Columbia University
Department of Physics
QUALIFYING EXAMINATION

Friday, January 18, 2013
1:00PM to 3:00PM
General Physics (Part I)
Section 5.

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 6 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 5 (General Physics), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on General Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. A vertical cylinder jar has height H and radius R. $3/4$ of its volume $V = \pi R^2 H$ is filled with water. The cylinder steadily rotates about its axis with angular velocity ω. Find the maximum ω above which the water would spill out of the jar.

2. Consider an idealized Sun and Earth as blackbodies in otherwise empty space. The Sun has a surface temperature $T_S = 6,000$ K, and heat transfer processes on the Earth are effective enough to keep the Earth's surface temperature uniform. The radius of the Earth is $R_E = 6.4 \times 10^6$ m, the radius of the Sun is $R_S = 7.0 \times 10^8$ m, and the Earth-Sun distance is $d = 1.5 \times 10^{11}$ m. The mass of the Sun is $M_S = 2.0 \times 10^{30}$ kg.

   (a) Find the temperature of the Earth.

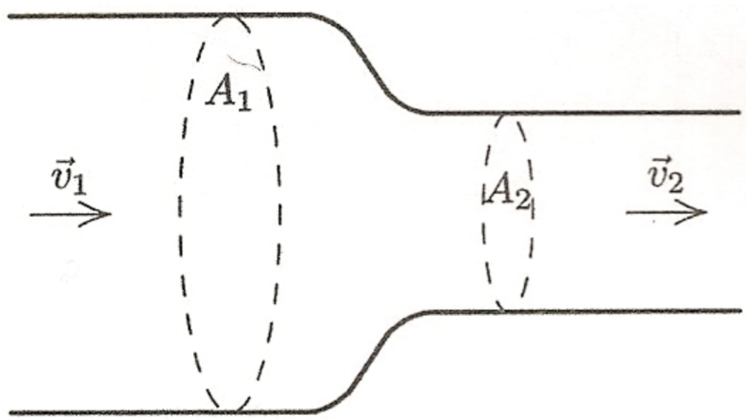
   (b) Find the radiation force on Earth.

   (c) Compare these results with those for an interplanetary granule in the form of a spherical, perfectly conducting blackbody with a radius $R = 0.1$ cm, moving in a circular orbit around the Sun at a radius equal to the Earth-Sun distance d.

   (d) At what distance from the Sun would a metallic particle melt if its melting temperature $T_m = 1,550$ K.

   (e) For what size particle would the radiation force calculated in c) be equal to the gravitational force from the Sun at a distance d?

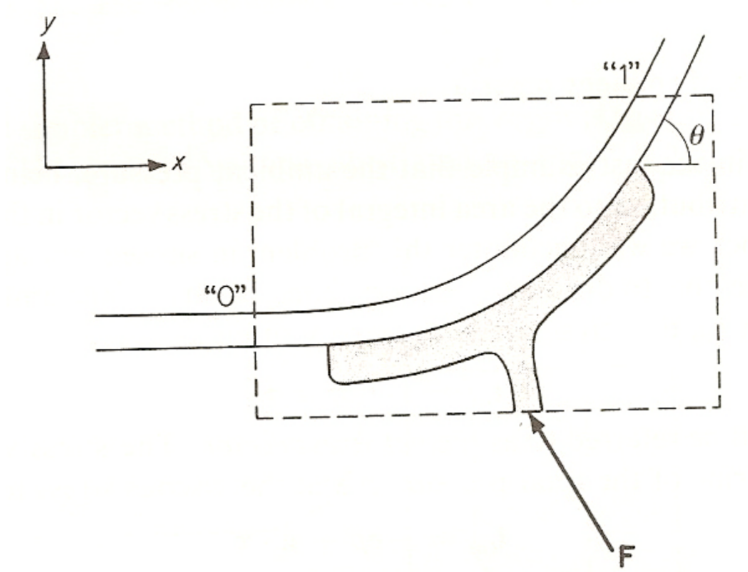
3. An incompressible fluid flows along the pipe shown below.



