Columbia University
Department of Physics
QUALIFYING EXAMINATION

Friday, January 13, 2012
3:10PM to 5:10PM
General Physics (Part II)
Section 6.

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 6 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 6 (General Physics), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" × 11" paper (double-sided) you have prepared on General Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

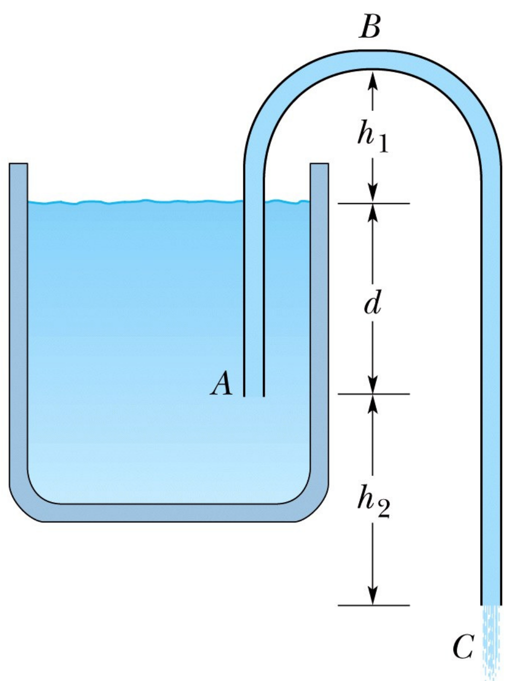
1. The figure shows a siphon, which is a device for removing liquid from a container. Tube $ABC$ must initially be filled, but once this has been done, liquid will flow through the tube until the liquid surface in the container is level with the tube opening at $A$. The liquid has density $\rho$ and negligible viscosity.

   (Bernoullis equation: $p + \frac{1}{2}\rho\nu^2 + \rho g h = \text{constant}$)

   (a) At what speed does the liquid emerge from the tube at $C$?

   (b) With respect to atmospheric pressure, what is the pressure in the liquid at the topmost point $B$?

   (c) What is the greatest possible height $h_1$ for which the siphon will operate?



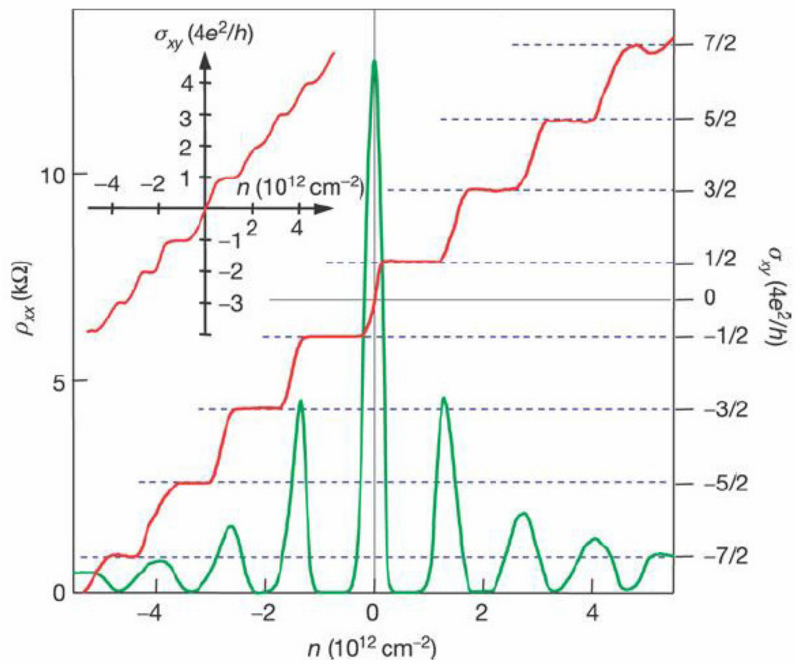
2. The hydrogen atom states $2S$ and $2P$ (*i.e.* their orbital angular momenta are 0 and 1, respectively) are split by a small "Lamb shift" energy $\hbar\omega_0 = 2\pi\hbar \times 1.06$ GHz.

