+m},$$

$$E[I_i|I_j = 1] = P(I_i = 1|I_j = 1) = \frac{n-1}{n+m-1} \quad \text{and}$$

$$E[I_iI_j] = E[E[I_iI_j|I_j]] = E[I_jE[I_i|I_j]] = P(I_j = 1)E[I_i|I_j = 1] = \frac{n}{n+m} \times \frac{n-1}{n+m-1}.$$

Using these, we get

$$E[X] = kP(I_1 = 1) = \frac{kn}{n+m},$$

$$E[X^2] = kE[I_1^2] + k(k-1)E[I_1I_2] = \frac{kn}{n+m} + \frac{k(k-1)n(n-1)}{(n+m)(n+m-1)}, \quad \text{and}$$

$$V(X) = E[X^2] - E[X]^2 = \frac{knm(n+m-k)}{(n+m)^2(n+m-1)}.$$

Problem 1.9 Assume that the results of each pairing are independent with each of the players being equally likely to win. For each permutation $i_1, \cdots, i_n$ of 1, 2, $\cdots$, $n$ define an indicator variable $I_{(i_1, \cdots, i_n)}$ equal to 1 if that permutation is a Hamiltonian and 0 if it is not. Then

$$E[\text{Number of hamiltonians}] = E[\sum I_{(i_1, \cdots, i_n)}] = n!E[I_{(1,2,\cdots,n)}] = \frac{n!}{2^{n-1}}.$$ 

Hence, for at least one outcome the number of Hamiltonians must be at least $\frac{n!}{2^{n-1}}$.

Problem 1.18 Let $N$ be the number of flips that are made until a string of $r$ heads in a row.

Define $T$ as the number of trials until the first tails. Then we have

$$E[N|T = k] = \begin{cases} k + E[N] & \text{if } k \leq r \\ r & \text{if } k > r. \end{cases}$$
Using the fact that $T$ has geometric distribution,

\[ E[N] = E[E[N|T]] = \sum_{k=1}^{\infty} E[N|T = k] P(T = k) \]

\[ = \sum_{k=1}^{r} (k + E[N])(1 - p)p^{k-1} + \sum_{k=r+1}^{\infty} r(1 - p)p^{k-1} \]

\[ = (1 - p) \sum_{k=1}^{r} (k + E[N])p^{k-1} + \sum_{k=r+1}^{\infty} \frac{r}{1 - p} p^{k-1} \]

\[ = (1 - p) \sum_{k=1}^{r} k p^{k-1} + E[N](1 - p) \sum_{k=1}^{r} p^{k-1} + rp^{r} \]

\[ = \frac{1 - (r + 1)p^{r} + rp^{r+1}}{1 - p} + E[N](1 - p^{r}) + rp^{r} \]

\[ = \frac{1 - p^{r}}{1 - p} + E[N](1 - p^{r}) \]

\[ = \frac{1 - p^{r}}{(1 - p)p^{r}}. \]

**Problem 1.20** Let $L$ be the left hand point of the first interval. Note that \( \{N(x)|L = y\} = \{1 + N(y) + N(x - y - 1)\} \). If $x > 1$,

\[ M(x) = E[N(x)] = E[E[N(x)|L]] \]

\[ = \int_{0}^{x-1} E[N(x)|L = y] \frac{dy}{x - 1} \]

\[ = \frac{1}{x - 1} \int_{0}^{x-1} E[1 + N(y) + N(x - y - 1)]dy \]

\[ = 1 + \frac{1}{x - 1} \int_{0}^{x-1} E[N(y) + N(x - y - 1)]dy \]

\[ = 1 + \frac{1}{x - 1} \int_{0}^{x-1} (M(y) + M(x - y - 1))dy \]

\[ = 1 + \frac{2}{x - 1} \int_{0}^{x-1} M(y)dy. \]

**Problem 1.23** Let $i \rightarrow j$ be the event that the particle moves from $i$ to $j$ in one step. Let $i \Rightarrow j$ be the event that the particle ever reaches $j$ starting $i$. Conditioning on the random variable denoting the first movements of the particle, $I$,

(a)

\[ \alpha = P(0 \Rightarrow 1) \]

\[ = E[P(0 \Rightarrow 1|I)] \]

\[ = P(0 \rightarrow 1)P(1 \Rightarrow 1) + P(0 \rightarrow -1)P(-1 \Rightarrow 1) \]
\[ = p \times 1 + (1 - p)P(-1 \Rightarrow 1) \]
\[ = p + (1 - p)P(-1 \Rightarrow 0, 0 \Rightarrow 1) \]
\[ = p + (1 - p)P(-1 \Rightarrow 0)P(0 \Rightarrow 1) \]
\[ = p + (1 - p)P(0 \Rightarrow 1)^2 \]
\[ = p + (1 - p)\alpha^2 . \]