(a) If the velocity v_1 is known, what is the velocity v_2?

(b) If the velocity of v_1 is unknown, but the pressure in pipe 1 and pipe 2 (P_1 and P_2 respectively), what is the velocity v_2?

(c) Consider a liquid jet against a turbine blade shown below with steady, incompressible flow.



Find the $\hat{x}$ and $\hat{y}$ components of the fluid flow force $\vec{F}$. (Assume a fluid density ρ, an entrance and exit area A_0 and an incoming velocity v_0.)

4. The core of a neutron star may be considered to be a perfect gas of N neutrons compressed to such a *uniform* density that the energy of each neutron is much larger than its rest energy. The energy is related to the momentum by

$$\epsilon = cp. \tag{1}$$

   (a) Find the pressure P at absolute zero in terms of the number density N/V where V is the volume of the gas.

   (b) The Fermi temperature T_F of neutrons inside a neutron star is 3×10^{12} K. Calculate the speed of sound in the star. The Fermi temperature is related to the Fermi energy ϵ_F by $\epsilon_F = kT_F$, where $k = 1.38 \times 10^{-23}$ J/K.

5. Consider a one-dimensional chain consisting of N molecules. Each molecule can exist in one of two configurations, A and B, with energy E_A and E_B and length l_A and l_B respectively (see Fig. 1).

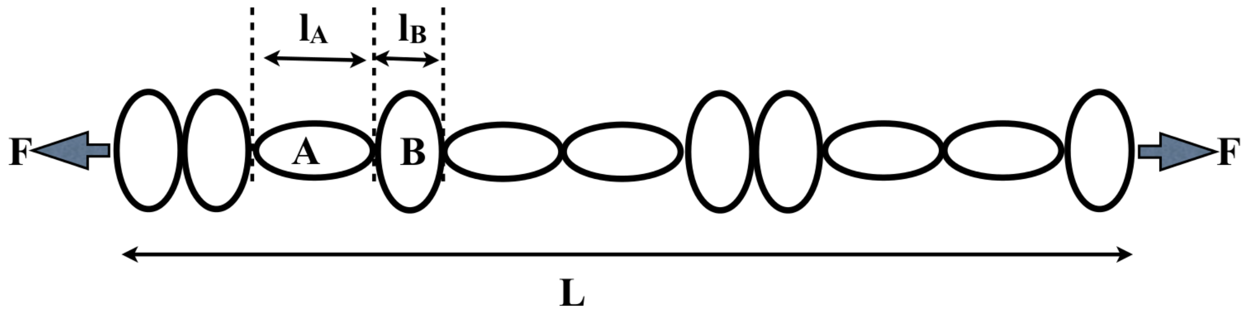


Figure 1: Example configuration of linear chain molecule with two orientations and corresponding energies and lengths indicated, along with applied tensile force (part c).

(a) Write down the partition function for the system at temperature T, assuming $F = 0$.

(b) Assume that $E_A > E_B$ and $l_A > l_B$ (still with $F = 0$). Give an expression for the average length L of the molecule as a function of temperature at a fixed, large N. Sketch the result, give the high and low T limits and indicate what is the characteristic temperature at which the length changes from the high-T to the low-T limits.

(c) Calculate the linear response function $\chi = \partial \langle L \rangle / \partial F$ showing how the length changes if a small tensile force is applied to the two ends of the chain as shown in the figure.

6. A large isolated box of volume V is filled with blackbody radiation in thermodynamic equilibrium. The radiation has an initial total energy U_0 (relative to that of a vacuum in the same volume). A cyclic engine pulls heat from the boxed radiation which is subsequently dumped into a large thermodynamic reservoir whose temperature remains at a fixed T_R.
   (a) What is the minimum value of work (W) that must be expended on the cyclic engine to bring the radiation in the box arbitrarily close to being a vacuum if the boxed radiation temperature was originally T_R?
   (b) What is the average energy in the lowest frequency mode ($\omega_0 = 2\pi f_0$) at arbitrary temperature (T)?
   (c) What is the probability that it is empty of photons?
   (d) What is changed if it is not the lowest frequency mode?