   (a) Find the eigenenergies of these states in an applied electric field $\vec{\mathcal{E}} = \mathcal{E}\hat{z}$. Write down the low-field and the high-field approximations. Sketch these eigenenergies vs. the field strength $\mathcal{E}$. (Use the matrix element $\langle 2P|\, z\, |2S\rangle = 3a_0$, where the Bohr radius $a_0 \approx 0.053$ nm.)

   (b) How large should the electric field be for the energy shifts (*i.e.* Stark shifts) to become linear?

   (c) What is the value of the linear Stark shift at large electric fields?

3.




The figure above shows the quantum Hall effect in graphene.

(a) Why are the units of quantization of the Hall conductivity $4e^2/h$ and not $e^2/h$?

(b) What is the difference between this Hall effect and the one in conventional 2D systems?

(c) Find, from the plot, the magnitude of the magnetic field at which this measurement was performed.

4. A hollow sphere of radius $R$ has a conducting boundary wall. The different possible normal modes of electromagnetic fields within it have (angular) frequencies

$$\omega_0 \leq \omega_1 \leq \omega_2 \cdots \leq \omega_s \leq \cdots \omega_\infty .$$

The container wall and the photons within the hollow sphere are in thermal equilibrium at temperature $T$.

(a) What is the probability for finding just <u>one</u> photon in the mode with frequency $\omega_s$?

(b) What is the probability to for finding <u>no</u> photons in that mode?

(c) What is the average total energy in photons in that mode?

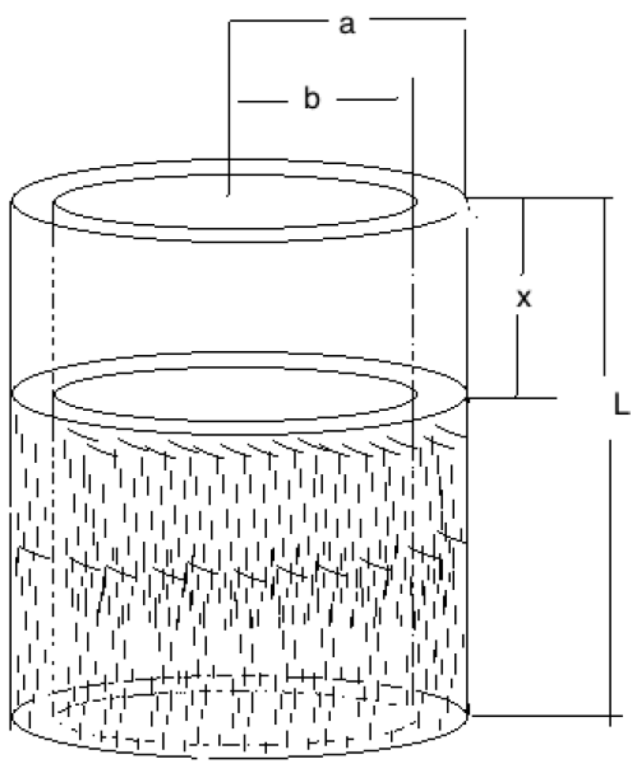
(d) What is the most probable total photon energy in this mode?

(e) What is the probability for finding a total of zero photons in the two lowest frequency modes?

(f) Suppose the spherical container wall is replaced by a perfectly reflecting mirror. The spherical mirror radius is then slowly changed from $R$ to $\alpha R$. How does the average number of photons in the $s^{\text{th}}$ mode change?

(g) How does the photon temperature within the container change?

5. A capacitive levelmeter relies on the difference in dielectric constants of a liquid $\epsilon_L$ and a gas $\epsilon_g$ to measure the height of the liquid in a given volume. Consider two concentric metallic cylindrical shells, with radius $a$ and $b$ ($a > b$) and length $L$ ($L \gg a,\ b$). Determine the height of the liquid in the levelmeter as a function of the capacitance. As the liquid fills the space between the shells in the levelmeter, the pressure of the gas trapped above the liquid level increases and the dielectric constant of the gas changes as a power law. Discuss the behavior of the capacitance as a function of the height of the liquid in the column for different exponents.



