Columbia University
Department of Physics
QUALIFYING EXAMINATION

Friday, January 14, 2011
1:00PM to 3:00PM
General Physics (Part I)
Section 5.

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 6 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 5 (General Physics), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" × 11" paper (double-sided) you have prepared on General Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

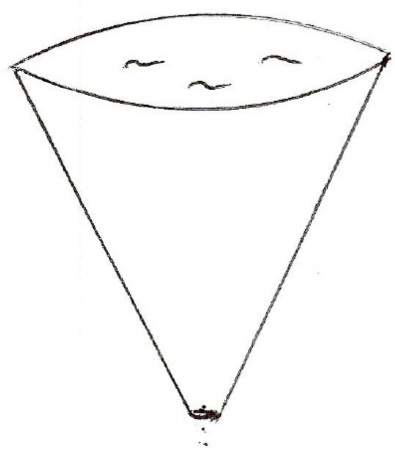
Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. An open inverted cone has a height h = 10cm, and has a radius R = 3cm at its larger (upper) end. It is filled with water of density ρ = 1.0 g/cm^3. A small hole of radius r = 1mm is opened at the bottom, and the water starts to flow out of the hole.

   How long does it take for all of the water to run out? (Assume that the water velocity at the top is zero, and that viscosity can be ignored.)



2. The Large Hadron Collider (LHC) has recently started operations, with an initial center-of-mass energy of 7 TeV (= 7000 GeV). Consider the process pp $\rightarrow$ H + Z + X, depicting production at the LHC of the Standard Model Higgs boson (H) in association with a Z boson. Consider the limit where you neglect X (which denotes whatever else is produced in the same proton-proton collision) and make the approximation that all of the available energy goes into the H+Z.

   (a) Calculate the momentum of the Higgs boson in the lab frame.

   (b) Assume the Higgs immediately decays into a *b*-quark and an anti-*b* quark (i.e. H $\rightarrow b\bar{b}$). If the *b*-quark lives for 1 ps, as measured in its rest frame, calculate the maximum distance that it travels in the lab frame before decaying.

   (Assume the following mass values: m(H) = 130 GeV, m(Z) = 90 GeV, m(*b*) = 5 GeV.)

3. As shown in Figure 1, a small light bulb is suspended at distance d_1 above the surface of the water in a swimming pool where the water depth is d_2. The bottom of the pool is a large mirror. How far below the mirror surface is the image of the bulb? (Assume all distances are large enough that $\sin\theta = \theta$ is a valid approximation, and that the index of refraction of air equals 1.) Express your answer in terms of n_{water}, d_1, and d_2.

1

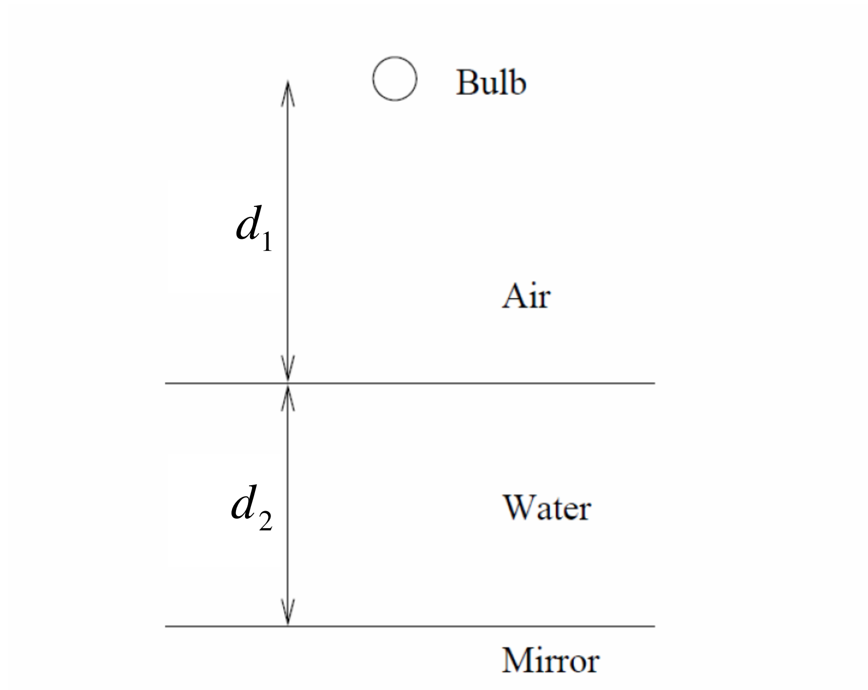


Figure 1: A bulb suspended over a swimming pool with a mirror.

