Columbia University
Department of Physics
QUALIFYING EXAMINATION

Friday, January 13, 2012
1:00PM to 3:00PM
General Physics (Part I)
Section 5.

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 6 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 5 (General Physics), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" × 11" paper (double-sided) you have prepared on General Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. Consider a material whose index of refraction, $n(y)$, changes gradually along one direction. Suppose an incoming ray enters the material almost parallel to the y-direction, which is the direction of the normal.

   (a) Find a general expression for the radius of curvature of the ray as it moves in the medium.

   (b) In particular, suppose the index of refraction changes as

   $$n(y) = n_0 e^{\alpha y}$$

   Discuss how the ray moves through the medium.

2. In an experiment, an electron of mass m and 3-momentum p_e collides with a positron (of same mass m) at rest. They annihilate, producing two photons. If one of the photons is emitted at an angle of 30 degrees with respect to the incoming electron's direction, what is its energy? (Ignore spins.)

3. We have a simple 3-dimensional metal which has a carrier density of n cm^{-3} and an effective mass m^*. We consider plasma oscillation by using a simple electron gas model, where a group of electrons moves collectively against a background lattice approximated by uniform and static positive charges.

   (a) Show that the frequency ω_p of the collective oscillation of electrons satisfies the following relationship
   $$\omega_p^2 \propto n/m^*$$

   (b) A basic property of metals "shining in light" is related to the plasma oscillation. Explain how and why.

   (c) When a voltage is applied to this metal and a current passes through it, electrons in this metal are scattered with an average time constant τ. Show that the DC-conductivity of this metal is proportional to $[n/m^*] \times \tau$. This is Ohm's law.

   (d) At temperatures sufficiently higher than the Debye temperature, resistivity of this metal, due to phonons, is proportional to the temperature T. Briefly explain the reasons why.

   (e) At low temperatures, much lower than the Debye temperature, the resistivity due to lattice vibration deviates from the linear T relationship due to the effects of (1) imperfection, and (2) reduction of the Umklapp process. Briefly how these effects modify the T-dependence of resistivity.

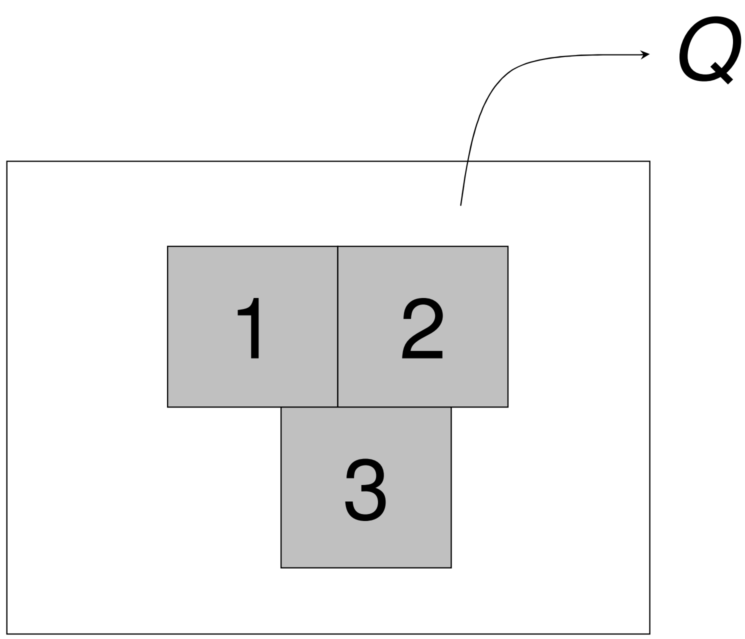
4.

(a) Assume that everybody on Earth will consume as much energy as we do in the US by 2050. Can the Earth's need for energy be satisfied by covering the Sahara Desert with solar panels? Does it make sense to cover the Sahara? What will it do to the global climate?

(b) What is the heat output of a human? Justify your answer.

(c) Show that if we increase the kinetic energy of an electron and a proton in a homogeneous magnetic field then orbital radii will get closer to each other's.