6. A compact object emits a steady spherically-symmetric hot wind. The wind has an unknown Lorentz factor $\gamma(r) \gg 1$, where $r$ is radius (distance from the central source). The wind is made of protons, electrons, and photons, all in common thermodynamic equilibrium in the local rest-frame of the wind. The wind is adiabatic (no energy losses). Its energy density $U$ (measured in the rest frame and including rest-mass energy) is dominated by photons and related to the wind temperature $T(r)$ by $U = aT^4$ where $a$ is the radiation constant.

   (a) How is the entropy density in the wind (measured in its rest frame) related to temperature $T$?

   (b) Let $n_\gamma(r)$ and $n_p(r)$ be the photon and proton number densities measured in the rest frame. How does the ratio $n_\gamma/n_p$ scale with radius?

   (c) How does the Lorentz factor of the wind scale with radius? [Hint: use conservation of proton number and conservation of energy in the adiabatic outflow.]

   You can use the following relations without derivation: radiation pressure (measured in the wind rest frame) is $P = U/3$, energy density in the lab frame is $\gamma^2 U$, and proton density in the lab frame is $\gamma n_p$.

Quals Problem Fluids:

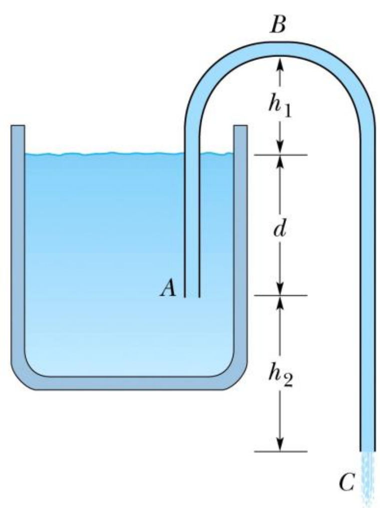
The figure shows a siphon, which is a device for removing liquid from a container. Tube *ABC* must initially be filled, but once this has been done, liquid will flow through the tube until the liquid surface in the container is level with the tube opening at *A*. The liquid has density ρ and negligible viscosity.

(Bernoulli's equation: $p + \frac{1}{2}\rho v^2 + \rho g h = constant$)

(a) With what speed does the liquid emerge from the tube at *C*?

(b) With respect to atmospheric pressure, what is the pressure in the liquid at the topmost point *B*?

(c) What is the greatest possible height $h_1$ for which the siphon will operate?



Solution:

a) Applying Bernoulli's equation to points $D$ (top of container liquid) and $C$, we obtain

$p_D + \frac{1}{2}\rho v_D^2 + \rho g h_D = p_C + \frac{1}{2}\rho v_C^2 + \rho g h_C$ which leads to

$v_C = \sqrt{\frac{2(p_D - p_C)}{\rho} + 2g(h_D - h_C) + v_D^2} \approx \sqrt{2g\,(d + h_2)}$ since $p_D = p_C = p_{\text{air}}$ and $v_D/v_C \approx 0$.

b) Using points $B$ and $C$ with $p_B + \frac{1}{2}\rho v_B^2 + \rho g h_B = p_C + \frac{1}{2}\rho v_C^2 + \rho g h_C$ and $v_B = v_C$ by equation of continuity plus $p_C = p_{\text{air}}$ gives

$p_B = p_C + \rho g(h_C - h_B) = p_{\text{air}} - \rho g\,(h_1 + h_2 + d)$

c) Since $p_B \geq 0$ for the siphon to operate, $p_{\text{air}} - \rho g(h_1 + d + h_2) \geq 0$ means that

$h_1 \leq \frac{p_{\text{air}}}{\rho g} - d - h_2$

## GENERAL PHYSICS – ATOMIC

### Hydrogen atom in an electric field.

The hydrogen atom states $2S$ and $2P$ (i.e. their orbital angular momenta are 0 and 1, respectively) are split by a small "Lamb shift" energy $\hbar\omega_0 = 2\pi\hbar \times 1.06$ GHz.

(a) Find the eigenenergies of these states in an applied electric field $\vec{\mathcal{E}} = \mathcal{E}\hat{z}$. Write down the low-field and the high-field approximations. Sketch these eigenenergies vs. the field strength $\mathcal{E}$. (Use the matrix element $\langle 2P|z|2S\rangle = 3a_0$, where the Bohr radius $a_0 \approx 0.053$ nm.)

