Columbia University
Department of Physics
QUALIFYING EXAMINATION

Wednesday, January 16, 2013
3:10PM to 5:10PM
Modern Physics
Section 4. Relativity and Applied Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 4 (Relativity and Applied QM), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. Clock A has rest mass $m = 10$ kg and moves with constant speed $v = (4/5)c$ on a circular orbit of radius $r = 10$ km. (Speed of light $c = 3 \times 10^8$ m/s.)

   (a) What is the orbital period measured by clock A?

   (b) Clock A inelastically collides with another clock B which is at rest and has the same mass m. Find their common speed v' after the collision. (Bear in mind that the rest mass of A+B after the collision differs from $2m$ due to dissipated heat.)

   (c) Heat released in this collision is radiated in photons distributed isotropically in the rest frame of A+B. How much energy is radiated? Would the answer change if the emission were anisotropic?

2. Consider the reaction $\pi^+ + n \to K^+ + \Lambda^0$. The rest masses of the particles are $m_\pi c^2 = 140$ MeV; $m_n c^2 = 940$ MeV; $m_K c^2 = 494$ MeV; $m_\Lambda c^2 = 1115$ MeV. What is the threshold kinetic energy of the π to create a K moving at an angle of 90° to the initial direction of the π in the lab frame, in which the n is at rest?

3. Consider the decay of a massive particle (e.g. a neutral pion) into two massless ones (e.g. two photons). Suppose for simplicity that the two final state particles are distinguishable, say photon 1 and photon 2. In the pion rest frame, the decay process is isotropic, in the sense that the decay rate (=probability per unit time) is independent of the propagation direction of photon 1,

$$\frac{d\Gamma}{d\Omega_1} = \text{const.} \tag{1}$$

Show that, regardless of the pion velocity, in the lab frame the decay has a constant rate per unit energy of photon 1,

$$\frac{d\Gamma'}{dE'_1} = \text{const.} \tag{2}$$

(Primes denote lab frame quantities.)

4. Consider a two-dimensional square lattice in which non-interacting electrons move. We will treat the motion of the electrons in the nearly free motion approximation. We consider the effect of the lattice using a two dimensional potential. Two possibilities to describe this potential are:

   Choice A,

$$V(x,y) = V_0 \cos\left(\frac{2\pi x}{a}\right)\cos\left(\frac{2\pi y}{a}\right) \tag{3}$$

   Choice B,

$$V(x,y) = V_0 \left(\cos\left(\frac{2\pi x}{a}\right) + \cos\left(\frac{2\pi y}{a}\right)\right) \tag{4}$$

   here a is the lattice constant and V_0 is the strength of the potential.

   Let us ask whether these potentials will open a band gap in the electronic states at the point in momentum space located at $\left(\frac{\pi}{a}, \frac{\pi}{a}\right)$.

   (a) In the limit where V_0 goes to zero, what is the degeneracy of the lowest energy state at the point $(k_x, k_y) = \left(\frac{\pi}{a}, \frac{\pi}{a}\right)$?

   (b) In the presence of a non-zero value of V_0, which one of the two potentials will open a band gap at the point $\left(\frac{\pi}{a}, \frac{\pi}{a}\right)$? Justify.

   (c) Sketch the band structure along the $(0,0)$ - $\left(\frac{\pi}{a}, \frac{\pi}{a}\right)$ direction for each potential.

5. A spinless particle of charge $-e$ and mass m is constrained to move in the x-y plane. There is a constant magnetic field B along the direction normal to the plane. Assume that the field derives from a vector potential that has a single component along the x-direction given by $A_x = -By$.

   (a) Write the expression for the Hamiltonian of one particle.

   (b) To find the solutions to the Schroedinger equation for the stationary states consider the wavefunctions

$$\psi(x, y) = f(x)\phi(y) \tag{5}$$

   where

$$f(x) = \exp\left[(i/\hbar)p_x x\right] \tag{6}$$

   and p_x is the x-component of the momentum.