(b) Solving the previous quadratic equation, we get two solutions, 1 and \( \frac{1}{1-p} \). The condition \( \frac{p}{1-p} < 1 \) implies \( p < 1/2 \). Hence if \( p \geq 1/2 \), \( \alpha \) should be 1. For \( p < 1/2 \), the strong law of large numbers says that the particle ever goes to the negative infinity with probability 1. If \( \alpha = 1 \), then the starting position would be reached infinitely often, which contradicts to the strong law of large numbers. Hence
\[
\alpha = \begin{cases} 
1 & \text{if } p \geq 1/2 \\
\frac{p}{1-p} & \text{if } p < 1/2 .
\end{cases}
\]

(c) \[ P(0 \Rightarrow n) = P(0 \Rightarrow 1) \times \cdots \times P(n - 1 \Rightarrow n) \]
\[ = P(0 \Rightarrow 1) \times \cdots \times P(0 \Rightarrow 1) \]
\[ = P(0 \Rightarrow 1)^n \]
\[ = \alpha^n . \]

(d) \[ P(i \rightarrow i + 1|i \Rightarrow n) = \frac{P(i \rightarrow i + 1, i \Rightarrow n)}{P(i \Rightarrow n)} \]
\[ = \frac{P(i \Rightarrow n|i \rightarrow i + 1)P(i \rightarrow i + 1)}{P(i \Rightarrow n)} \]
\[ = \frac{P(i + 1 \Rightarrow n)p}{P(i \Rightarrow n)} \]
\[ = \frac{\alpha^{n-i-1}p}{\alpha^n} \]
\[ = \frac{p}{\alpha} \]
\[ = 1 - p . \]

Problem 1.24 Let \( T_{(i \Rightarrow j)} \) the number of steps to reach \( j \) first time starting \( i \). Then we have an apparent arithmetic like \( T_{(-1 \Rightarrow 1)} = T_{(-1 \Rightarrow 0)} + T_{(0 \Rightarrow 1)} \) and a distributional identity like \( T_{(-1 \Rightarrow 0)} \overset{d}{=} T_{(0 \Rightarrow 1)} \). We also know that \( T_{(-1 \Rightarrow 0)} \) and \( T_{(0 \Rightarrow 1)} \) are independent because of the independence of every transition. That is, \( T_{(-1 \Rightarrow 0)} \) and \( T_{(0 \Rightarrow 1)} \) are iid. Using the notation in the book, \( T \equiv T_{(0 \Rightarrow 1)} \),
\[ \mathbf{E}[T_{(-1 \Rightarrow 1)}] = 2\mathbf{E}[T] , \]
\[ \mathbf{V}(T_{(-1 \Rightarrow 1)}) = 2\mathbf{V}(T) . \]
Let the random variable \( X \) denote the particle’s location after the first move.

(a) Conditioning on \( X \) gives

\[
E[T] = E[E[T|X]] = E[T|X = 1]P(X = 1) + E[T|X = -1]P(X = -1) = 1 \times p + (1 + E[T_{(-1\rightarrow 1)}])(1-p)
\]

\[
= 1 + 2(1-p)E[T].
\]

Hence, \( E[T] = \infty \) if \( p \leq 1/2 \). If we can show that \( E[T] < \infty \) when \( p > 1/2 \), we obtain in this case that

\[
E[T] = \frac{1}{2p-1}.
\]

Now let’s show that \( E[T] < \infty \) if \( p > 1/2 \): Let \( p^{(n)} \) denotes the probability that the particle reaches 1 by \( n \)-transitions starting 0. Then \( n \) should be odd. That is, only \( p^{(2n+1)} \) is nonzero. Now we have an upper bound on this probability:

\[
p^{(2n+1)} \leq \binom{2n}{n} p[p(1-p)]^n \sim p \frac{[4p(1-p)]^n}{\sqrt{\pi n}}
\]

using an approximation, due to Stirling, which asserts that

\[
n! \sim n^{n+1/2}e^{-n} \sqrt{2\pi}
\]

where \( a_n \sim b_n \) denotes \( \lim_{n \to \infty} \frac{a_n}{b_n} = 1 \). Now it is easy to verify that if \( a_n \sim b_n \), then \( \sum_n a_n < \infty \) if, and only if, \( \sum_n b_n < \infty \). Hence \( E[T] < \infty \) if

\[
\sum_{n=0}^{\infty} (2n+1)p \frac{[4p(1-p)]^n}{\sqrt{\pi n}} < \infty
\]

which is true if \( 4p(1-p) < 1 \) or \( p \neq 1/2 \).

