Columbia University
Department of Physics
QUALIFYING EXAMINATION

Monday, January 14, 2013
1:00PM to 3:00PM
Classical Physics
Section 1. Classical Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 1 (Classical Mechanics), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

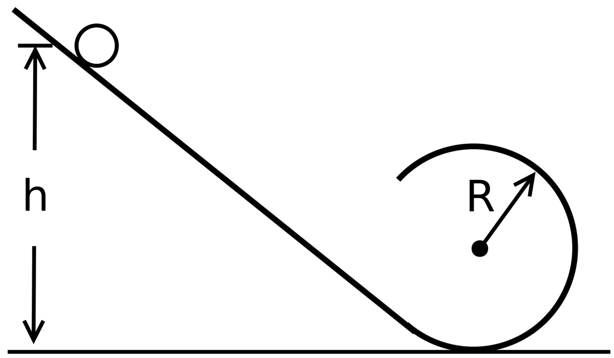
Good Luck!

1. Consider a rocket that has a constant fuel burn rate (i.e. mass consumption rate), α, and a constant exhaust velocity, u.

   (a) If the rocket starts from rest in free space by emitting mass, at what fraction of the initial mass is its momentum a maximum?

   (b) If the rocket launches from the surface of the Earth, what is the minimum u such that the rocket will lift off immediately after firing? What is the rocket's velocity in the early stages of the ascent? Assume a vertical assent.

2. As shown in the figure, a solid brass ball of mass 0.271 g will roll smoothly along a loop-the-loop track when released from rest along a straight section. The circular loop has radius $R = 0.05$ m, and the ball has radius $r \ll R$. What is h if the ball is on the verge of leaving the track when it reaches the top of the loop?



3. A child of mass m on a swing raises her center of mass by a small distance b every time the swing passes the vertical position, and lowers her center of mass by the same amount b at each extremal position (i.e. at the end-point of every swing).

   (a) Assuming small oscillations, calculate the work done by the child per period of oscillation.

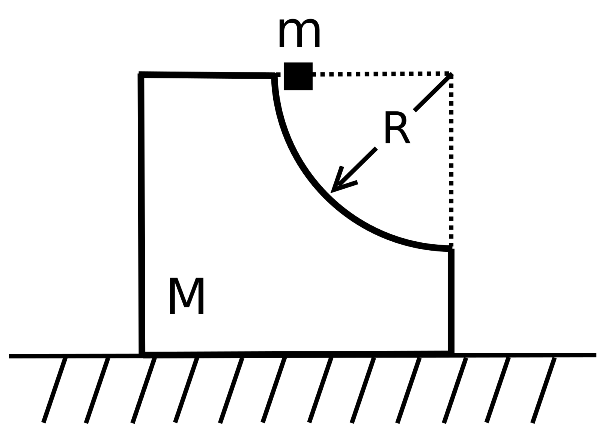
   (b) Show that the energy of the swing grows exponentially according to

   $$\frac{dE}{dt} = \alpha E \tag{1}$$

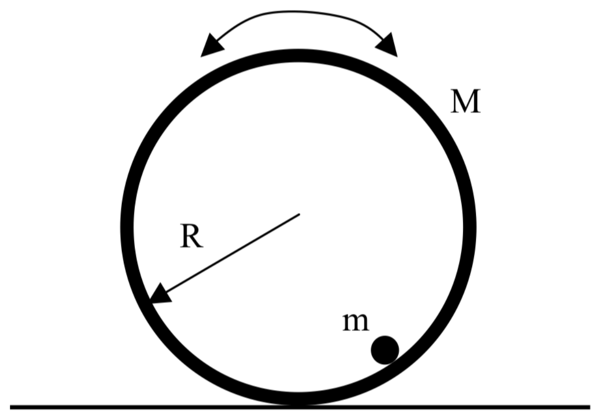
   and determine the constant α.

4. A mass m slides down a circularly curved surface on an object with mass M as shown in the diagram below. Mass M is free to slide on a frictionless surface.

   What are the final speeds of the two masses after m separates from M?



5. A point particle of mass m slides without friction within a hoop of radius R and mass M. The hoop is free to roll without slipping along a horizontal surface. What is the frequency of small oscillations of the point mass, when it is close to the bottom of the hoop?



Columbia University Department of Physics
Mechanics Quals Problem

Prof. Aprile

November 5, 2012

Problem: Rocket Motion

Consider a rocket that has a constant fuel burn rate (i.e. mass consumption rate), α, and a constant exhaust velocity, u.