5. A system of three identical blocks of temperatures T_1, T_2, and T_3 have a heat capacity at constant volume $C_v = KT^{1/2}$. The three blocks are partially insulated from their surroundings and are brought into good thermal contact with each other. Assume that there is virtually no change in the volume of the blocks as they approach thermal equilibrium, but that heat Q leaks out of the system. Find an expression for the equilibrium temperature T_f of the three blocks.



6. Planck postulated that at temperature T a collection of $N \gg 1$ atomic oscillators of frequency ν share at random an integer number, $M \gg 1$ of energy packets $h\nu$, where h is some new universal constant. Then $U = E/N = Mh\nu$ is the average energy of each oscillator.

   (a) Use the Boltzmann/Planck definition of entropy in terms of the total number, $\Omega(N, M)$, of degenerate configurations to compute the entropy per oscillator $S(U, h\nu)$. (Hint: use the Stirling approximation $\log n! \approx n \log n$ for any $n \gg 1$.)

   (b) Use the Clausius thermodynamic relation between differential changes of the entropy and the internal energy to deduce the temperature $T(U, h\nu)$ in terms of U and $h\nu$, and then $U(h\nu, T)$.

   (c) Einstein's postulated that $h\nu$ should be interpreted instead as the energy of a corpuscular photon instead of mechanical oscillators. Show that part (b) also follows from the Bose-Einstein partition function of photons.

3 Non-uniform index of refraction

Consider a material whose index of refraction, $n(y)$, changes gradually along one direction. Suppose an incoming ray enters the material almost parallel to that axis. a) Find a general expression for the radius of curvature of the ray as it moves in the medium. b) In particular, suppose the index of refraction changes as

$$n(y) = n_0 exp(\alpha x) \tag{13}$$

Discuss how the ray moves through the medium.

4 Solution

Consider a small layer of medium with index of refraction $n(y)$ and thickness t. An incident ray with angle θ_1 refracts in the layer into an angle θ_I and finally exists the layer with an angle θ_2. All these angles are measured with respect to the normal, in this case the y-axis, and all three are small so we can use the approximation $sin\theta \approx \theta$. Snell's law relates the index of refraction above and below the layer n_1 and n_2 respectively.

$$n_1\theta_1 = n_2\theta_2 \tag{14}$$

$$n_1\theta_1 = \left(n_1 + \frac{dn}{dy} \cdot t\right) \cdot \theta_2 \tag{15}$$

where we used a Taylor expansion.

$$n\Delta\theta = \frac{dn}{dy} \cdot t \cdot \theta \tag{16}$$

where we dropped the i. Now the radius of curvature and $\Delta\theta$ are related by

$$\Delta\theta = \frac{t}{r} \tag{17}$$

which leads to

$$\frac{1}{r} = \frac{d}{dy} ln\left(n(y)\right) \tag{18}$$

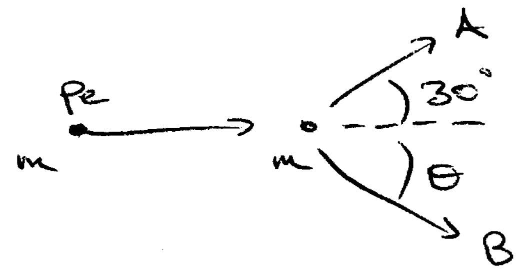
For the particular case of

$$n(y) = n_0 \cdot exp(\alpha \cdot y) \tag{19}$$

The radius of curvature ends up to be constant so we have a circular path.

Quals Section 5 Problem 2 : Corrected Solution

(Dodd)



Energy conservation :

$$\sqrt{p_e^2c^2 + m^2c^4} + mc^2 = E_A + E_B$$

Momentum conservation :

$$p_e = \frac{E_A}{c}\frac{\sqrt{3}}{2} + \frac{E_B}{c}\cos\theta \quad \text{(horiz.)}$$

$$0 = \frac{E_A}{c}\frac{1}{2} + \frac{E_B}{c}\sin\theta \quad \text{(vert.)}$$

$$\Rightarrow E_B^2\cos^2\theta = p_e^2c^2 - p_eE_Ac\sqrt{3} + \frac{3}{4}E_A^2$$

