Columbia University
Department of Physics
QUALIFYING EXAMINATION
Wednesday, January 16, 2008
11:10 AM – 1:10 PM

Modern Physics
Section 4. Relativity and Applied Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 4 (Relativity and Applied QM), Question 2; Section 4(Relativity and Applied QM) Question 3, etc.).

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

1. The Schrödinger equation for helium cannot be solved exactly. However, if we replace the Coulomb force with a spring force, the system can be solved exactly. As an example, consider the Hamiltonian in 3-dimensional space given by:

$$H = -\frac{\hbar^2}{2m}(\nabla_1^2 + \nabla_2^2) + \frac{1}{2}m\omega^2(r_1^2 + r_2^2) - \frac{\lambda}{4}m\omega^2 |\vec{r}_1 - \vec{r}_2|^2$$

assuming $\lambda < 1$.

(a) Setting first $\lambda = 0$, write the ground state energy of the two (uncoupled) oscillators. Also write the normalized ground state wavefunction (use the notation $\alpha \equiv m\omega/\hbar$).

(b) Based on the above ground state wavefunction, estimate the ground state energy of the system for $\lambda \neq 0$.

(c) What can you say about the sign of the error of your approximation? That is, do you expect the approximate answer to be larger or smaller than the exact result?

(d) By a suitable change of variables, transform the full H into two independent 3-dimensional harmonic oscillators.

(e) What is the exact ground state energy of the system? Does it fulfill your expectation of part (c)?

The following integral may be useful:

$$\int_0^\infty x^{2n} e^{-ax^2} = \begin{cases} \dfrac{1}{2}\sqrt{\dfrac{\pi}{a}} & for\ n = 0 \\ \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^{n+1} a^n}\sqrt{\dfrac{\pi}{a}} & for\ n = 1, 2, \ldots \end{cases}$$

2. A long straight wire is carrying a steady current I. A charge q at a distance r is moving with velocity v parallel to the wire. For the special case where the velocity v is the same as that for the charge carriers in the wire, calculate the following quantities in the rest frame of q:

(a) the charge density of the wire;

(b) the electrical force the wire exerts on q. Transform this force to the rest frame of the wire and show that it is precisely the same as the magnetic force exerted on the moving charge in this frame.

Hint: the wire is electrically neutral in its rest frame.

3. The cosmic microwave background radiation has a temperature of 3 K. A cosmic ray proton of sufficient energy can collide with a microwave background photon and form a pion through the process $p + \gamma \rightarrow p + \pi$. Assuming a photon of average energy, what is the minimum proton energy (in electron volts) needed for this process to occur?

(Hint: the mass of a pion is about 1/6 the mass of a proton.)

4. The π^- meson interacts with the deuteron from an s orbital state to form two neutrons.

$$\pi^- + d \rightarrow n + n$$

(a) Determine the allowed L, S, J values for the neutrons.

(b) From this process, deduce the parity of the pion.

(A deuteron has J^P (spin$^{\text{parity}}$) = 1^+, the neutron has $J^P = \frac{1}{2}^+$ and pions have zero spin.)

5. Figure 1 shows C_V, the specific heat at constant volume, of H_2 gas as a function of temperature in units of R, where R is the universal gas constant. We know that $R = k_B N_A$, where $k_B = 1.38\times10^{-23}$ J/K is the Boltzmann constant and $N_A = 6.02\times10^{23}$ is Avogadro's number. From this data for C_V, and the value of the Bohr radius, $a_0 = 0.053$ nm, estimate the value of Planck's constant.