Part 1: Free Space

If the rocket starts from rest in free space by emitting mass, at what fraction of the initial mass is its momentum a maximum?

Part 2: Ascent under Gravity

If the rocket launches from the surface of Earth, what is the minimum u such that the rocket will lift off immediately after firing? What is the rocket's velocity in the early stages of the ascent? Assume a vertical ascent.

Solution

Part 1

We assume there are no external forces acting on the rocket, and we choose a closed system so Newton's 2nd Law applies. Then, part 1 is simply a conservation of linear momentum problem.

First, we must find the velocity of the rocket. At a time t, let the instantaneous mass of the rocket be m, and its velocity be v relative to some inertial reference frame. Then, during a time dt, a mass dm' is ejected with speed $-u$ with respect to the rocket. So, conservation of linear momentum gives:

$$\begin{aligned} p(t) &= p(t+dt) \\ mv &= (m - dm')(v + dv) + dm'(v - u) \\ &\Rightarrow mdv = udm' \\ &\Rightarrow dv = -u\frac{dm}{m} \end{aligned}$$

using $dm = -dm'$ and dropping the product of differentials.

Let the initial mass be m_0 and the initial speed be 0. Then, integrating the above equation gives:

$$\begin{aligned} \int_0^v dv &= -u\int_{m_0}^{m} \frac{dm}{m} \\ v &= u \ln\left(\frac{m_0}{m}\right) \end{aligned}$$

and hence $p = mu \ln\left(\frac{m_0}{m}\right)$.

To find when p is maximized, we take the derivative and set it to 0:

$$\begin{aligned} \dot{p} &= \alpha u \ln\left(\frac{m_0}{m}\right) + mu\left(\frac{m}{m_0}\right)\left(\frac{-m_0\alpha}{m^2}\right) = 0 \\ &\Rightarrow \ln\left(\frac{m_0}{m}\right) = 1 \\ &\Rightarrow \boxed{m = \frac{m_0}{e}} \end{aligned}$$

Part 2

Now there is an external force, so

$$F_{ext} = \frac{dp}{dt} \Rightarrow F_{ext}dt = dp = p(t+dt) - p(t)$$

From part 1, we know

$$p(t+dt) - p(t) = mdv + udm$$

We assumed $\dot{m} = \alpha$, and since in a vertical ascent at the surface of the earth, $F_{ext} = -mg$

$$\begin{aligned} F_{ext}dt &= -mgdt = mdv + udm \\ &\Rightarrow dv = \left(-g - \frac{\alpha u}{m}\right) dt \end{aligned}$$

Using $\dot{m} = \alpha$ once again to eliminate dt

$$dv = \left(\frac{-g}{\alpha} - \frac{u}{m}\right) dm$$

To ensure $dv > 0$ when the engine fires, it is required that $\left(\frac{-g}{\alpha} - \frac{u}{m_0}\right) > 0$, and hence

$$\boxed{u > -\frac{gm_0}{\alpha}}$$

To find v at any time shortly after liftoff (so we can assume the acceleration due to gravity is well approximated to be g), we must integrate the above equation for dv, as in part 1

$$\begin{aligned} \int_0^v dv &= \int_{m_0}^m \left(\frac{-g}{\alpha} - \frac{u}{m}\right) dm \\ v &= -\frac{g}{\alpha}(m - m_0) + u \ln\left(\frac{m_0}{m}\right) \end{aligned}$$

Integrating $dm = \alpha dt$ gives $m - m_0 = \alpha t$, and hence we obtain

$$\boxed{v = -gt + u \ln\left(\frac{m_0}{m_0 + \alpha t}\right)}$$

1 Mechanics Problem

In Figure 1, a solid brass ball of mass 0.271 g will roll smoothly along a loop-the-loop track when released from rest along a straight section. The circular loop has radius R = 0.05 m, and the ball has radius r $\ll$ R. What is h if the ball is on the verge of leaving the track when it reaches the top of the loop?

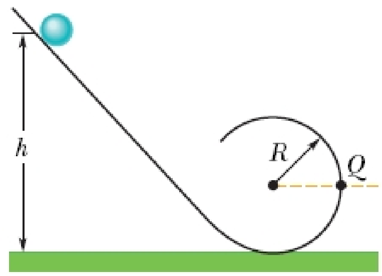


Figure 1: A ball rolling down a "loop-the-loop" track.