General:
A vertical cylindrical jar has height H and radius R. 3/4 of its volume $V = \pi R^2 H$ is filled with water. The cylinder steadily rotates about its axis with angular velocity ω. Find the maximum ω above which the water would spill out of the jar.

Solution:
In the rotating frame (rest frame of the water and the cylinder), the effective potential is given by

$$\Phi(r, z) = -gz + \frac{\omega^2 r^2}{2} + const,$$

where $r \leq R$ is the distance from the axis and z is the vertical coordinate. The water surface is equipotential, $\Phi(r, z) = const$, and hence the surface is described by

$$z(r) = z_0 + \frac{\omega^2}{2g}\, r^2,$$

where $z_0 = z(0)$. Water will not spill out if $z(R) < H$, i.e. the critical ω satisfies the condition,

$$z(R) = z_0 + \frac{\omega^2}{2g}\, R^2 = H. \tag{1}$$

Water is nearly incompressible and occupies the constant volume $(3/4)V$,

$$\int_0^R z(r)\, 2\pi r\, dr = \frac{3}{4} V \quad \Rightarrow \quad \pi R^2 z_0 + \frac{\pi\omega^2}{4g}\, R^4 = \frac{3}{4}\, \pi R^2 H. \tag{2}$$

Equations (1) and (2) determine the critical ω and the corresponding z_0,

$$\omega^2 = \frac{gH}{R^2}, \qquad z_0 = \frac{H}{2}.$$

2013 Quals General Question (Dodd)

Consider an idealized Sun and Earth as blackbodies in otherwise empty space. The Sun has a surface temperature T_S = 6,000 K, and heat transfer processes on the Earth are effective enough to keep the Earth's surface temperature uniform. The radius of the Earth is $R_E = 6.4 \times 10^6$ m, the radius of the Sun is $R_S = 7.0 \times 10^8$ m, and the Earth-Sun distance is $d = 1.5 \times 10^{11}$ m. The mass of the Sun is $M_S = 2.0 \times 10^{30}$ kg.

a). Find the temperature of the Earth.

b). Find the radiation force on the Earth.

c). Compare these results with those for an interplanetary granule in the form of a spherical, perfectly conducting blackbody with a radius R = 0.1 cm, moving in a circular orbit around the Sun at a radius equal to the Earth-Sun distance d.

d). At what distance from the Sun would a metallic particle melt if its melting temperature T_m = 1,550 K?

e). For what size particle would the radiation force calculated in c). be equal to the gravitational force from the Sun at a distance d?

NOTE: Need values for σ (Stefan-Bolztmann constant) and G for this problem.

Solution:

a). Total power radiated from the Sun is:

$$P_S = (4\pi R_S^2)\sigma\varepsilon T_S^4$$

where ε = 1 for a blackbody.

Of this, the fraction that hits the Earth is $\frac{\pi R_E^2}{4\pi d^2}$, and in equilibrium, with the Earth also radiating as a blackbody, then:

$$(4\pi R_S^2)\sigma T_S^4\left(\frac{\pi R_E^2}{4\pi d^2}\right) = (4\pi R_E^2)\sigma T_E^4$$

so:

$$T_E = \sqrt{\frac{R_S}{2d}}T_S = 290 \text{ K}$$

b). The radiation pressure is:

$$P_{rad} = \frac{4\sigma}{3c} T_S^4 \left(\frac{4\pi R_S^2}{4\pi d^2}\right) = 7.1 \times 10^{-6}\ \mathrm{N}/m^2$$

with the corresponding force on the Earth being:

$$F_E = (\pi R_E^2) P_{rad} = 9.1 \times 10^8\ \mathrm{N}$$

c). For the small granule, the temperature will be the <u>same</u> since it the same distance from the Sun as the Earth is. The radiation pressure on the granule is:

$$F_{gr} = (\pi R^2) P_{rad} = 2.2 \times 10^{-11}\ \mathrm{N}$$

d). From the result of part a). we can write for the distance d_m at which the granule will melt:

$$d_m = \frac{1}{2} R_S \left(\frac{T_S}{T_m}\right)^2 = 5.2 \times 10^9\ \mathrm{m}$$

e). Assume a spherical particle of mass *m* and radius *r*, then when the radiation and gravitational forces balance we have:

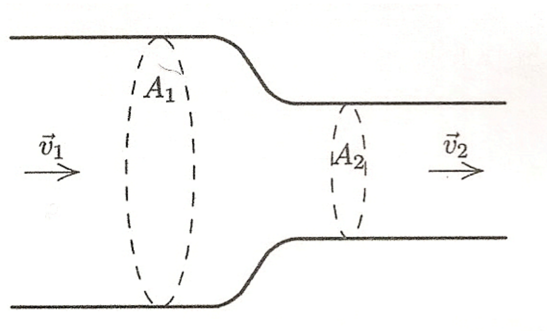
$$\frac{G M_S m}{d^2} = (\pi r^2) P_{rad}$$

where $m = \frac{4}{3}\pi r^3 \rho$ and we can assume a density of say $\rho = 5.0 \times 10^3$ kg/m^3, then:

$$r = \frac{3}{4} \frac{P_{rad} d^2}{G M_S \rho} = 8.6 \times 10^{-7}\ \mathrm{m}$$

2. General Physics I (continuous medium, hydrodynamics, etc.)

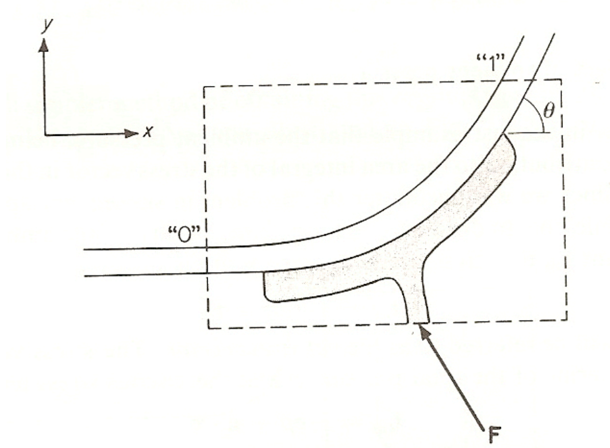
An incompressible fluid flows along the pipe shown below.



(a) If the velocity of v_1 is known, what is the velocity v_2?

(b) If the velocity of v_1 is unknown, but the pressures in pipe 1 and pipe 2, (P_1 and P_2, respectively) are known, what is the velocity v_2?

(c) Consider a liquid jet against a turbine blade shown below with steady, incompressible flow.



Find the $\hat{x}$ and $\hat{y}$ components of the fluid flow force $\dot{F}$? (Assume a fluid density ρ, an entrance and exit area A_0 and an incoming velocity v_0.)

ANSWER:

(a) vA is a constant, so $v_2 = \frac{A_1}{A_2} v_1$

(b) By Bernoulli's equation,

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2 = P_1 + \frac{1}{2}\rho\left(\frac{A_2}{A_1} v_2\right)^2$$

Therefore $v_2 = \sqrt{\dfrac{2(P_1 - P_2)}{\rho\left(1 - \dfrac{A_2^2}{A_1^2}\right)}}$

(c) The force, $\vec{F}$, is given by

$$\vec{F} = \int (\rho\vec{v})\vec{v} \cdot \hat{n}\, dA$$

Therefore,

$$-\rho v_0^2 A_0 + \rho v_1^2 A_0 \cos\theta = F_x$$

$$\rho v_1 A_0 \sin\theta = F_y$$

But, since the entrance and exit areas are the same and the fluid is ideal, the entrance and exit velocities are the same.