$$E_B^2\sin^2\theta = \frac{1}{4}E_A^2$$

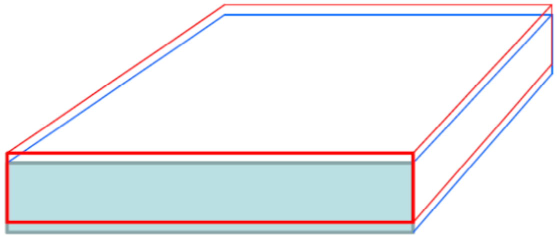
$$\Rightarrow E_B^2 = p_e^2c^2 - p_eE_Ac\sqrt{3} + E_A^2 = \left(\sqrt{p_e^2c^2+m^2c^4} + mc^2 - E_A\right)^2$$

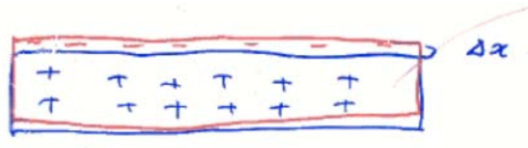
i.e. $p_e^2c^2 - p_eE_Ac\sqrt{3} + E_A^2 = p_e^2c^2 + m^2c^4 + 2mc^2\sqrt{p_e^2c^2+m^2c^4}$
$\quad - 2E_A\sqrt{p_e^2c^2+m^2c^4} + m^2c^4 - 2mc^2E_A + E_A^2$

$$\Rightarrow E_A = \frac{2mc^2\left(mc^2 + \sqrt{p_e^2c^2+m^2c^4}\right)}{2mc^2 + 2\sqrt{p_e^2c^2+m^2c^4} - p_ec\sqrt{3}}$$

$$= \frac{mc^2\left(mc^2 + \sqrt{p_e^2c^2+m^2c^4}\right)}{mc^2 + \sqrt{p_e^2c^2+m^2c^4} - p_ec\cos 30^\circ}$$

(or more elegantly , using 4-vectors ...)





$$\sigma = -n \cdot e \cdot \Delta x$$

$$E = 4\pi\, n \cdot e \cdot \Delta x$$

(a) The above figure shows positively charged lattice in blue and electrons in red. Suppose a vertical displacement of Δx is made. σ denotes the surface charge density and n is the 3 dimensional carrier density. The small displacement makes an effective capacitor, whose electric field E is given as above. Electrons with mass m and density n feel this electric field and oscillate as

$$n m \frac{d^2 x}{dt^2} = -n e E$$

$$= -4\pi n^2 e^2 x$$

$$x = e^{-i\omega_p t}$$

$$\omega_p = \sqrt{\frac{4\pi n e^2}{m}}$$

Points: 3 point

(b) This implies that to an ac electromagnetic field, electron cloud can respond by making collective displacement, as long as the frequency is smaller than the plasma frequency. This leads to total reflection of light and shining metallic surface. Electrons cannot respond to the light of higher frequency, which will go through the metal.

Give 2 points up to here (without math)

More precisely, if there is an ac electric field E with frequency ω,

suppose $\tau \to \infty$

$$m \frac{d^2}{dt^2} x = -eE \boxed{\cos(\omega t)}$$

$$m\omega^2 x = eE$$

$$x = \frac{eE}{m\omega^2}$$

Polarization

$$P = -nex$$

$$= -\frac{ne^2}{m\omega^2} E$$

ε : Dielectric Constant

$$\varepsilon = 1 + 4\pi P/E$$

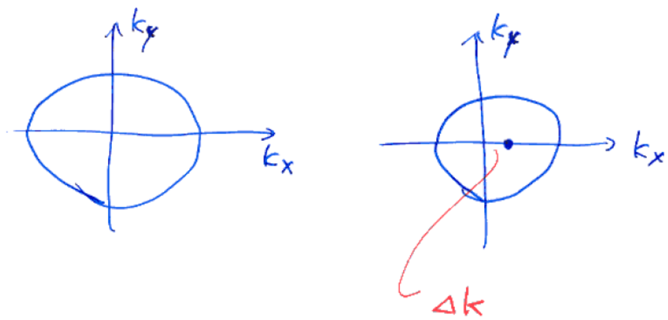
$$= 1 - \frac{4\pi ne^2}{m\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}$$

cf N (refraction index) $= \sqrt{\varepsilon}$

$\therefore$ $\omega < \omega_p$ .... Total Reflection

Give one more point for this kind of discussion with math. Total 3 points for (b).