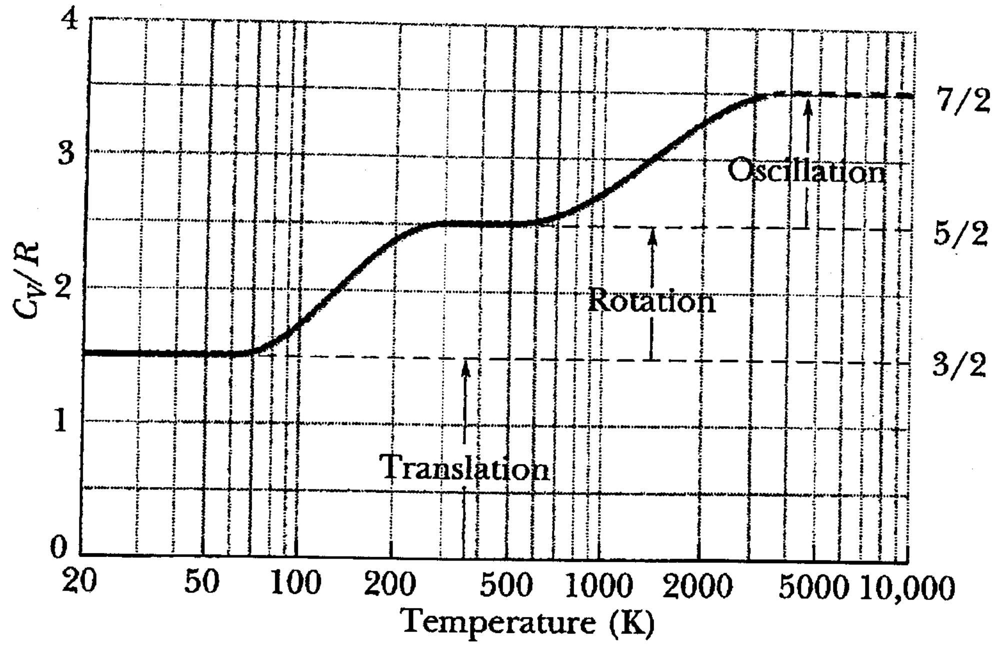

Figure 1.

Qualifying exam
Quantum mechanics
Eduardo Pontón

1. The Schrödinger equation for helium cannot be solved exactly. However, if we replace the Coulomb force with a spring force, the system can be solved exactly. As an example, consider the Hamiltonian in 3-dimensional space given by:

$$H = -\frac{\hbar^2}{2m}(\nabla_1^2 + \nabla_2^2) + \frac{1}{2}m\omega^2(r_1^2 + r_2^2) - \frac{\lambda}{4}m\omega^2|\vec{r}_1 - \vec{r}_2|^2 \, , \tag{1}$$

assuming $\lambda < 1$.

   (a) Setting first $\lambda = 0$, write the ground state energy of the two (uncoupled) oscillators. Also write the normalized ground state wavefunction (use the notation $\alpha \equiv m\omega/\hbar$).

   (b) Based on the above ground state wavefunction, estimate the ground state energy of the system for $\lambda \neq 0$.

   (c) What can you say about the sign of the error of your approximation? That is, do you expect the approximate answer to be larger or smaller than the exact result?

   (d) By a suitable change of variables, transform the full H into two independent 3-dimensional harmonic oscillators.

   (e) What is the exact ground state energy of the system? Does it fulfill your expectation of part (c)?

The following integral may be useful:

$$\int_0^\infty x^{2n} e^{-ax^2} = \begin{cases} \frac{1}{2}\sqrt{\frac{\pi}{a}} & \text{for } n = 0 \\ \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}} & \text{for } n = 1, 2, \ldots \end{cases}$$

Solutions

① a) Ground state wavefunction for $\lambda = 0$:

$$\Psi_0(\vec{r}_1, \vec{r}_2) = \left(\frac{\alpha}{\pi}\right)^{3/2} e^{-\alpha(\vec{r}_1^{\,2} + \vec{r}_2^{\,2})/2} \quad , \quad \alpha = \frac{m\omega}{\hbar}$$

with energy $E_0 = 3\hbar\omega$

b) Use the variational method with trial wavefunction

$$\Psi_\eta \equiv \left(\frac{\eta}{\pi}\right)^{3/2} e^{-\eta(\vec{r}_1^{\,2} + \vec{r}_2^{\,2})/2} \quad , \quad \langle \Psi_\eta | \Psi_\eta \rangle = 1$$