   Write the Schroedinger equation for $\phi(y)$ and obtain an expression for the spectrum of energy levels E_n (Landau levels) in the field B. What are the quantum numbers that correspond to a Landau level?

   (c) Assume that the area of the plane is given by the product of two lengths $L_x L_y$ that are along the x- and y-directions. Also assume that the function $f(x)$ satisfies the boundary condition

$$f(x = 0) = f(x = L_x) \tag{7}$$

   Find the degeneracy of a Landau level as a function of magnetic field for $L_x = L_y = L$.

Columbia University
Department of Physics
QUALIFYING EXAMINATION

Wednesday, January 16, 2013
3:10PM to 5:10PM
Modern Physics
Section 4. Relativity and Applied Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 4 (Relativity and Applied QM), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. Clock A has rest mass $m = 10$ kg and moves with constant speed $v = (4/5)c$ on a circular orbit of radius $r = 10$ km. (Speed of light $c = 3 \times 10^8$ m/s.)

   (a) What is the orbital period measured by clock A?

   (b) Clock A inelastically collides with another clock B which is at rest and has the same mass m. Find their common speed v' after the collision. (Bear in mind that the rest mass of A+B after the collision differs from $2m$ due to dissipated heat.)

   (c) Heat released in this collision is radiated in photons distributed isotropically in the rest frame of A+B. How much energy is radiated? Would the answer change if the emission were anisotropic?

2. Consider the reaction $\pi^+ + n \rightarrow K^+ + \Lambda^0$. The rest masses of the particles are $m_\pi c^2 = 140$ MeV; $m_n c^2 = 940$ MeV; $m_K c^2 = 494$ MeV; $m_\Lambda c^2 = 1115$ MeV. What is the threshold kinetic energy of the π to create a K moving at an angle of $90°$ to the initial direction of the π in the lab frame, in which the n is at rest?

3. Consider the decay of a massive particle (e.g. a neutral pion) into two massless ones (e.g. two photons). Suppose for simplicity that the two final state particles are distinguishable, say photon 1 and photon 2. In the pion rest frame, the decay process is isotropic, in the sense that the decay rate (=probability per unit time) is independent of the propagation direction of photon 1,

$$\frac{d\Gamma}{d\Omega_1} = \text{const.} \tag{1}$$

Show that, regardless of the pion velocity, in the lab frame the decay has a constant rate per unit energy of photon 1,

$$\frac{d\Gamma'}{dE'_1} = \text{const.} \tag{2}$$

(Primes denote lab frame quantities.)

4. Consider a two-dimensional square lattice in which non-interacting electrons move. We will treat the motion of the electrons in the nearly free motion approximation. We consider the effect of the lattice using a two dimensional potential. Two possibilities to describe this potential are:

   Choice A,

$$V(x,y) = V_0 \cos\left(\frac{2\pi x}{a}\right)\cos\left(\frac{2\pi y}{a}\right) \tag{3}$$

   Choice B,

$$V(x,y) = V_0 \left(\cos\left(\frac{2\pi x}{a}\right) + \cos\left(\frac{2\pi y}{a}\right)\right) \tag{4}$$

   here a is the lattice constant and V_0 is the strength of the potential.

   Let us ask whether these potentials will open a band gap in the electronic states at the point in momentum space located at $\left(\frac{\pi}{a}, \frac{\pi}{a}\right)$.

   (a) In the limit where V_0 goes to zero, what is the degeneracy of the lowest energy state at the point $(k_x, k_y) = \left(\frac{\pi}{a}, \frac{\pi}{a}\right)$?

   (b) In the presence of a non-zero value of V_0, which one of the two potentials will open a band gap at the point $\left(\frac{\pi}{a}, \frac{\pi}{a}\right)$? Justify.

   (c) Sketch the band structure along the $(0,0)$ - $\left(\frac{\pi}{a}, \frac{\pi}{a}\right)$ direction for each potential.

5. A spinless particle of charge $-e$ and mass m is constrained to move in the x-y plane. There is a constant magnetic field B along the direction normal to the plane. Assume that the field derives from a vector potential that has a single component along the x-direction given by $A_x = -By$.