4. Provide order-of-magnitude estimates for the following questions:

   (a) We have heard it said by journalists that we may be able to identify extraterrestrial intelligence by observing visible light signals from aliens. The light would be transmitted by lasers. Assuming that we need a photon per minute in our detectors installed on a 2m telescope at Earth, provide an estimate on the power requirements of the alien laser. Is this feasible?

   (b) Estimate the minimum speed that you, a typical person, would need in order to run on a water surface and not submerge.

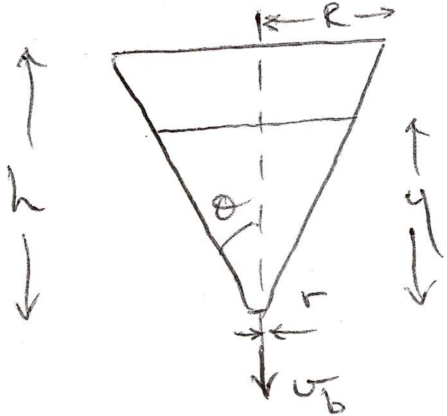
   (c) Suppose you are being sold a pocket size 20 megapixel digital camera with a 3mm diameter lens, and a 35mm focal length. Does such a purchase make sense from the optical point of view? Justify your answer.

5. An otherwise empty cavity contains electromagnetic radiation (photons) in thermodynamic equilibrium (temperature T) with the cavity walls. The angular frequencies ($\omega = 2\pi\nu$) of the electromagnetic waves' normal modes are $\omega_1 < \omega_2 < \omega_3 < \ldots < \omega_s < \ldots < \infty$.
   (a) What is the most probable number of photons in the s^{th} mode?
   (b) What is the probability for finding n_s photons in the s^{th} mode?
   (c) What is the average (or "expected") number of photons in this mode?
   (d) Which mode has the largest average photon number?
   (e) Suppose the cavity were to be filled with a transparent liquid which, in the relevant frequency regime, has a frequency-independent dielectric constant (ϵ) and a magnetic permeability (μ). How would the total photon energy in the container change? (Assume all container dimensions to be large compared to the relevant $\lambdabar = c/\omega$.)

6. Suppose you discover a substance whose equation of state is given by $p = \frac{5T^3}{V}$ where p is the pressure, V is the volume, and T is the temperature (note that the units for the constant "5" must be Nm/K^3). Your colleagues have carried out another experiment where they have found that the internal energy is a function of V and T given by $U(V,T) = BT^x \ln\left(\frac{V}{V_0}\right) + f(T)$ where B, x and V_0 are constants, and $f(T)$ is an unmeasured function that only depends on T. What are the values of B and x? (In terms of constants you already know).

Solution:

First step is to find the water flow velocity as it exits the hole; from Bernoulli :



$$\Rightarrow \underbrace{P_y + \frac{1}{2}\rho v_y^2 + \rho g y}_{\text{at height } y} = \underbrace{P_b + \frac{1}{2}\rho v_b^2 + \rho g y_b}_{\text{bottom}}$$

where $P_y = P_b = P_{atm}$; $v_y \sim 0$; $y_b = 0$

$$\Rightarrow \cancel{\rho} g y = \frac{1}{2}\cancel{\rho} v_b^2$$

$$\Rightarrow v_b = \sqrt{2gy}$$

The volume of a cone of height h and radius R (at its large end) is:

$$V = \pi \int_0^h \left(\frac{R}{h} y\right)^2 dy = \frac{\pi R^2}{h^2} \int_0^h y^2 \, dy$$

$$= \frac{\pi R^2}{h^2} \left[\frac{y^3}{3}\right]_0^h = \frac{1}{3} \pi R^2 h$$

so the volume of water corresponding to height y is:

$$V_y = \frac{1}{3} \pi (y \tan\theta)^2 y = \frac{1}{3} \pi y^3 \tan^2\theta$$

with θ defined as shown above, and $\tan\theta = \frac{R}{h}$.