We need

$$\langle \Psi_\eta | \frac{\hat{p}_1^2}{2m} | \Psi_\eta \rangle = \frac{1}{2m} \| \hat{p}_1 | \Psi_\eta \rangle \|^2$$

$$= \frac{\hbar^2}{2m} \left(\frac{\eta}{\pi}\right)^{3/2} \int d^3 r_1 \left| \nabla_1 e^{-\eta \vec{r}_1^{\,2}/2} \right|^2$$

$$= \frac{\hbar^2}{2m} \left(\frac{\eta}{\pi}\right)^{3/2} 4\pi \int_0^\infty r_1^2 dr_1 \left(\eta^2 r_1^2 e^{-\eta r_1^2}\right)$$

$$= \frac{\hbar^2}{2m} \left(\frac{\eta}{\pi}\right)^{3/2} 4\pi \eta^2 \frac{3}{2^3 \eta^2} \sqrt{\frac{\pi}{\eta}}$$

$$= \frac{3}{2} \eta \frac{\hbar^2}{2m}$$

$$\langle \Psi_\eta | \tfrac{1}{2} m\omega^2 r_1^2 | \Psi_\eta \rangle = \frac{3}{2} \eta^{-1} \frac{1}{2} m\omega^2$$

$$\langle \Psi_\eta | -\frac{\lambda}{4} m\omega^2 (\vec{r}_1 - \vec{r}_2)^2 | \Psi_\eta \rangle =$$

(angular integration)

$$-\frac{\lambda}{4} m\omega^2 \left(\frac{\eta}{\pi}\right)^3 \int d^3 r_1 \, r_2^2 dr_2 \, d\varphi_2 \, d\cos\theta_2 \left\{ r_1^2 + r_2^2 - \underbrace{2 r_1 r_2 \cos\theta_2}_{0} \right\} e^{-\eta(r_1^2 + r_2^2)}$$

$$= -\frac{\lambda}{4} m\omega^2 \left(\frac{\eta}{\pi}\right)^{3/2} \times 2 \int d^3 r \, r^2 e^{-\eta r^2}$$

$$= -\frac{3}{2} \eta^{-1} \frac{\lambda}{2} m\omega^2$$

Then

$$0 = \frac{d}{d\eta} \frac{\langle \Psi_\eta | H | \Psi_\eta \rangle}{\langle \Psi_\eta | \Psi_\eta \rangle} = \frac{3}{2} \frac{d}{d\eta} \left\{ \frac{\hbar^2}{m} \eta + m\omega^2 \eta^{-1} - \frac{\lambda}{2} m\omega^2 \eta^{-1} \right\}$$

$$\Rightarrow \quad \eta = \sqrt{1 - \frac{\lambda}{2}} \; \frac{m\omega}{\hbar} = \sqrt{1 - \frac{\lambda}{2}} \; \alpha$$

and

$$E_0^{estimate} = \frac{\langle \Psi_\eta | H | \Psi_\eta \rangle}{\langle \Psi_\eta | \Psi_\eta \rangle} = 3 \sqrt{1 - \frac{\lambda}{2}} \; \hbar\omega$$

c) $E_0^{true} \leqslant E_0^{estimate}$

d) Do
$$\vec{u} = \frac{1}{\sqrt{2}} (\vec{r}_1 + \vec{r}_2)$$
$$\vec{v} = \frac{1}{\sqrt{2}} (\vec{r}_1 - \vec{r}_2)$$

to obtain

$$H = \left[-\frac{\hbar^2}{2m} \nabla_{\vec{u}}^2 + \frac{1}{2} m\omega^2 u^2 \right] + \left[-\frac{\hbar^2}{2m} \nabla_{\vec{v}}^2 + \frac{1}{2} m\omega^2 (1-\lambda) v^2 \right]$$

e)
$$E_0^{true} = \frac{3}{2} \hbar\omega + \frac{3}{2} \hbar\omega \sqrt{1-\lambda}$$
$$= \frac{3}{2} \hbar\omega \left\{ 1 + \sqrt{1-\lambda} \right\}$$