   (a) Write the expression for the Hamiltonian of one particle.

   (b) To find the solutions to the Schroedinger equation for the stationary states consider the wavefunctions

$$\psi(x, y) = f(x)\phi(y) \tag{5}$$

   where

$$f(x) = \exp\left[(i/\hbar)p_x x\right] \tag{6}$$

   and p_x is the x-component of the momentum.

   Write the Schroedinger equation for $\phi(y)$ and obtain an expression for the spectrum of energy levels E_n (Landau levels) in the field B. What are the quantum numbers that correspond to a Landau level?

   (c) Assume that the area of the plane is given by the product of two lengths $L_x L_y$ that are along the x- and y-directions. Also assume that the function $f(x)$ satisfies the boundary condition

$$f(x = 0) = f(x = L_x) \tag{7}$$

   Find the degeneracy of a Landau level as a function of magnetic field for $L_x = L_y = L$.

Relativity:
Clock A has rest mass $m = 10$ kg and moves with constant speed $v = (4/5)c$ on a circular orbit of radius $r = 10$ km. (Speed of light $c = 3 \times 10^8$ m/s.)

(a) What is the orbital period measured by clock A?

(b) Clock A inelastically collides with another clock B, which is at rest and has the same mass m. Find their common speed v' after the collision. Bear in mind that the rest mass of A+B after the collision differ from 2m due to dissipated heat.

(c) All heat released in this collision is radiated in photons, isotropically in the rest frame of A+B. How much energy is radiated? Would the answer change if the emission was anisotropic?

Solution:
(a) $\gamma = (1 - v^2/c^2)^{-1/2} = 5/3$. Orbital period measured by A:

$$P = \frac{2\pi r}{v\,\gamma} = 1.57 \times 10^{-4} \text{ s.}$$

(b) Let m' be the rest mass of A+B after the collision (it includes the dissipated heat and therefore differs from $2m$). Energy and momentum conservation laws give

$$\gamma' m' = \gamma m + m, \qquad v'\gamma' m' = v\gamma m.$$

Dividing the two equations, one finds

$$v' = \frac{v\,\gamma}{\gamma + 1} = \frac{c}{2}.$$

(c) Isotropic emission has no rocket effect on A+B, and after heat is radiated away A+B move with the same $v' = c/2$. The final energy of A+B is $2mc^2\gamma'$, where $\gamma' = 2/\sqrt{3}$. The radiated energy is

$$E = \gamma mc^2 + mc^2 - \gamma' 2mc^2 = \frac{4}{3}(2 - \sqrt{3})mc^2 \approx 3.2 \times 10^{17} \text{ J.}$$

Anisotropic emission would affect the velocity of A+B and hence would change the emitted energy E.

Consider the reaction $\pi^+ + n \to K^+ + \Lambda^0$.
The rest masses of the particles are
$m_\pi c^2 = 140\,\text{MeV}$; $m_n c^2 = 940\,\text{MeV}$,
$m_K c^2 = 494\,\text{MeV}$; $m_\Lambda c^2 = 1115\,\text{MeV}$.
What is the threshhold kinetic energy of the π to create a K at an angle of 90° in the lab, in which the n is at rest?

Solution, Quarks, Relativity, Harley

$c=1$ $\quad P_\pi + P_n = P_K + P_\Lambda \quad$ 4-vectors

$$P_\pi + P_n - P_K = P_\Lambda \qquad P_\pi = (E_\pi, p_\pi)$$

$$P_n = (m_n, 0)$$

$$|\vec{P}|^2 = -m^2 \qquad P_K = (E_K, p_K)$$

$$P_\Lambda = (E_\Lambda, p_\Lambda)$$

$$(P_\pi + P_n - P_K)^2 = P_\Lambda^2 = -m_\Lambda^2$$

$$-m_\Lambda^2 = P_\pi^2 + P_n^2 + 2P_\pi \cdot P_n - 2P_K \cdot P_\pi - 2P_n \cdot P_K + P_K^2$$

$$-m_\Lambda^2 = -m_\pi^2 - m_n^2 - m_K^2 - 2E_\pi m_n + 2E_K m_n - 2P_K \cdot P_\pi$$

$$P_K \cdot P_\pi = -E_K E_\pi + \vec{P}_K \cdot \vec{P}_\pi \qquad \vec{P}_K \cdot \vec{P}_\pi = 0$$