The volume of water changes at a rate:

$$\frac{dV_y}{dt} = \frac{d}{dt}\left(\frac{1}{3}\pi y^3 \tan^2\theta\right) = \frac{1}{3}\pi \tan^2\theta \frac{dy^3}{dt}$$

$$\text{ie. } \frac{dV_y}{dt} = \pi \tan^2\theta \, y^2 \frac{dy}{dt}$$

We can also define the volume flow of water at the hole:

$$\frac{dV}{dt} = v_b A_{hole} = \sqrt{2gy} \cdot \pi r^2$$

and equate these two volume flows, with a negative sign denoting a decrease in volume for positive v_b, viz:

$$\sqrt{2gy} \cdot \cancel{\pi} r^2 = -\cancel{\pi} \tan^2\theta \, y^2 \frac{dy}{dt}$$

$$\Rightarrow -\int_0^T \frac{\sqrt{2g} \cdot r^2}{\tan^2\theta} dt = \int_h^0 y^{\frac{3}{2}} dy$$

where T is the time taken for all of the water to run out.

$$\Rightarrow -\frac{\sqrt{2g}\cdot r^2}{\tan^2\theta}\left[t\right]_0^T = \left[\frac{2}{5}y^{\frac{5}{2}}\right]_h^0$$

$$-\frac{\sqrt{2g}\cdot r^2 T}{\tan^2\theta} = -\frac{2}{5}h^{\frac{5}{2}}$$

$$\Rightarrow T = \frac{2h^{\frac{5}{2}}\tan^2\theta}{5r^2\sqrt{2g}} = \frac{2h^{\frac{5}{2}}\left(\frac{R}{h}\right)^2}{5r^2\sqrt{2g}} = \frac{2R^2h^{\frac{1}{2}}}{5r^2\sqrt{2g}}$$

Substituting numerical values:

$$\Rightarrow T = \frac{2}{5}\left(\frac{3\times10^{-2}\,m}{1\times10^{-3}\,m}\right)^2\frac{(0.1\,m)^{\frac{1}{2}}}{\sqrt{2(9.81\,m/s^2)}} = \underline{26\,s.}$$

Answer:

(a) Conservation of momentum $\Rightarrow$ p(H) = p(Z) $\equiv$ p
Conservation of E then $\Rightarrow$ E = E(H) + E(Z) = sqrt(p^2 + M^2) + sqrt(p^2 + m^2)

where E = 7000, M=m(H)=130 and m = m(Z)=90

Solve for p, after some algebra: p = sqrt[(E^2 + m^2 – M^2)^2/(4 E^2) – m^2]

Plug in numbers to get p = 3498 GeV.

(b) First consider Higgs rest frame, in which b and bbar have equal momentum, called p_b.

In this frame,

E_b = M/2 = 65 GeV
p_b = sqrt[(M/2)^2 – m_b^2] which solves to 64.8 GeV.

To get to lab frame, need to boost, with gamma = E(H)/M = sqrt (p^2 + M^2)/M ~ 26.9.

MAXIMUM momentum of b in lab frame is when boost is in direction of b in rest frame, in which case p_b_lab = gamma (p_b + beta E_b), which is ~3495 GeV.

Now calculate flight distance using delta(s) = gamma beta c tau

gamma = E/m = sqrt(3495^2 + 5^2)/5 ~ 700
beta ~ 1

So, delta(s) = 700 (1) 3E8 (1E-12) m ~ 0.21 m.

1.1 Solution

Figure 2 shows the refracted and reflected rays. Applying the law of refraction yields

$$\frac{\sin\theta}{\sin\theta'} = \frac{n_w}{n_{air}}, \tag{1}$$

where n_w and n_air are the indices of refraction of water and air respectively. This reduces to $\theta' = \theta/n_w$.