Therefore,

$$F_x = -\rho v_0^2 A_0 (1 - \cos\theta)$$

$$F_y = \rho v_0^2 A_0 \sin\theta$$

Question

The core of a neutron star may be considered to be a perfect gas of N neutrons compressed to such a *uniform* density that the energy of each neutron is much larger than its rest energy. The energy is related to the momentum by

$$\epsilon = cp. \tag{1}$$

1. Find the pressure P at absolute zero in terms of the number density N/V where V is the volume of the gas.

2. The Fermi temperature T_F of neutrons inside a neutron star is 3×10^{12} K. Calculate the speed of sound in the star. The Fermi temperature is related to the Fermi energy ϵ_F by $\epsilon_F = kT_F$, where $k = 1.38 \times 10^{-23}$ J/K.

Solution

(1) Consider a cube with sides of length L within the neutron star. When a single particle strikes a side of the cube perpendicular to the z axis, it transfers a momentum $2p_z$. The particle strikes the side at a rate $v_z/2L$, so the rate of momentum transfer per particle per unit area is $p_z v_z/L$. Since all sides of the cube are equivalent, we have,

$$< p_x v_x > = < p_y v_y > = < p_z v_z > = < \vec{p} \cdot \vec{v} > /3. \tag{2}$$

Thus, integrating over all particles within a volume $V = L^3$, we find the pressure P to be

$$P = \frac{N}{3V} \ < \vec{p} \cdot \vec{v} > . \tag{3}$$

But for relativistic particles, we have $< \vec{p} \cdot \vec{v} > = cp = \epsilon$, so

$$P = \frac{N\epsilon}{3V}. \tag{4}$$

(2) In an ideal gas, we know that

$$PdV + VdP = NkdT, \tag{5}$$

and that the specific heat at constant volume is related to the change in temperature and energy of the system:

$$\begin{aligned} dU &= C_v dT. \\ \Rightarrow dT &= \tfrac{dU}{C_v} = -\tfrac{PdV}{C_v} \\ \Rightarrow PdV + VdP &= -\left(\tfrac{Nk}{C_v}PdV\right) \end{aligned}$$

Using the fact that the adiabatic index $\gamma = C_p/C_v$ and $C_p = C_v + Nk$, this simplifies to

$$\gamma\frac{dV}{V} = -\frac{dP}{P}, \tag{6}$$

and solving this differential equation yields,

$$\begin{aligned} PV^{\gamma} &= K \\ \Rightarrow \tfrac{\partial P}{\partial V} &= -\gamma K V^{-\gamma-1}. \end{aligned}$$

The speed of sound v_s in a material is related to its bulk modulus B and its density ρ. We can express the bulk modulus in terms of the pressure, and note that for relativistic gases, $\gamma = 4/3$.

$$\begin{aligned} v_s &= \sqrt{B/\rho}, \\ \text{and } B &= -V(\tfrac{\partial P}{\partial V})_{ad} = \gamma P \\ \Rightarrow v_s &= \sqrt{\gamma P/\rho} = \sqrt{\tfrac{\gamma N\epsilon}{3V\rho}} \sim \sqrt{\tfrac{\gamma k T_F}{3m_p}} \\ \Rightarrow v_s &= \sqrt{\tfrac{4}{3}\tfrac{1.38\cdot 10^{-23}\ \mathrm{J/K}\cdot 3\cdot 10^{12}\ \mathrm{K}}{3\cdot 1.67\cdot 10^{-27}\ \mathrm{kg}}} \\ \Rightarrow v_s &= 1.05\cdot 10^{8}\ \mathrm{m/s}. \end{aligned}$$