(90° scatter)

$$-m_\Lambda^2 = -m_\pi^2 - m_n^2 - m_K^2 - 2m_n E_\pi + 2m_n E_K + 2E_\pi E_K$$

$$E_\pi = \frac{m_\Lambda^2 - m_\pi^2 - m_n^2 - m_K^2 + 2m_n E_K}{2(m_n - E_K)}$$

For threshhold $E_K = m_K$

$$E_\pi^{Thresh} = \frac{m_\Lambda^2 - m_\pi^2 - m_n^2 - m_K^2 + 2m_n m_K}{2(m_n - m_K)}$$

$$E_\pi^{Thresh} \simeq 1150 \text{ MeV}$$

2 Relativity: neutral pion decay

Consider the decay of a massive particle (e.g., a neutral pion) into two massless ones (e.g., two photons). Suppose for simplicity that two final particles are distinguishable, say photon 1 and photon 2. In the pion rest frame, the decay process is isotropic, in the sense that the decay rate (=probability per unit time) is independent of the propagation direction of photon 1,

$$\frac{d\Gamma}{d\Omega_1} = \text{const} \; . \tag{4}$$

Show that, regardless of the pion velocity, in the lab frame the decay has a constant rate per unit energy of photon 1,

$$\frac{d\Gamma'}{dE_1'} = \text{const} \; . \tag{5}$$

(Primes denote lab frame quantities.)

2 Relativity: neutral pion decay

Consider the decay of a massive particle (e.g., a neutral pion) into two massless ones (e.g., two photons). Suppose for simplicity that two final particles are distinguishable, say photon 1 and photon 2. In the pion rest frame, the decay process is isotropic, in the sense that the decay rate (=probability per unit time) is independent of the propagation direction of photon 1,

$$\frac{d\Gamma}{d\Omega_1} = \text{const} \ . \tag{14}$$

Show that, regardless of the pion velocity, in the lab frame the decay has a constant rate per unit energy of photon 1,

$$\frac{d\Gamma'}{dE_1'} = \text{const} \ . \tag{15}$$

(Primes denote lab frame quantities.)

Solution

For simplicity, let's use units in which $c = 1$. The energy and momentum conservation laws, applied to the decay of a neutral pion into two photons, are enough to determine uniquely the energies of the photons in the center-of-mass frame (the rest frame of the original pion):

$$E_1 = E_2 = \frac{M}{2} \ , \tag{16}$$

with obvious notation. The final momenta are also identical in magnitude,

$$|\vec{p}_1| = |\vec{p}_2| = \frac{M}{2} \ . \tag{17}$$

They have opposite direction

$$\vec{p}_2 = -\vec{p}_1 \ , \tag{18}$$

and so the only unknown can be taken to be the propagation direction of photon 1, identified by polar and azimuthal angles θ and ϕ, defined with respect to some arbitrary z-axis. The decay is isotropic in the center-of-mass frame:

$$\frac{d\Gamma}{d\Omega} = \text{const} \ , \tag{19}$$

where the solid angle element is $d\Omega = d\cos\theta\, d\phi$.