1.1 Solution

Take V=0 at ground level, then,

$$V_{releasepoint} = V_{toploop} + K_{toploop}, \tag{1}$$

$$mgh = mg2R + \frac{1}{2}mv_{com}^2 + \frac{1}{2}I\omega^2 \tag{2}$$

where $I = \frac{2}{5}mr^2$ and $v_{com} = \omega r$. We have

$$gh = 2gR + \frac{7}{10}v_{com}^2. \tag{3}$$

Since at the top, the ball is on the verge of losing contact with the track, the normal force is 0 there. Therefore $g = \frac{v_{com}^2}{R-r} \approx \frac{v_{com}^2}{R}$. We find that $h = \frac{27}{10}R = 13.5cm$.

2013 Quals Question: Mechanics (Dodd)

A child of mass *m* on a swing raises her center of mass by a small distance *b* every time the swing passes the vertical position, and lowers her center of mass by the same amount *b* at each extremal position (i.e. at the end-point of every swing).

a). Assuming small oscillations, calculate the work done by the child per period of oscillation.

b). Show that the energy of the swing grows exponentially according to:

$$\frac{dE}{dt} = \propto E$$

and determine the constant α.

Solution:

Examine one half-period (one half-swing) from one extremum to the other. Let the length of the swing be *l*.

From the first extremum (denote the initial angle to the vertical as ϑ_i) to the vertical, the child's center of mass is lowered by *b*, therefore using energy conservation for this part of the swing:

$$mg(l+b)(1-\cos\vartheta_i) = \frac{1}{2}mv_{ci}^2$$

where v_{ci} is the incoming velocity at the vertical/center position. Given small oscillations, we can approximate $(1-\cos\vartheta_i) \approx \frac{\vartheta_i^2}{2}$ so:

$$mg(l+b)\frac{\vartheta_i^2}{2} = \frac{1}{2}mv_{ci}^2$$

At the vertical position, the child raises her center of mass by *b*. Angular momentum is conserved during this transition, so:

$$m(l+b)v_{ci} = m(l-b)v_{cf}$$

where v_{cf} is the outgoing velocity from the vertical/center position. Then to the other extremum (denote the final angle to the vertical as ϑ_i), energy is again conserved, and making again the small angle approximation:

$$mg(l-b)\frac{\vartheta_f^2}{2} = \frac{1}{2}mv_{cf}^2 = \frac{1}{2}mv_{ci}^2\left(\frac{l+b}{l-b}\right)^2 = mg(l+b)\frac{\vartheta_i^2}{2}\left(\frac{l+b}{l-b}\right)^2$$

so:

$$\vartheta_f^2 = \left(\frac{l+b}{l-b}\right)^3 \vartheta_i^2$$

Given that *b* is small (compared to *l*), we can approximate this to:

$$\vartheta_f \approx \left(1 + 3\frac{b}{l}\right)\vartheta_i$$

and so:

$$\Delta\vartheta = 3\frac{b}{l}\vartheta_i$$

a). Now to find the work done per period, this is just the energy change per period, viz.:

$$\Delta E = 2mgl\frac{1}{2}\left(\vartheta_f^2 - \vartheta_i^2\right) \approx 2mgl\vartheta_i\Delta\vartheta = 6mgb\vartheta_i^2 = 12E_i\frac{b}{l}$$

where E_i is the initial energy.

b). And to find an expression for *dE/dt*:

$$\frac{dE}{dt} \approx \frac{\Delta E}{\frac{2\pi}{\omega_i}} \approx 6\frac{\omega_i}{\pi}\frac{b}{l}E_i$$

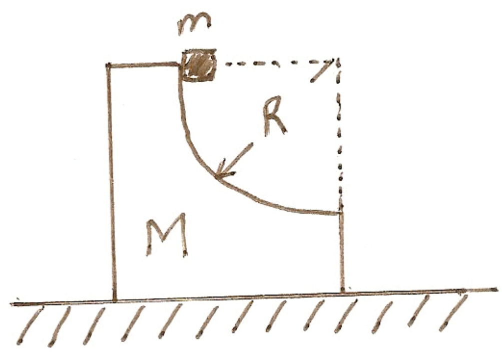
where $\omega_i = \sqrt{\frac{g}{l}}$. This has the stated form, with:

$$\alpha = 6\frac{\omega_i}{\pi}\frac{b}{l}$$

1. Classical Mechanics

A mass *m* slides down a circularly curved surface on an object with mass *M* as shown in the diagram below. Mass *M* is free to slide on a frictionless surface.