We are looking for a distance d below the mirror where the image I of the object O is formed. In the triangle OAB we have

$$|AB| = d_1 \tan\theta \approx d_1\theta \tag{2}$$

and in the triangle CBD

$$|BC| = 2d_2 \tan\theta' \approx 2d_2\theta' \approx \frac{2d_2\theta}{n_w}. \tag{3}$$

Finally then, in the triangle ACI we have $|AI| = d + d_2$ and therefore

$$d = |AI| - d_2 = \frac{|AC|}{\tan\theta} - d_2 \approx \frac{|AB| + |BC|}{\theta} - d_2 = d_1 + \frac{2d_2}{n_w} - d_2 \tag{4}$$

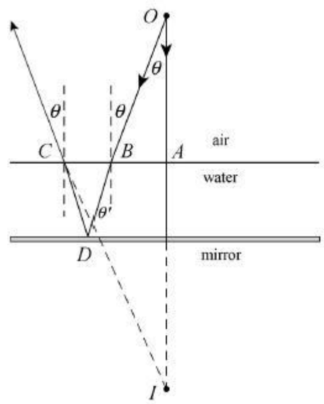


Figure 2: A bulb suspended over a swimming pool with a mirror.

General - #4 – Solution Hints

(a) We have heard it said by journalists that we may be able to identify extraterrestrial intelligence by observing visible light signals from aliens. The light would be transmitted by lasers. Assuming that we need a photon per minute in our detectors installed on a 2m telescope at Earth, provide an estimate on the power requirements of the alien laser. Is this feasible?

There can be many different ways to solve this problem. In a nutshell, diffraction will limit the beam-size at the Earth, and it is better for the Aliens to make a bigger and brighter laser if they wish us to detect them. Assuming a typical d~10kpc (3×10^{17}km) Galactic distance, a~100 km (e.g., expanded by a huge telescope only aliens can build…) for the alien laser aperture (diffraction limiting the alien laser), λ~600nm light wavelength, we get $D_{@Earth}$= 1.22 (d λ) / (2 a) = 1.1 x 10^6 km. This is about 3×10^{17} times larger than the surface area of our telescope at Earth. This means, that the alien laser must emit 5×10^{15} photons per second, that is equivalent of 1.6 mJ/s. Not totally crazy. If they use a much smaller aperture and more energy to signal us then the numbers change a little, 166kJ/s for a 10m aperture laser…

(b) Estimate the minimum speed that you, a typical person, would need in order to run on a water surface and not submerge.


United States Patent [19]
Tomlinson
[11] Patent Number: 4,787,871
[45] Date of Patent: Nov. 29, 1988
[54] WATER SURFACE RUNNING FINS FOR THE FEET

There can be many proper answers to this question. The classic answer is: The speed must be faster than the orbital velocity at the surface of the Earth. Of course this is huge and lower speeds are possible. (However running that fast is impossible.)

Some basilisk lizard species can run on water. Assume that they run with ~1.5m/s, and their height is about 3in. Lets assume that the exerted force by someone running on water is proportional to v*A, v being their velocity and A is the size of their foot. Assuming that A scales with l^2, l being the linear scale of the body, while weight scales with l^3. Keeping the exerted force over weight constant requires v to be proportional to l, i.e. a 6 foot giant "basilisk" has to run with ~36m/s.
People waterskiing barefoot do about 20 m/s according to the internet… BBC tells us that a human should run with the speed of 29 m/s to replicate the trick of the basilisk…

;-) ->

(c) Suppose you are being sold a pocket size 20 megapixel digital camera with a 3mm diameter lens, and a 35mm focal length. Does such a purchase make sense from the optical point of view? Justify your answer.