Let's choose the z-axis to be the pion's propagation direction in the lab frame. We thus have that the energy of photon 1 in the lab frame is

$$E_1' = \gamma\big(E_1 + \beta p_1^z\big) = \frac{1}{2}\gamma M(1+\beta\cos\theta) \ , \tag{20}$$

where γ and β are the pion's boost parameters in the lab frame. In this equation, the only variable that depends on the decay process is $\cos\theta$—everything else is a constant. We thus have

$$dE_1' = \frac{1}{2}\gamma\beta M\, d\cos\theta \tag{21}$$

Since Γ is a probability per unit *time*, there is a time-dilation effect one has to take into account in going from the rest frame to the lab frame, which is simply

$$d\Gamma' = \frac{1}{\gamma} d\Gamma \tag{22}$$

Putting everything together, one thus has

$$\frac{d\Gamma}{d\Omega} \propto \frac{d\Gamma'}{dE_1' d\phi} \ , \tag{23}$$

where the proportionality constants depend on the boost parameters γ and β, but not on the decay variables θ and ϕ (or E_1' and ϕ). Integrating in ϕ we finally get

$$\frac{d\Gamma'}{dE_1'} = \text{const} \ . \tag{24}$$

Solid State

Consider a two-dimensional square lattice in which non-interacting electrons move. We will treat the motion of the electrons in the nearly free motion approximation. We consider the effect of the lattice using a two dimensional potential. Two possibilities to describe this potential are:
Choice A

$$\mathrm{V}(\mathrm{x},\mathrm{y}) = \mathrm{V}_0 \cos\left(\frac{2\pi \mathrm{x}}{\mathrm{a}}\right)\cos\left(\frac{2\pi \mathrm{y}}{\mathrm{a}}\right)$$

Choice B

$$\mathrm{V}(\mathrm{x},\mathrm{y}) = \mathrm{V}_0 \left(\cos\left(\frac{2\pi \mathrm{x}}{\mathrm{a}}\right) + \cos\left(\frac{2\pi \mathrm{y}}{\mathrm{a}}\right)\right)$$

here a is the lattice constant and V_0 is the strength of the potential.
Let us ask whether these potentials will open a band gap in the electronic states at the point in momentum space located at $(\frac{\pi}{a},\frac{\pi}{a})$.
(a) In the limit where V_0 goes to zero, what is the degeneracy of the lowest energy state at the point $(\mathrm{k_x},\mathrm{k_y}) = \left(\frac{\pi}{a},\frac{\pi}{a}\right)$?
(b) In the presence of a non-zero value of V_0, which one of the two potentials will open a band gap at the point $\left(\frac{\pi}{a},\frac{\pi}{a}\right)$? Justify.
(c) Sketch the band structure along the (0,0) - $(\frac{\pi}{a},\frac{\pi}{a})$ direction for each potential.

Answer:
(a) The wavefunction $\psi(x,y)$ has to satisfy the periodicity of the lattice (Bloch's theorem). We can look for solutions to the wavefunctions as combinations of plane waves that have lattice periodicity:

$$\psi_{kx,ky}(x,y) = \sum_{m,n} c_{mn} e^{ik_{mx}x} e^{ik_{ny}y}$$

where

$$k_{mx} = k_x + \frac{2\pi m}{a}$$

Stick into Schrodinger equation:

$$\sum_{n,m} c_{mn} e^{ik_{mx}x} e^{ik_{ny}y} \left(\frac{\hbar^2(k_{mx}^2 + k_{ny}^2)}{2m} + V(x,y) - E(k_x,k_y)\right) = 0$$

For a given potential, this equation can be solved by Fourier decomposition.
In the limit of an infinitesimally small potential, the Schrodinger equation becomes

$$\sum_{n,m} c_{mn} e^{ik_{mx}x} e^{ik_{ny}y} \left(\frac{\hbar^2(k_{mx}^2 + k_{ny}^2)}{2m} - E(k_x,k_y)\right) = 0$$