(b) How large should the electric field be for the energy shifts (i.e. Stark shifts) to become linear?

(c) What is the linear Stark shift at large electric fields?

**GENERAL PHYSICS – ATOMIC**

**Hydrogen atom in an electric field. SOLUTION.**

(a) In the absence of the field, the Hamiltonian is

$$H_0 = \hbar \begin{pmatrix} \omega_0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{1}$$

The perturbation due to the electric field is

$$H_1 = e\mathcal{E}z = \hbar \begin{pmatrix} 0 & V \\ V & 0 \end{pmatrix}, \tag{2}$$

where $V = 3ea_0\mathcal{E}/\hbar$ (since the dipole moment operator $ez$ couples states of opposite parity, i.e. $S$ and $P$).

This is a two-state problem, and the eigenvalues are

$$\lambda_\pm = \frac{\hbar\omega_0}{2}\left(1 \pm \sqrt{1 + \frac{4V^2}{\omega_0^2}}\right). \tag{3}$$

Limiting cases:

$$\begin{aligned} |V| \ll \omega_0 : \quad & \lambda_\pm \simeq \frac{\hbar\omega_0}{2}\left(1 \pm \left(1 + \frac{2V^2}{\omega_0^2}\right)\right) \\ & \Rightarrow \begin{cases} \lambda_+ \simeq \hbar(\omega_0 + V^2/\omega_0) \\ \lambda_- \simeq -\hbar V^2/\omega_0. \end{cases} \end{aligned} \tag{4}$$

$$\begin{aligned} |V| \gg \omega_0 : \quad & \lambda_\pm \simeq \frac{\hbar\omega_0}{2}\left(1 \pm \frac{2|V|}{\omega_0}\right) \\ & \Rightarrow \begin{cases} \lambda_+ \simeq \hbar(\omega_0/2 + |V|) \\ \lambda_- \simeq \hbar(\omega_0/2 - |V|) \end{cases} \end{aligned} \tag{5}$$

The Stark shift is quadratic at low fields and linear at high fields. This is sketched in Fig. 1.

(b) From part (a), the critical value of the electric field $\mathcal{E}_0$ occurs when $\lambda_\pm$ change the behavior from quadratic to linear in $V$:

$$\begin{aligned} & \frac{4V_0^2}{\omega_0^2} \simeq 1, \quad V_0 \simeq \frac{\omega_0}{2}, \\ & \mathcal{E}_0 \simeq \boxed{\frac{\hbar\omega_0}{6ea_0}} = \frac{(6.63 \times 10^{-34}\mathrm{J \cdot s})(1.06 \times 10^9 \mathrm{s}^{-1})}{6(1.60 \times 10^{-19}\mathrm{C})(5.29 \times 10^{-9}\mathrm{cm})} \approx \boxed{140\ \mathrm{V/cm.}} \end{aligned} \tag{6}$$