(c) Suppose the dc electric field is applied along the x direction. The figure below shows the Fermi sphere in the momentum space. The electric field will generate a small shift of the momentum, but the scattering will lead to equilibrium with the drift velocity Δv



$$\hbar \Delta k = \Delta p = F \cdot \tau$$

$$F = eE$$

$$\longrightarrow \quad \frac{\hbar \Delta k}{m} = \Delta v = \frac{eE}{m}\tau$$

$$n \cdot e \cdot \Delta v = j = \frac{e^2 E n}{m}\tau$$

n: Carrier density $/cm^{-3}$

j: current density

$$j = S\,E$$

$$S = \frac{n}{m} e^2 \tau$$

S: Conductivity

Give 3 points for (c).

(d) Scattering of electrons by phonons at temperatures well above the Debye temperature. Debye temperature provides an energy scale of phonons (which are effective in scattering electrons with a large momentum change). Phonons are bosons. The number of phonons excited at T is proportional to the Bose Einstein distribution factor (often called occupation factor <n>)

$$= \frac{1}{\exp\left(\frac{\hbar\omega}{kT}\right) - 1} = \begin{cases} \simeq \exp\left(-\frac{\hbar\omega}{kT}\right), & \text{for } \hbar\omega \gg kT \\ \simeq \frac{kT}{\hbar\omega} & \text{for } \hbar\omega \ll kT \end{cases}$$

Planck Distribution

So, for the temperature higher than the energy sale of relevant excitation (Debye frequency), the occupation factor of phonons <n> is proportional to kT. This is the origin of the linear-T dependence of resistivity at the high temperature region.

(3 points for (d)).

(e) When there are multiple processes for scattering of electrons, the resistivity scales to the sum of the scattering rates $1/\tau$. At low temperatures, the phonon population is very small due to the above mentioned pupolation factor being proportional to exp(-E/kT). A larger contribution to the scattering rate comes from the T-independent scattering due to imperfection.

Resistivity ρ

$$\frac{1}{\tau} = \frac{1}{\tau_{phonon}} + \frac{1}{\tau_{impurity}}$$

$$\rho = \rho_{ph} + \rho_{imp}$$

This effect leads to T-independent resistivity at low T.

To be precise, the scattering of electrons by phonons effective in resistivity is the scattering with a large momentum transfer, which is represented by Umklapp process. The Umklapp scattering requires corresponding large energy transfer, because phonons have a linear dispersion relation between momentum and energy transfers. If the temperature becomes much lower than this “Umklapp phonon energy scale”, there are no longer sufficient phonons available for effective large-angle scattering of electrons. The Umklapp phonon temperature can roughly sale with Debye temperature. This is another reason why resistivity of metals due to phonons deviate from the linear-T behavior at low T.

(total 3 points for (e): give 2 point if either of the impurity or Umklapp process is correctly described: give 3 points when both of them are properly discussed.)

(a) Assume that everybody on Earth will consume as much energy as we do in the US by
2050. Can the Earth's need for energy be satisfied by covering the Sahara Desert with solar panels? Does it make sense to cover the Sahara? What will it do to the global climate?

N–Expected population of the Earth in 2050: **9 billion** (this may be too optimistic on the low side..)

W-Per capita Energy use in 2050 in the US: **12,000kWh/year** (currently it is ~12,800kWh/year, but declining … This may not be a number that one would know by heart. However we all look at our electricity bills – which could give an estimate, e.g. $150/ month at ~15 cents per kWh would give 12,000kWh consumption for a full year.)