(b) Noting that \( T|\{X = 1\} = 1 \) and \( T|\{X = -1\} = 1 + T_{(-1\rightarrow 1)} \) which is 1 plus the convolution of two independent random variables both having the distribution of \( T \). Therefore,

\[
E[T|X = 1] = 1, \quad E[T|X = -1] = 1 + 2E[T]
\]

\[
V(T|X = 1) = 0, \quad V(T|X = -1) = 2V(T)
\]

and thus

\[
V(E[T|X]) = V(E[T|X] - 1) = 4E[T]^2p(1-p) = \frac{4p(1-p)}{(2p-1)^2}
\]

\[
E[V(T|X)] = 2(1-p)V(T)
\]

By the conditional variance formula

\[
V(T) = 2(1-p)V(T) + \frac{4p(1-p)}{(2p-1)^2}
\]

which gives the result.
(c) \( T_{(0 \Rightarrow n)} = T_{(0 \Rightarrow 1)} + \cdots + T_{(n-1 \Rightarrow n)} = \sum_{i=1}^{n} T_i \) where \( T_i \) are iid having distribution of \( T \).

Hence
\[
E[T_{(0 \Rightarrow n)}] = nE[T].
\]

(d) By the same reasoning as in (c),
\[
V(T_{(0 \Rightarrow n)}) = nV(T).
\]

Problem 1.28 The MGF is given by \( \phi(t) = \frac{\lambda}{\lambda - t} \). So
\[
\phi'(t) = \frac{\lambda}{(\lambda - t)^2} \quad \text{and} \quad \phi''(t) = \frac{2\lambda}{(\lambda - t)^3}.
\]

Hence,
\[
E[X] = \phi'(0) = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}
\]
\[V(X) = E[X^2] - E[X]^2 = \phi''(0) - \frac{1}{\lambda^2} = \frac{2\lambda}{\lambda^3} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.
\]

Problem 1.29 The MGF of an exponential random variable, \( X \), is \( \phi_X(t) = \frac{\lambda}{\lambda - t} \). Then
\[
\phi_{\sum_{i=1}^{n} X_i}(t) = E[e^{t \sum_{i=1}^{n} X_i}]
\]
\[= \prod_{i=1}^{n} E[e^{t X_i}] = \phi_X(t)^n = \left( \frac{\lambda}{\lambda - t} \right)^n
\]
which is an MGF of a Gamma distribution with parameter \((n, \lambda)\). Hence the result follows from the uniqueness of MGF.

Problem 1.31
\[
P(\min\{X, Y\} > a | \min\{X, Y\} = X) = P(X > a | X < Y) = \frac{P(a < X, X < Y)}{P(X < Y)}.
\]
\[
P(a < X, X < Y) = \int_{a}^{\infty} P(Y > X | X = x) \lambda_1 e^{-\lambda_1 x} dx = \int_{a}^{\infty} e^{-\lambda_2 x} \lambda_1 e^{-\lambda_1 x} dx
\]
\[= \lambda_1 \int_{a}^{\infty} e^{-(\lambda_1 + \lambda_2)x} dx = \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-(\lambda_1 + \lambda_2)a},
\]
\[
P(X < Y) = \int_{0}^{\infty} P(Y > X | Y = y) \lambda_2 e^{-\lambda_2 y} dy = \int_{0}^{\infty} (1 - e^{-\lambda_1 y}) \lambda_2 e^{-\lambda_2 y} dy
\]
\[= 1 - \frac{\lambda_2}{\lambda_1 + \lambda_2} = \frac{\lambda_1}{\lambda_1 + \lambda_2}.
\]
Hence,
\[ P(\min\{X,Y\} > a| \min\{X,Y\} = X) = e^{-(\lambda_1 + \lambda_2)a}. \]

**Problem 1.43**

\[ P(X \geq a) = P(X^t \geq a^t) \leq \frac{E[X^t]}{a^t} \]

with the inequality following from the Markov inequality. Let \( X \) be exponential with rate 1, and let \( a = t = n \) in the preceding, to obtain that

\[ e^{-n} \leq \frac{n!}{n^n}. \]

(Of course, the above inequality could also be shown by noting that it is equivalent to the statement that \( P(Y = n) \leq 1 \) where \( Y \) is Poisson with mean \( n \).)