Problem 4.1 Let $D_n$ be the random demand of time period $n$. Clearly $D_n$ is i.i.d. and independent of all $X_k$ for $k < n$. Then we can represent $X_n + 1$ by

$$X_{n+1} = \max\{0, X_n \cdot 1_{[s, \infty)}(X_n) + S \cdot 1_{[0,s)}(X_n) - D_{n+1}\}$$

which depends only on $X_n$ since $D_{n+1}$ is independent of all history. Hence $\{X_n, n \geq 1\}$ is a Markov chain. It is easy to see assuming $\alpha_k = 0$ for $k < 0$,

$$P_{ij} = \begin{cases} 
\alpha_{s-j} & \text{if } i < s, j > 0 \\
\sum_{k=S}^{\infty} \alpha_k & \text{if } i < s, j = 0 \\
\alpha_{i-j} & \text{if } i \geq s, j > 0 \\
\sum_{k=i}^{\infty} \alpha_k & \text{if } i \geq s, j = 0 
\end{cases}$$

The following three problems (4.2, 4.4, 4.5) needs a fact:

$$P(A \cap B|C) = P(A|B \cap C)P(B|C)$$

which requires a proof to use. Try to prove it by yourself.

Problem 4.2 Let $S$ be the state space. First we show that

$$P(X_{n+1} = j|X_n = i_1, \cdots, X_k = i_k) = P(X_{n+1} = j|X_k = i_k)$$

by the following: Let $A = \{X_{n+1} = j\}$, $B = \{X_{n1} = i_1, \cdots, X_k = i_k\}$ and $B_b, b \in I$ are elements of $\{(X_l, l \leq n_k, l \neq n_1, \cdots, l \neq n_k) : X_l \in S\}$.

$$P(A|B) = \sum_{b \in I} P(A \cap B_b|B)$$

$$= \sum_{b \in I} P(A|B_b \cap B)P(B_b|B)$$

$$= \sum_{b \in I} P(A|X_{n_k} = i_k)P(B_b|B)$$

$$= P(A|X_{n_k} = i_k) \sum_{b \in I} P(B_b|B)$$

$$= P(A|X_{n_k} = i_k)P(\Omega|B)$$

$$= P(X_{n+1} = j|X_k = i_k) .$$
We consider the mathematical induction on \( l \equiv n - m \). For \( l = 1 \), we just showed. Now assume that the statement is true for all \( l \leq l^* \) and consider \( l = l^* + 1 \):

\[
\begin{align*}
P(X_n = j | X_{n_1} = i_1, \ldots, X_{n_k} = i_k) &= \sum_{i \in S} P(X_n = j | X_{n-1} = i, X_{n_1} = i_1, \ldots, X_{n_k} = i_k) P(X_{n-1} = i | X_{n_1} = i_1, \ldots, X_{n_k} = i_k) \\
&= \sum_{i \in S} P(X_n = j | X_{n-1} = i) P(X_{n-1} = i | X_{n_k} = i_k) \quad \text{By } l \leq l^* \text{ cases} \\
&= \sum_{i \in S} P(X_n = j, X_{n-1} = i) P(X_{n-1} = i | X_{n_k} = i_k) \\
&= \sum_{i \in S} P(X_n = j | X_{n_k} = i_k) P(X_{n} = j) \\
&= P(X_n = j | X_{n_k} = i_k)
\end{align*}
\]

which completes the proof for \( l = l^* + 1 \) case.

**Problem 4.3** Simply by *Pigeon hole principle* which saying that if \( n \) pigeons return to their \( m(< n) \) home (through hole), then at least one home contains more than one pigeon.

Consider any path of states \( i_0 = i, i_1, \ldots, i_n = j \) such that \( P_{i_k}i_{k+1} > 0 \). Call this a path from \( i \) to \( j \). If \( j \) can be reached from \( i \), then there must be a path from \( i \) to \( j \). Let \( i_0, \ldots, i_n \) be such a path. If all of values \( i_0, \ldots, i_n \) are not distinct, then there must be a subpath from \( i \) to \( j \) having fewer elements (for instance, if \( i, 1, 2, 4, 1, 3, j \) is a path, then so is \( i, 1, 3, j \)). Hence, if a path exists, there must be one with all distinct states.

**Problem 4.4** Let \( Y \) be the first passage time to the state \( j \) starting the state \( i \) at time 0.

\[
P^n_{ij} = P(X_n = j | X_0 = i) = \sum_{k=0}^{n} P(X_n = j, X_0 = i) = \sum_{k=0}^{n} P(X_n = j | Y = k) P(Y = k | X_0 = i) = \sum_{k=0}^{n} P(X_n = j | X_k = j) P(Y = k | X_0 = i) = \sum_{k=0}^{n} P^n_{i_j} P^n_{ji}
\]

**Problem 4.5 (a)** The probability that the chain, starting in state \( i \), will be in state \( j \) at time \( n \) without ever having made a transition into state \( k \).
(b) Let $Y$ be the last time leaving the state $i$ before first reaching to the state $j$ starting the state $i$ at time 0.

$$
P^n_{ij} = P(X_n = j | X_0 = i)$$

$$= \sum_{k=0}^{n} P(X_n = j, Y = k | X_0 = i)$$

$$= \sum_{k=0}^{n} P(X_n = j, Y = k, X_k = i | X_0 = i)$$

$$= \sum_{k=0}^{n} P(X_n = j, Y = k | X_k = i) P(X_k = i | X_0 = i)$$

$$= \sum_{k=0}^{n} P(X_n = j, X_l \neq i, l = k + 1, \ldots, n - 1 | X_k = i) P^k_{ii}$$

$$= \sum_{k=0}^{n} P_{ij/i}^{n-k} P^k_{ii}$$

**Problem 4.12** If we add the irreducibility of $P$, it is easy to see that $\pi = \frac{1}{n} \mathbf{1}$ is a (and the unique) limiting probability.