U-Useful surface fraction of solar power stations: 50% -just a guess – design dependent

T-Efficiency of transport: 50% - assumes major advances

S-Average solar power incident per sq meter: The solar constant is 1.36kW/m^2. The Earth receives a total amount of radiation determined by its cross section (πr^2), but as it rotates this energy is distributed across the entire surface area ($4\pi r^2$). The average incoming solar radiation on the lit portion of the Earth, taking into account the angle at which the rays strike and that at any one moment half the planet does not receive any solar radiation, is one-fourth the solar constant (~ 340 W/m^2). Of this roughly half reaches the surface, the rest gets reflected or absorbed in the atmosphere. **S~170W/m^2**

E-Efficiency of conversion: about 25% (although by 2050 it may be twice as much)

D-Fraction of useful sunlight during daylight hours: about 80% (assume tracking of sun and good weather)

A-Area of Sahara: 3,600,000 square miles ~9,400,000km^2 (may estimate using the radius of Earth and estimate the size of Sahara compared to Earth surface)

Upper limit of power available from the Sahara < D A U S E T = $0.8 * 9.4 \times 10^{12} m^2 * 0.5 * 170 W/m^2 * 0.25 * 0.5 \approx$ **8×10^{13} W**

Lower limit of power needed in 2050 > N W = $9 \times 10^9 * 1.2 \times 10^7$ Wh/year * (1year / (365*24hour) $\approx$ **1.2×10^{13}W**

Based on assumptions above, solar panels on a vast portion of Sahara, can cover our energy need in the future. Should we do it? Well, just Google the topic and you will find the current status of the project plans and much discussion of its pros and cons…
(i.e., any reasonable, proper, and logical analysis/argument will be accepted)

(b) What is the heat output of a human? Justify your answer.

An average person consumes about 2500 Calories per day - based on food label readings. That actually is 2500kcal. (Note: when referring to food calories, the common practice in the US is to refer to them in multiples of 1,000. When you read 500 calories on a food label it actually means 500 kilocalories.)

P= 2500 kcal/day = (2,500,000 cal/day) x (4.184 J/1cal) x (1day/86400s)=121 J/s
=120W
(like a light bulb)

(c) Show that if we increase the kinetic energy of an electron and a proton in a homogeneous magnetic field then orbital radii will get closer to each other's.

The Larmor radius of a particle is

R = m*v / (|q|*B)

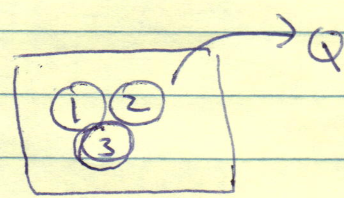
We are interested in the difference in the orbital radii of a proton and an electron:

$dR = R_p – R_e = (m_p*v_p – m_e*v_e) / (|q|*B)$
Assume we increase the kinetic energy of the electron and the proton with the same amount (E). The new velocities will be
$v_p' = sqrt[(1/2*m_p*v_p^2 + E)/(1/2*m_p*v_p^2)]*v_p = sqrt[1+2*E/(m_pv_p^2)]*v_p$
and the similar expression for electrons.
It is enough to show the statement for $E<<m_pv_p^2$ from which the general case will automatically follow. In this case,
$v_p' = v_p + E/(m_pv_p)$
$v_e' = v_e + E/(m_ev_e)$
And therefore the change in the difference between their orbital radii will be
$dR' – dR = (m_p* E/(m_pv_p) – m_e* E/(m_ev_e)) / (|q|*B)$
$= E * (1/v_p –1/v_e) / (|q|*B)$
Therefore dR' – dR > 0 for any small energy increase (we assume that electrons go faster than protons). Now if we assume that the energy of the proton and the electron are comparable, than dR < 0, and therefore a positive dR'-dR means that dR' is closer to zero than dR. Q.E.D.

Hailey: Thermo 1

1

A system of 3 identical blocks of temperatures T_1, T_2 and T_3 have a heat capacity at constant volume $C_V = K T^{1/2}$.
The 3 blocks are partially insulated from their surroundings and are brought into good thermal contact with each other. Assume that there is virtually no change in the volume of the blocks as they ~~reach~~ approach thermal equilibrium, but that heat Q leaks out of the system.