The function

$$f(\lambda) = \sqrt{1 - \frac{\lambda}{2}} - \frac{1}{2} \left\{ 1 + \sqrt{1-\lambda} \right\}$$

has a minimum at $\lambda = 0$, where $f(0) = 0$, and is otherwise positive (for $\lambda \leqslant 1$) as expected in part c).

Problem: A long straight wire is carrying a steady current I. A charge q at a distance r is moving with velocity v parallel to the wire. For the special case where the velocity v is the same as that for the charge carriers in the wire, calculate the following quantities in the rest frame of q: a) the charge density of the wire and b) the electrical force the wire exerts on q. Transform this force to the rest frame of the wire and show that it is precisely the same as the magnetic force exerted on the moving charge in this frame. **Hint:** the wire is electrically neutral in its rest frame.

Solution: (I'll treat the charge carriers as positive, to avoid an irrelevant minus sign.)

Write $\lambda^0_\pm$ for the charge density of positive and negative charges when at rest in the wire. The presence of the current and the neutrality of the wire requires

$$\frac{\lambda^0_+}{\sqrt{1-\beta^2}} + \lambda^0_- = 0 \quad ; \quad \beta = v/c \quad . \tag{1}$$

In the rest frame of q, $\lambda^0_+/\sqrt{1-\beta^2} \to \lambda^0_+$ and $\lambda^0_- \to \lambda^0_-/\sqrt{1-\beta^2}$, so q sees a line charge density

$$\lambda_{NET} = \lambda^0_+ + \frac{\lambda^0_-}{\sqrt{1-\beta^2}} \tag{2}$$

$$= \lambda^0_-(-\sqrt{1-\beta^2} + \frac{1}{\sqrt{1-\beta^2}}) \tag{3}$$

$$\Rightarrow E(r) = \frac{\lambda_{NET}}{2\pi\epsilon_0 r} = \frac{\lambda^0_-}{2\pi\epsilon_0 r}(-\sqrt{1-\beta^2} + \frac{1}{\sqrt{1-\beta^2}}) \quad , \tag{4}$$

so the electrical force in q's rest frame is $F_q = qE(r)$, directed towards the wire.

To transform back to the wire frame, write

$$F_q = \frac{dp'_\perp}{dt'} = \frac{1}{\sqrt{1-\beta^2}} \frac{dp_\perp}{dt} \tag{5}$$

so that the force F_w in the wire frame is

$$F_w(r) = \sqrt{1-\beta^2}F_q(r) = \sqrt{1-\beta^2}\frac{q\lambda_-^0}{2\pi\epsilon_0 r}(-\sqrt{1-\beta^2}+\frac{1}{\sqrt{1-\beta^2}}) = \frac{q\lambda_-^0}{2\pi\epsilon_0 r}(-1+\beta^2+1) = \frac{q\lambda_-^0 v^2}{2\pi\epsilon_0 c^2 r} \quad . \tag{6}$$

This can be rewritten as

$$F_w(r) = \frac{q\lambda_-^0 v^2}{2\pi\epsilon_0 c^2 r} = qv\mu_0\frac{\lambda_-^0 v}{2\pi r} = qv\frac{\mu_0 I}{2\pi r} = qvB(r) \quad , \tag{7}$$

with a direction (towards the wire) consistent with the right-hand-rule.