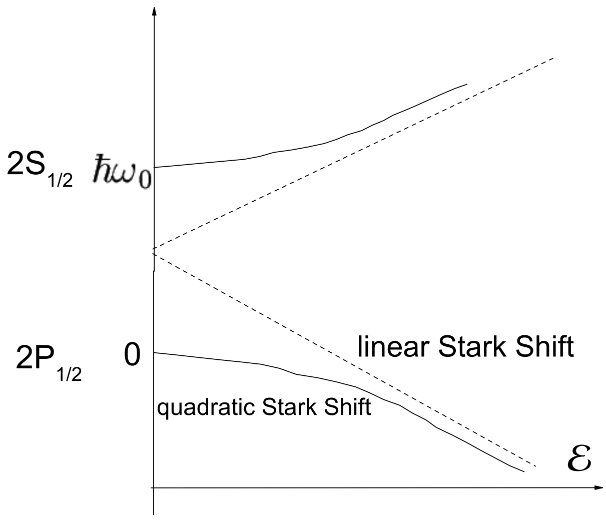


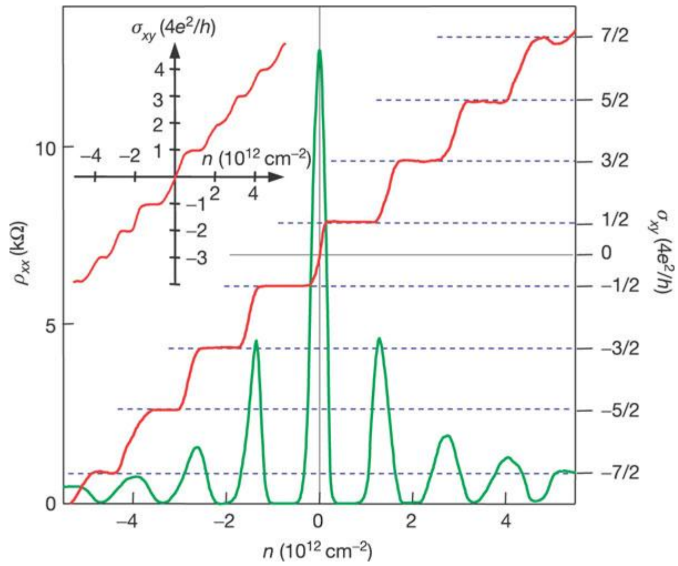

Figure 1: The hydrogen dc Stark shifts.

(c) From part (a), the large-field eigenvalues are

$$\lambda_\pm = \hbar\left(\frac{\omega_0}{2} \pm |V|\right), \tag{7}$$

hence the Stark shift (in Hz per unit electric field) is

$$\frac{1}{2\pi}\frac{dV}{d\mathcal{E}} = \boxed{\frac{3ea_0}{h}} = \frac{3(1.60\times 10^{-19}\mathrm{C})(5.29\times 10^{-9}\mathrm{cm})}{6.63\times 10^{-34}\mathrm{J}\cdot\mathrm{s}} \approx \boxed{3.8\ \mathrm{MHz/(V/cm).}} \tag{8}$$




Condensed matter problem:

The figure above shows the quantum Hall effect in graphene (K. S. Novoselov, A. K. Geim, S. V. Morozov, D. Jiang, M. I. Katsnelson, I. V. Grigorieva, S. V. Dubonos and A. A. Firsov Nature, 2006)

1) Why the units of quantization of the Hall conductivity are 4e^2/h and e^2/h.

2) What is the difference between this Hall effect and the one in conventional 2D systems?

3) Find from the plot the magnitude of the magnetic field at which this measurement was performed.

Answers:

1. Valley and spin degeneracy of the Landau levels.
2. The plateaus are quantized in half-integer value;
3. The density difference for the first and second plateaus is \delta n=10^{12} cm^{-2}; On the other hand \delta n 2\pi \lambda_b^2 = 4, Where lambda_B=\sqrt{c\hbar/(eB)} is the magnetic length. The result is 14 Tesla.

Problem for Statistical Mechanics

M. Ruderman

A hollow sphere of radius R has a conducting boundary wall: the different possible normal modes of electromagnetic fields ~~field~~ within it have (angular) frequencies

$$\omega_0 \leq \omega_1 \leq \omega_2 \ldots \leq \omega_s \leq \ldots \omega_\infty$$

The container wall and the photons within the hollow are in thermal equilibrium at temperature T.

a) What is the probability for finding just <u>one</u> photon in the mode with frequency $\omega_s$?

b) What is the probability for finding <u>no</u> photons in that mode?

c) What is the average total energy in photons in that mode?

d) What is the most probable total photon energy in this mode?

e) What is the probability for finding a total of zero photons in the two lowest frequency modes?

f) Suppose the spherical container wall is replaced by a perfectly reflecting mirror. The spherical mirror radius is then slowly changed from R to $\alpha R$.

How does the average number of photons in the $s^{th}$ mode change?

g) How does the photon temperature within the container change?