Find an expression for the equilibrium temperature T_F of the three blocks

Soln: $dQ = C_v dT + P dV \qquad dV = 0$

$$-Q = \sum_i \int_{T_i}^{T_f} C_v \, dT$$

$$-Q = \sum_{i=1}^{3} \int_{T_i}^{T_f} K T^{1/2} \, dT = \frac{3}{2} K \sum_i \left(T_f^{3/2} - T_i^{3/2}\right)$$

$$\left\{ \frac{1}{3} \left[T_1^{3/2} + T_2^{3/2} + T_3^{3/2} - \frac{2Q}{3K} \right] \right\}^{2/3} = T_f \qquad \text{Ans}_2$$

1. General section Quals 2012 Thermo and Stat Mech:

   Planck postulated that at temperature T a collection of $N \gg 1$ atomic oscillators of frequency ν share at random an integer number, $M \gg 1$ of energy packets $h\nu$, where h is some new universal constant. Then $U = E/N = Mh\nu$ is the average energy of each oscillator.

   a) Use the Boltzmann/Planck definition of entropy in terms of the total number, $\Omega(N, M)$, of degenerate configurations to compute the entropy per oscillator $S(U, h\nu)$. (Hint: use the Stirling approximations $\log n! \approx n \log n$ for any $n \gg 1$)

   b) Use the Clausius thermodynamic relation between differential changes of the entropy and the internal energy to deduce the temperature $T(U, h\nu)$ in terms of U and $h\nu$, and then $U(h\nu, T)$.

   c) Einstein's postulated that $h\nu$ should be interpreted instead as the energy of a corpuscular photon intead of mechanical oscillators. Show that part b also follows from the Bose-Einstein partition function of photons.

General (Thermo + Stat)

MG: GpL

a) $\Omega = \frac{(N+M-1)!}{(N-1)!\,M!}$ combinatoric # ways of partitiony M packets (of $h\nu$) into N boxes (oscillators)

For $N \gg 1$ and $M \gg 1$

we can use Stirling $\log N! = N \log N + O(N)$

$$\log \Omega \cong (N+M)\log(N+M) - N\log N - M \log M$$

$$= N\left(1+\frac{M}{N}\right)\left(\log N + \log\left(1+\frac{M}{N}\right)\right)$$

$$- N \log N - N\left(\frac{M}{N}\right)\left(\log N + \log M/N\right)$$

$$= N\left\{\left(1+\frac{E/h\nu}{E/u}\right)\log\left(1+\frac{u}{h\nu}\right) - \frac{u}{h\nu}\log\frac{u}{h\nu}\right\}$$

$\Rightarrow$ entropy per oscillator $s = \frac{S_N}{N} = \frac{k_B}{N}\log\Omega$

$$\frac{1}{k_B} s[u, h\nu] = \left(1+\frac{u}{h\nu}\right)\log\left(1+\frac{u}{h\nu}\right) - \frac{u}{h\nu}\log\frac{u}{h\nu}$$

b) Clausius $dS_N = \frac{1}{k_B T}\, dE_N$

$$ds = \frac{1}{k_B T}\, du = \frac{1}{h\nu}\left(\log\left(1+\frac{u}{h\nu}\right) - \log u/h\nu\right)$$

$$\Rightarrow \quad T(u, h\nu) = \frac{h\nu}{k_B}\left(\log\left(\frac{h\nu}{u}+1\right)\right)^{-1}$$

General thermo + Stat

MG: Gp2

c) $\frac{h\nu}{k_BT} = \log\left(\frac{h\nu}{u}+1\right)$

$$e^{h\nu/k_BT} = \frac{h\nu}{u} + 1$$

$$\Rightarrow \boxed{u = h\nu \frac{1}{e^{h\nu/k_BT}-1}}$$

d) $Z = \sum_{n=0}^{\infty} e^{-nh\nu/k_TT} = \frac{1}{1-e^{-h\nu/k_BT}}$

$$u = -\frac{\partial}{\partial(1/kT)} \log Z = \frac{(-1)(-1)\left(-(-h\nu)e^{-h\nu/kT}\right)}{1-e^{-h\nu/kT}}$$

$$= h\nu\left(\frac{1}{e^{h\nu/kT}-1}\right)$$

Bose + Einstein also assumed 2 independent polarization d.o.f. of photons in accord with Maxwell

$$\Rightarrow Z \to Z^2 \qquad u \to 2u$$

or u is the mean energy per polarization state