QUALS 2013

AJM

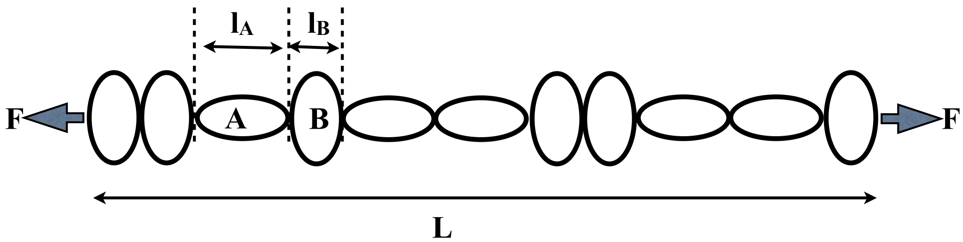


FIGURE 1. Example configuration of linear chain molecule with two orientations and corresponding energies and lengths indicated, along with applied tensile force (part c)

Molecular Chain

Consider a one-dimensional chain consisting of N molecules. Each molecule can exist in one of two configurations, A and B, with energy E_A and E_B and length l_A and l_B respectively (see Fig.)

(a) Write down the partition function for the system at temperature T.

(b) Assume that $E_A > E_B$ and $l_A > l_B$. Give an expression for the average length L of the molecule as a function of temperature at a fixed, large, N. Sketch the result, give the high and low T limits and indicate what is the characteristic temperature at which the length changes from the high-T to the low-T limits.

(c) Calculate the linear response function $\chi = \partial < L > /\partial F$ showing how the length changes if a small tensile force is applied

Solutions Chain

Sec 5 - 5

(a)

$$Z = \left(e^{-E_A/T} + e^{-E_B/T}\right)^N \tag{3}$$

(b) Define $\Delta = E_A - E_B > 0$.

$$< L >= N\frac{l_A e^{-\Delta/T} + l_B}{1 + e^{-\Delta/T}} \tag{4}$$

At T=0, $< L >= N l_B$. As $T \to \infty$, $< L >\to (l_A + l_B)/2$. Crossover when $T \approx \Delta$.

(c) *may want to add an intermediate step such as asking for the partition function at given F.* Adding an elastic force changes $\Delta \to \Delta - F(L_A - L_B)$. Putting this in and differentiating with respect to F gives

$$\chi = \frac{(L_A - L_B)^2}{4T\ cosh^2 \frac{\Delta}{2T}} \tag{5}$$

Solutions Two state

General (Thermo/statistical) Ruderman

(Points out of 15)

A large isolated box of volume V is filled with black body radiation in thermodynamic equilibrium. The radiation has an initial total energy U_0 (relative that of a vacuum in that same volume). A cyclic engine pulls heat from the boxed radiation which is subsequently dumped into a large thermal reservoir whose temperature remains at a fixed T_R.

(5) a) What is the minimum work (W) that must be expended on the cyclic engine to bring the radiation in the box arbitrarily close to being a vacuum if the boxed radiation temperature was originally T_R?

(5) b) What is the average energy in a ~~the~~ lowest frequency mode ($\omega_0 = 2\pi f_0$) at arbitrary temperature (T)?

(4) c) What is the probability that it is empty of photons?

(1) d) What is changed if it is not the lowest frequency mode?

Answers

a) $\overset{\swarrow \text{Into Res}}{Q_R} = U_0 + W$

$\Delta S_R \uparrow$ by $\frac{Q_R}{T_R}$

$S_{BB} = \int_0^{T_R} \frac{dU}{dT} \frac{1}{T} dT =$ $\quad$ BB rad $\quad U = \left(\frac{T}{T_R}\right)^4 U_0$

$= \frac{4}{3} U_0 / T_R$

$\frac{U_0 + W}{T_R} = \frac{4}{3} \frac{U_0}{T_R}$ $\quad \boxed{W = \frac{U_0}{3}}$

b) $\bar{E}_0 = \frac{\hbar \omega_0}{e^{\hbar\omega_0/k_BT} - 1}$ (Planck ~~law~~)

c) $P(n) = \frac{e^{-\hbar\omega_0 n/kT}}{\sum_{n=0}^{\infty} e^{-\hbar\omega_0 n/kT}} = \frac{1}{\Sigma} = \left(1 - e^{-\hbar\omega_0/k_BT}\right)$ $\quad n=0$

d) nothing.