What are the final speeds of the two masses after *m* separates from M?



SOLUTION:

Let v = velocity of mass *m*

and V = velocity of mass *M*

There are 2 external forces on the system of M and m, namely gravity, which is conservative, and the normal force of the table, which does no work.

Therefore the sum of the kinetic and gravitational potential energies is conserved:

$$mgR = \frac{1}{2}m\mathrm{v}^2 + \frac{1}{2}MV^2$$

The external forces have no horizontal components so the horizontal component of the momentum is conserved.

$$m\mathrm{v} - \mathrm{MV} = 0$$

Combining these gives:

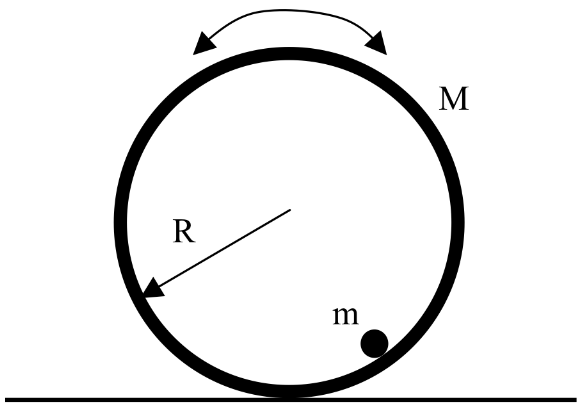
$$\mathrm{v} = \sqrt{\frac{2\mathrm{gR}}{1+\frac{\mathrm{m}}{\mathrm{M}}}}$$

and

$$V = \frac{m}{M}\sqrt{\frac{2gR}{1+\frac{m}{M}}}$$

Mechanics Problem: M. Shaevitz

A point particle of mass m slides without friction within a hoop of radius R and mass M. The hoop is free to roll without slipping along a horizontal surface. What is the frequency of small oscillations of the point mass, when it is close to the bottom of the hoop?



Solution :

$$X = R\theta \quad I_{hoop} = MR^2 \quad x = R\theta + R\sin\phi \quad y = -R\cos\phi$$

$$T = \tfrac{1}{2}\left(MR^2\dot{\theta}^2 + I\dot{\theta}^2\right) + \tfrac{1}{2}m\left(R\dot{\theta} + R\cos\phi\dot{\phi}\right)^2 + \tfrac{1}{2}m\left(R\sin\phi\dot{\phi}\right)^2$$

$$= \left(M + \tfrac{1}{2}m\right)R^2\dot{\theta}^2 + \tfrac{1}{2}mR^2\dot{\phi}^2 + mR^2\cos\phi\dot{\theta}\dot{\phi}$$

$$= \left(M + \tfrac{1}{2}m\right)R^2\dot{\theta}^2 + \tfrac{1}{2}mR^2\dot{\phi}^2 + mR^2\dot{\theta}\dot{\phi}$$

$$U = -mgR\cos\phi = \tfrac{1}{2}mgR\phi^2$$

$$L = T - U = \left(M + \tfrac{1}{2}m\right)R^2\dot{\theta}^2 + \tfrac{1}{2}mR^2\dot{\phi}^2 + mR^2\dot{\theta}\dot{\phi} - \tfrac{1}{2}mgR\phi^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = \frac{d}{dt}\left(2\left(M + \tfrac{1}{2}m\right)R^2\dot{\theta} + mR^2\dot{\phi}\right) = 0$$

$$(2M + m)\ddot{\theta} + m\ddot{\phi} = 0 \Rightarrow \ddot{\theta} = -\frac{m}{(2M + m)}\ddot{\phi}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \frac{\partial L}{\partial\phi} = \frac{d}{dt}\left(mR^2\dot{\phi} + mR^2\dot{\theta}\right) + mgR\phi = 0$$

$$R\ddot{\phi} + R\ddot{\theta} + g\phi = R\ddot{\phi} - R\frac{m}{(2M + m)}\ddot{\phi} + g\phi = 0$$

$$\ddot{\phi} + \left(\frac{2M + m}{2M}\right)\frac{g}{R}\phi = 0 \Rightarrow \omega = \sqrt{\left(\frac{2M + m}{2M}\right)\frac{g}{R}}$$