The diffraction limited spot diameter on the CCD is about 2 * 1.22 (f λ) / (D) = 2 * 1.22 (35mm 650nm) / 3mm = 18.5 μm. Therefore, it does not make sense to have a pixel that is smaller than ~20μm. This means that the area of the 20 megapixel CCD should be around 9cm x 9cm. Usually, the CCD is an order of magnitude smaller than this. Consequently, you should probably be suspicious about the performance of this digital camera. You would also need a huge pocket to hold it…

Statistical Mechanics Answers (Mal Ruderman)

a) $\boxed{\text{Zero photons}}$

~~(constant~~ probabilities are proportional to $\exp(-n_s \hbar\omega_s \beta)$ $\quad \beta = 1/k_B T$

b)

$$P(n_s) = \frac{e^{-n_s \hbar\omega_s \beta}}{\sum_{n_s'=1}^{\infty} e^{-n_s' \hbar\omega_s}}$$

$$= e^{-n_s \hbar\omega_s \beta}\left(1 - e^{-\hbar\omega_s \beta}\right)$$

c)

$$\langle n_s \rangle = \frac{\sum P(n_s)\, n_s}{\sum P(n_s) = 1}$$

$$= \frac{1}{e^{\hbar\omega_s \beta} - 1} \qquad \text{(Planck)}$$

d) $k = 1$ (with smallest $\hbar\omega_s$)

e)

$$U = \int_0^\infty \frac{\hbar\omega}{e^{\hbar\omega/kT} - 1}\, 8\pi k^2 dk \text{ Volume}$$

$$k = \sqrt{\epsilon\mu}\, \omega/c$$

index of refraction $n = (\epsilon\mu)^{1/2}$ and $k = n\omega/c$

$$\therefore U_{\epsilon\mu} = (\epsilon\mu)^{3/2}\, U_{\epsilon=\mu=1}$$

Quals Thermo Solution

Sunday, November 21, 2010
2:40 PM

$$dS = \frac{dQ}{T} = \frac{dU + p\,dV}{T} \text{ is a perfect differential}$$

$$= \frac{\left(\frac{\partial U}{\partial V}\Big|_T dV + \frac{\partial U}{\partial T}\Big|_V dT\right) + p\,dV}{T}$$

but $U(V,T) = BT^x \ln(V/V_0) + f(T)$

$$\therefore \quad \frac{\partial U}{\partial V}\Big|_T = \frac{BT^x}{V}, \quad \frac{\partial U}{\partial T}\Big|_V = BxT^{x-1}\ln(V/V_0) + f'(T)$$

and $p\,dV = \frac{5T^3}{V}dV$

$$\therefore \quad dS = \frac{BT^x}{TV}dV + \frac{BxT^{x-1}\ln(V/V_0)}{T}dT + \frac{f'(T)}{T}dT + \frac{5T^3}{TV}dV$$

$$= \left[\frac{BT^{x-1} + 5T^2}{V}\right]dV + \left[BxT^{x-2}\ln(V/V_0) + \frac{f'(T)}{T}\right]dT$$

But since

$$dS = \frac{\partial S}{\partial V}\Big|_T dV + \frac{\partial S}{\partial T}\Big|_V dT$$

then $\frac{\partial S}{\partial V}\Big|_T = \frac{BT^{x-1} + 5T^2}{V}$

and $\frac{\partial S}{\partial T}\Big|_V = BxT^{x-2}\ln(V/V_0) + \frac{f'(T)}{T}$

However $\frac{\partial}{\partial T}\left(\frac{\partial S}{\partial V}\right) = \frac{\partial}{\partial V}\left(\frac{\partial S}{\partial T}\right)$

$$\Rightarrow \quad \frac{\partial}{\partial T}\left[\frac{BT^{x-1} + 5T^2}{V}\right] = \frac{\partial}{\partial V}\left[BxT^{x-2}\ln(V/V_0) + \frac{f'(T)}{T}\right]$$

or $$\frac{B(x-1)T^{x-2} + 10T}{V} = \frac{BxT^{x-2}}{V}$$

or $$\frac{B(x-1)T^{x-2}+10T}{V} = \frac{BxT^{x-2}}{V}$$

$$\therefore \quad B(x-1)T^{x-2} + 10T = BxT^{x-2}$$

$$-BT^{x-2} + 10T = 0$$

$$-BT^{x-3} + 10 = 0$$

$$\Rightarrow \boxed{x = 3, \quad B = 10}$$