At the point $\left(\frac{\pi}{a},\frac{\pi}{a}\right)$, the lowest energy state is $\frac{\hbar^2\pi^2}{ma^2}$. In the plane wave expansion above, four plane waves have the same energy, corresponding to the coefficients $c_{00}, c_{01}, c_{10}, c_{11}$.
A useful way to write the Schrodinger equation after Fourier decomposition, is as a matrix equation. For the case of infinitesimally small potential at the point $\left(\frac{\pi}{a},\frac{\pi}{a}\right)$, this matrix equation is:

$$\begin{pmatrix} E_0 - E & 0 & 0 & 0 \\ 0 & E_0 - E & 0 & 0 \\ 0 & 0 & E_0 - E & 0 \\ 0 & 0 & 0 & E_0 - E \end{pmatrix} \begin{pmatrix} c_{00} \\ c_{01} \\ c_{10} \\ c_{11} \end{pmatrix} = 0$$

with $E_0 = \frac{\hbar^2 \pi^2}{ma^2}$

(b) For choice A, the Schrodinger equation is:

$$\sum_{n,m} c_{mn} e^{ik_{mx}x} e^{ik_{ny}y} \left(\frac{\hbar^2 (k_{mx}^2 + k_{ny}^2)}{2m} + \frac{V_0}{4} \left(e^{\frac{i2\pi x}{a}} + e^{-\frac{i2\pi x}{a}} \right) \left(e^{\frac{i2\pi y}{a}} + e^{-\frac{i2\pi y}{a}} \right) - E(k_x, k_y) \right) = 0$$

We see that the potential links the plane wave centered at $(0,0)$ to the plane wave centered at $\left(\frac{2\pi}{a}, \frac{2\pi}{a}\right)$.
In matrix form, Schrodinger's equation at $\left(\frac{\pi}{a}, \frac{\pi}{a}\right)$ now becomes:

$$\begin{pmatrix} x & 0 & 0 & y \\ 0 & x & y & 0 \\ 0 & y & x & 0 \\ y & 0 & 0 & x \end{pmatrix} \begin{pmatrix} c_{00} \\ c_{01} \\ c_{10} \\ c_{11} \end{pmatrix} = 0$$

where $x = E_0 - E$ and $y = \frac{V_0}{4}$.
The eigenvalues of this matrix are given by the solutions of the equation:

$$(x^2 - y^2) = 0$$

or

$$E = E_0 \pm \frac{V_0}{4}$$

(with two eigenfunctions for each eigenvalue)
The eigenfunctions are given by $c_{00} = \pm c_{11}$, $c_{01} = \pm c_{10}$.

We see that this choice of potential opens a full energy gap at $\left(\frac{\pi}{a}, \frac{\pi}{a}\right)$

For choice B, the Schrodinger equation is

$$\sum_{n,m} c_{mn} e^{ik_{mx}x} e^{ik_{ny}y} \left(\frac{\hbar^2 (k_{mx}^2 + k_{ny}^2)}{2m} + \frac{V_0}{2} \left(\left(e^{\frac{i2\pi x}{a}} + e^{-\frac{i2\pi x}{a}} \right) + \left(e^{\frac{i2\pi y}{a}} + e^{-\frac{i2\pi y}{a}} \right) \right) - E(k_x, k_y) \right) = 0$$

In this case, the potential links the plane wave centered at $(0,0)$ to $\left(\frac{2\pi}{a}, 0\right)$ and $\left(0, \frac{2\pi}{a}\right)$.
In matrix form, Schrodinger's equation at $\left(\frac{\pi}{a}, \frac{\pi}{a}\right)$ now becomes:

$$\begin{pmatrix} x & y & y & 0 \\ y & x & 0 & y \\ y & 0 & x & y \\ 0 & y & y & x \end{pmatrix} \begin{pmatrix} c_{00} \\ c_{01} \\ c_{10} \\ c_{11} \end{pmatrix} = 0$$

