Problem 3.21 Let $X_i$ equal 1 if the gambler wins bet $i$, and let it be 0 otherwise. Also, let $N$ denote the first time the gambler has won $k$ consecutive bets. Then $X = \sum_{i=1}^N X_i$ is equal to the number of bets that he wins, and $X - (N - X) = 2X - N$ is his winnings. By Wald’s equation

$$E[X] = pE[N] = p \sum_{i=1}^k p^{-i}.$$ 

Thus

(a) $E[2X - N] = 2E[X] - E[N] = (2p - 1)E[N] = (2p - 1) \sum_{i=1}^k p^{-i}$

(b) $E[X] = p \sum_{i=1}^k p^{-i}$

Problem 3.22 (a)

$$E[T_{HHTTHH}] = E[T_{HH}] + p^{-4}(1 - p)^{-2}$$

$$= E[T_H] + p^{-2} + p^{-4}(1 - p)^{-2}$$

$$= p^{-1} + p^{-2} + p^{-4}(1 - p)^{-2}$$

(b) $E[T_{HHTTT}] = p^{-2}(1 - p)^{-3}$


(c) $P_A = (32 + 70 - 70)/(30 + 70) = 0.32$

(d) $E[M] = 32 - 30(0.32) = 22.4$

Problem 3.23 Let $H$ denote the first $k$ flips and $\Omega$ is the set of all possible $H$. Conditioning on $H$ gives:

$$E[\text{number until repeat}] = \sum_{H \in \Omega} E[\text{number until repeat}|H]P(H)$$

$$= \sum_{H \in \Omega} \frac{1}{P(H)}P(H) = |\Omega| = 2^k$$

1
Problem 3.25 (a) First note that

\[
\mathbb{E}[N_D(t) \mid X_1 = x] = \begin{cases} 
1 + \mathbb{E}[N(t-x)] & \text{if } x \leq t \\
0 & \text{if } x > t 
\end{cases}
\]

\[
m_D(t) = \mathbb{E}[N_D(t)] = \int_0^\infty \mathbb{E}[N_D(t) \mid X_1 = x] dG(x)
\]

\[
= \int_0^t (1 + \mathbb{E}[N(t-x)]) dG(x)
\]

\[
= G(t) + \int_0^t m(t-x) dG(x)
\]

(b) \[
\mathbb{E}[A_D(t)] = \mathbb{E}[A_D(t) \mid S_N(t) = 0] \bar{G}(t) + \int_0^t \mathbb{E}[A_D(t) \mid S_N(t) = s] \bar{F}(t-s) d\mu_D(s)
\]

\[
t \to \infty \quad \frac{1}{\mu} \int_0^\infty t \bar{F}(t) dt \quad \text{By key renewal theorem (Proposition 3.5.1(v))}
\]

\[
= \frac{1}{\mu} \int_0^\infty t \int_t^\infty dF(s) dt
\]

\[
= \frac{\int_0^\infty s^2 dF(s)}{2 \int_0^\infty s dF(s)}
\]

(c) \[
t \bar{G}(t) = t \int_t^\infty dG(x) \leq \int_t^\infty s dG(s) \xrightarrow{t \to \infty} 0 \text{ since } \int_0^\infty s dG(s) < \infty.
\]

(Here we used the so-called dominated convergence theorem.

\[
\int_n^\infty s dG(s) = \int_0^\infty s \mathbf{1}_{[n, \infty)}(s) dG(s)
\]

\[
n \to \infty \lim_{n \to \infty} \mathbf{1}_{[n, \infty)}(s) dG(s) = \int_0^\infty 0 dG(s)
\]

since \(s \mathbf{1}_{[n, \infty)}(s) \leq s \) and \(s\) is integrable with respect to \(G(\cdot)\) from \(\int_0^\infty s dG(s) < \infty\) and \(s \mathbf{1}_{[n, \infty)}(s) \to 0\) for each \(s\) in pointwise sense. (Check the conditions for the dominated convergence theorem.) Now we extend \(n\) to \(t\) using monotonicity of the integral. Wow!

This is a good example showing that if you are familiar with a little rigorous analysis, then it’s O.K. with only one line. But if not, you should practice the underlying logic whenever you encounter them.)

Problem 3.28 Using the uniformity of each Poisson arrival under given \(N(t)\),

\[
\mathbb{E}[\text{Cost of a cycle} \mid N(T)] = K + N(T) \times c \times \frac{T}{2}
\]

and so

\[
\frac{\mathbb{E}[\text{Cost}]}{\mathbb{E}[\text{Time}]} = \frac{K + \lambda c T^2/2}{T} = \frac{K}{T} + \frac{\lambda c T}{2}
\]
which is minimized at \( T^* = \sqrt{2K/\lambda c} \) and minimal average cost is thus \( \sqrt{2\lambda Kc} \). On the other hand the optimal value of \( N \) is (using calculus) \( N^* = \sqrt{2\lambda K/c} \) and the minimal average cost is \( \sqrt{2\lambda cK} - \frac{c}{2} \).

Problem 3.29 Let \( L \) denote the lifetime of a car with distribution function \( F(\cdot) \).

(a) Under the policy of replacements at \( A \),

\[
\text{Cost of cycle} = \begin{cases} 
C_1 + C_2 & \text{if } L \leq A \\
C_1 - R(A) & \text{if } L > A
\end{cases}
\]

and

\[
\text{Length of cycle} = \begin{cases} 
L & \text{if } L \leq A \\
A & \text{if } L > A
\end{cases}
\]

Then

\[
\frac{\mathbb{E}[	ext{Cost}]}{\mathbb{E}[	ext{Time}]} = \frac{C_1 + C_2 F(A) - R(A) \bar{F}(A)}{\int_0^A x dF(x) + A \bar{F}(A)}
\]

(Validate the final formula by yourself. If you are confusing, utilize the indicator to combine the if-clauses into one function as I said in the first recitation.)

(b) Condition on the life of the initial car.

\[
\mathbb{E}[	ext{Length of cycle}] = \int_0^\infty \mathbb{E}[	ext{Length}|L = x] dF(x)
\]

\[
= \int_0^A x dF(x) + \int_A^\infty (A + \mathbb{E}[	ext{Length}]) dF(x)
\]

\[
= \int_0^A x dF(x) + (A + \mathbb{E}[	ext{Length}]) \bar{F}(A)
\]

\[
= \int_0^A x dF(x) + A \bar{F}(A)
\]

and similarly

\[
\mathbb{E}[	ext{Cost of cycle}] = \int_0^\infty \mathbb{E}[	ext{Cost}|L = x] dF(x)
\]

\[
= \int_0^A (C_1 + C_2) dF(x) + (C_1 - R(A) + \mathbb{E}[	ext{Cost}]) \bar{F}(A)
\]

\[
= \frac{C_1 + C_2 F(A) - R(A) \bar{F}(A)}{F(A)}
\]

Then

\[
\frac{\mathbb{E}[	ext{Cost}]}{\mathbb{E}[	ext{Time}]} = \text{same as in (a)}.
\]