# Answers for Statistical Physics Problem of M. Ruderman

$$P_\ell(n) = \frac{e^{-n\hbar\omega_\ell/kT}}{\sum_{n=0}^{\infty} e^{-n\hbar\omega_\ell/kT}} = \frac{\text{Boltzmann factor}}{\text{Sum over states}}$$

a) $P_\ell(1) = e^{-\hbar\omega_\ell/kT}\left(1 - e^{-\hbar\omega_\ell/kT}\right)$

$$\left[\text{Note: } \frac{1}{\sum_{n=0}^{\infty} x^n} = 1 - x \quad \text{when } x < 1\right]$$

b) $P_\ell(0) = \left(1 - e^{-\hbar\omega_\ell/kT}\right)$

c) $\bar{E}_\ell = \sum_{n=0}^{\infty} n\hbar\, P_\ell(n) = \dfrac{\hbar\omega_\ell}{e^{\hbar\omega_\ell/kT} - 1}$

(Planck distribution)

d) $P_\ell(n)$ is a maximum for $n=0$. Therefore most probable photon energy $= 0$ (not counting zero-point energy)

e) $P_{\ell=0}(0) \times P_{\ell=1}(0) = \left(1 - e^{\hbar\omega_0/kT}\right)\left(1 - e^{\hbar\omega_1/kT}\right)$

f) $P_\ell(n)$ is unchanged!

g) $\dfrac{T(R)}{T(\alpha R)} = \dfrac{1}{\alpha}$

$\cdot\, \omega_\ell$ proportional to $\dfrac{R}{c}$

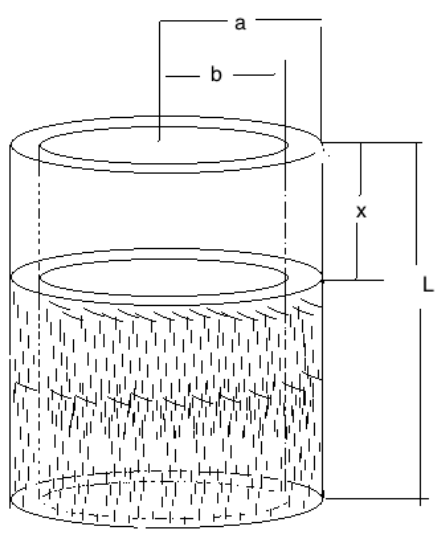
f) implies $\dfrac{\hbar\omega_\ell}{kT}$ is unchanged.

Quals-Problems-Aprile-Nov 2011

November 28, 2011

## 1 A model for a levelmeter

A capacitive levelmeter relies on the difference in dielectric constants of a liquid $\epsilon_L$ and a gas $\epsilon_g$ to measure the height of the liquid in a given volume. Consider two concentric metallic cylindrical shells, with radius $a$ and $b$ ($a > b$) and lenght $L$ ($L \gg a, b$). Determine the height of the liquid in the levelmeter as a function of the capacitance. As the liquid fills the space between the shells in the levelmeter, the pressure of the gas trapped above the liquid level increases and the dielectric constant of the gas changes as a power law. Discuss the behaviour of the capacitance as a function of the height of the liquid in the column for different exponents.



## 2 Solution

The capacitance of the system can be computed from the dependence of the electric potential on the charge. Consider a section of a cylinder of hight $L_S$ and radius $r$ sharing the same axis of symmetry of the shells but lying in between them. Let's start with Gauss' law for electric field on a closed surface.

$$\oint_S \vec{E} \cdot \hat{n} dS = \frac{Q_{enc}}{\epsilon_0} \tag{1}$$

On the surface described above, only the electric field that crosses the curved surface contributes to the integral. The enclosed charge is proportional to the fraction of $L_S$ to $L$ the total length of the system assuming the charge is uniformly distributed. The electric field in the region $b < r < a$ is then

$$E = \frac{1}{2\pi\epsilon_0} \cdot \frac{Q}{L} \cdot \frac{1}{r} \tag{2}$$

Now, the electric potential and electric field are related by

$$V = -\oint_P \vec{E} \cdot d\vec{s} \tag{3}$$

For our purposes, $P$ is a path from the surface at radius $b$ to the surface at radius $b$. Therefore

$$V = \frac{1}{2\pi\epsilon_0} \cdot \frac{Q}{L} \cdot ln\left(\frac{a}{b}\right) \tag{4}$$