The eigenvalues are now given by the solutions of the equation

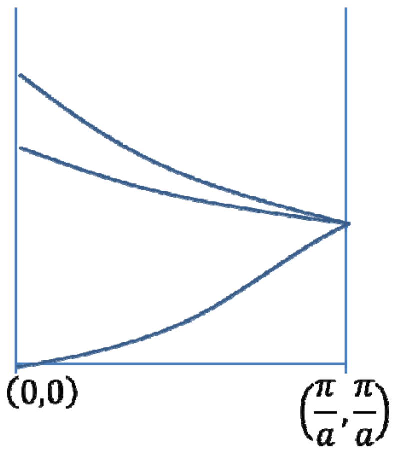
$$x^4 - 4x^2 y^2 = 0$$

or

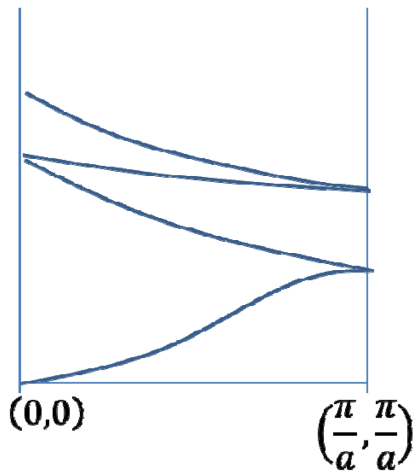
$$E = E_0, E_0, E_0 \pm \frac{V_0}{2}$$

we see that in this case, two eigenvalues are unaffected by the potential, so there is no energy gap at $\left(\frac{\pi}{a}, \frac{\pi}{a}\right)$

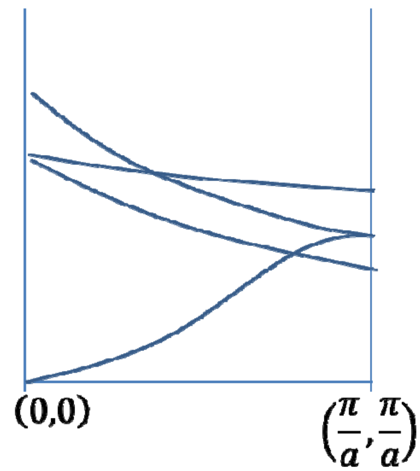
(c)



Limit of zero potential



Choice A



Choice B

General-Section 4: applied quantum mechanics

A spin-less particle of charge –e and mass m is constrained to move in the x-y plane. There is a constant magnetic field B along the direction normal to the plane. Assume that the field derives from a vector potential that has a single component along the x-direction given by $A_x = -By$.

(a) write the expression for the Hamiltonian of one particle.

(b) to find the solutions of the Schroedinger equation for the stationary states consider wavefunctions

$\psi(x,y)= f(x)\phi(y)$

where

$f(x)= \exp[(i/\hbar)p_x x]$

and p_x is the x-component of momentum.

Write the Schroedinger equation for $\phi(y)$ and obtain the expression for the spectrum of energy levels E_n (Landau levels) in the field B.
What are the quantum numbers that correspond to a Landau level?

(c) Assume that the area of the plane is given by the product of two lengths $L_x L_y$, that are along the x- and y-directions.
Also assume that the function f(x) satisfies the ‘obvious’ boundary condition

$f(x=0)=f(x=L_x)$

Find the degeneracy of a Landau level as function of magnetic field for $L_x = L_y = L$.

Solution

(a) $$H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2$$

$$\vec{A} \equiv (A_x, 0, 0) \qquad A_x = -By$$

$$H = \frac{1}{2m}\left(p_x + \frac{e}{c}By\right)^2 + \frac{1}{2m}p_y^2$$

(b) $$\frac{d^2\phi(y)}{dy^2} + \frac{2m}{\hbar^2}\left[E_n - \frac{1}{2}m\omega_c^2(y-y_0)^2\right]\phi(y) = 0$$

$$\omega_c = \frac{eB}{mc} \quad ; |y_0| = \frac{cp_x}{eB}$$

$$E_n = \left(n+\frac{1}{2}\right)\hbar\omega_c$$

(c) From the boundary condition

$$e^{(i/\hbar)p_x L} = 1 \text{ ; which implies}$$

$$p_x = m\,2\pi\frac{\hbar}{L} \quad ; \quad m \equiv \text{integer}$$

$$y_0 = 2\pi\frac{c\hbar}{eBL}m \leq L$$

The number of states is equal to m for the maximum value $y_0 = L$

$$N = (hc/eB)A \quad ; A = L^2$$