By definition, the capacitance $C$ and the potential $V$ are linearly related $Q = CV$ so

$$C_0 = \frac{2\pi\epsilon_0 L}{ln\left(\frac{a}{b}\right)} \tag{5}$$

The generalization of this result to arbitrary dielectric constant $\epsilon$ is straight forward.

$$C(L,\epsilon) = \frac{2\pi\epsilon\epsilon_0 L}{ln\left(\frac{a}{b}\right)} \tag{6}$$

$$C(L,\epsilon) = C_0\epsilon \tag{7}$$

The system can be treated as two capacitors in parallel since the share the same potential.

$$C_T = C(x,\epsilon_G) + C(L-x,\epsilon_L) \tag{8}$$

$$C_T = C_0\left[\epsilon_G \cdot \frac{x}{L} + \epsilon_L \cdot \left(1 - \frac{x}{L}\right)\right] \tag{9}$$

If the pressure and the dielectric constant are related by a power-law

$$\epsilon_G \propto P^\alpha \tag{10}$$

the dielectric constant will be related to the height of the gas column on the shells, so

$$\epsilon_G = \propto x^{-\alpha} \tag{11}$$

and the capacitance is then

$$C_T = C_0 \left[ B \cdot \left( \frac{x}{L} \right)^{-\alpha+1} + \epsilon_L \cdot \left( 1 - \frac{x}{L} \right) \right] \tag{12}$$

For $\alpha > 1$ as we increase the height of the gas column in the system, the capacitance increases (decreases) monotonically which makes it ideal to calibration purposes. If $\alpha < 1$ the system exhibits a minimum (maximum) in capacitance, which means that for most values of capacitance we have two possible column heights. However as the exponent increases the position of the maximum or minimum gets closer to $x = 0$ and again the calibration of the system is easy.

**General: astro**

A compact object emits a steady spherically-symmetric hot wind. The wind has an unknown Lorentz factor $\gamma(r) \gg 1$, where $r$ is radius (distance from the central source). The wind is made of protons, electrons, and photons, all in common thermodynamic equilibrium in the local rest-frame of the wind. The wind is adiabatic (no energy losses). Its energy density $U$ (measured in the rest frame and including rest-mass energy) is dominated by photons and related to the wind temperature $T(r)$ by $U = aT^4$ where $a$ is the radiation constant.

(a) How is the entropy density in the wind (measured in its rest frame) related to temperature $T$?

(b) Let $n_\gamma(r)$ and $n_p(r)$ be the photon and proton number densities measured in the rest frame. How does the ratio $n_\gamma/n_p$ scale with radius?

(c) How does the Lorentz factor of the wind scale with radius? [Hint: use conservation of proton number and conservation of energy in the adiabatic outflow.]

You can use the following relations without derivation: radiation pressure (measured in the wind rest frame) is $P = U/3$, energy density in the lab frame is $\gamma^2 U$, and proton density in the lab frame is $\gamma n_p$.

**Solution:**

**(a)** Entropy density $S$ may be calculated by considering heating from $T = 0$ at fixed volume $V = 1$,

$$TdS = d(aT^4) \quad \Rightarrow \quad S = \frac{4}{3}\,aT^3.$$

**(b)** Average energy of a Planckian photon $\bar{E}$ is proportional to $T$ ($\bar{E} \approx 2.7kT$) and hence

$$n_\gamma = \frac{U}{\bar{E}} \propto T^3 \propto S.$$

Entropy is conserved in the adiabatic flow, and the proton number is conserved, hence $S/n_p = const$ and $n_\gamma/n_p = const$.

**(c)** Write down Bernoulli equation (energy conservation) and proton conservation for the steady relativistic flow with the spherical cross section $4\pi r^2$,

$$\dot{E} = 4\pi r^2 c\,\gamma^2(U + P) = const \quad \Rightarrow \quad r^2\gamma^2 T^4 = const, \tag{1}$$

$$\dot{N} = 4\pi r^2 c\,\gamma n_p = const \quad \Rightarrow \quad r^2\gamma n_p = const. \tag{2}$$

Dividing the two equations and using $T^3/n_p \propto S/n_p = const$, one finds

$$\gamma T = const \tag{3}$$

From equations (1) and (3) one finds $\gamma \propto r$.