Weinberg (Relativistic)

The cosmic microwave background radiation has a temperature of 3 K. A cosmic ray proton of sufficient energy can collide with a microwave background photon and form a pion through the process

$$p + \gamma \rightarrow p + \pi$$

Assuming a photon of average energy, what is the minimum proton energy (in electron volts) needed for this process to occur?

Sec IV #3

Weinberg

DEC 1 2007

GZK Problem – Solution

We want $p + (CMB\gamma) \longrightarrow p + \pi$

$(p_1) \quad (k) \quad (p_2) \quad q$

$$\Rightarrow (p_1 + k)^2 = (p_2 + q)^2 > (m_p + m_\pi)^2$$

$$\rightarrow m_p^2 + 2p_1 \cdot k + 0$$

But $2p_1 \cdot k = 2\left(E_p k - \sqrt{E_p^2 - m^2}\, k \cos\theta\right)$

$$\approx 2 E_p k (1 - \cos\theta)$$

This is maximized by $\cos\theta = -1$

$$\Rightarrow \cancel{2}\, 4E_p k > (m_\pi^2 + 2m_p \cancel{2} m_\pi) = m_\pi(2m_p + m_\pi)$$

$$E_p > \frac{m_\pi(2m_p + m_\pi)}{4k}$$

k = avg E of CMB photons $= 2.7\,T$
$\approx 7 \times 10^{-4}$ eV

$$E_p > \frac{(1.4 \times 10^8 \text{eV})(2.0 \times 10^9 \text{eV})}{4(7.0 \times 10^{-4} \text{eV})} \approx 10^{20} \text{eV}$$

Sec 4 Question 5

Solution

Since quantum mechanics says that angular momentum is quantized, at low temperatures rotational motion of the H_2 molecule is not possible. With only translations, we have $C_V = \frac{3}{2}R$, as for a monatomic gas.

Around 100K, the graph shows that the heat capacity changes and this is due to rotational motion being excited by the average thermal energy. Equipartition then gives

$$E_{\text{rot}} = \frac{L^2}{2I} \sim k_B T \tag{1}$$

Since $L \sim \hbar$, we only need to know I to determine (roughly) Planck's constant.

We can estimate I from the Bohr radius, $a_0 = 0.053\text{nm}$, giving $I \sim 2ma_0^2$. This gives

$$I = 2 \times \frac{2 \times 10^{-3}\,\text{kg}}{6.02 \times 10^{23}} \times (5.3 \times 10^{-11}\,\text{m})^2 = 1.87 \times 10^{-47}\,\text{kg}\,\text{m}^2 \tag{2}$$

If we use $T = 100$ K, we find

$$\begin{aligned} \hbar^2 &\sim 2Ik_BT && (3)\\ &= 2 \times (1.87 \times 10^{-47}\,\text{kg}\,\text{m}^2) \times (1.38 \times 10^{-23}\,\text{J/K}) \times (100\,\text{K}) && (4)\\ &= 5.16 \times 10^{-68}\,\text{J}^2\,\text{s}^2 && (5) \end{aligned}$$

So we estimate

$$\hbar \sim 2 \times 10^{-34}\,\text{J}\,\text{s} \tag{6}$$

~~General~~ Physics Problem:

The π^- meson interacts with the deuteron from an s orbital state to form two neutrons.

$$\pi^- + d \rightarrow n + n$$

a) Explain why the neutrons necessarily need to be in the state L=1, S=1, and J=1.

b) From this process, deduce the parity of the pion.

(A deuteron has J^P ($\text{spin}^{\text{parity}}$) = 1^+, the neutron has $J^P = 1/2^+$ and pions have zero spin)

Solution:
Neutrons are identical fermions and need to be in an anti-symmetric state

a)

$$0^? + 1^+ \rightarrow \tfrac{1}{2}^+ + \tfrac{1}{2}^+$$

$S = 1 \qquad J = 1$

$\mathrm{L} = 0 \qquad \mathrm{L} = 0$ and $\mathrm{S} = 1$ not allowed since totally symmetric

$\mathrm{L} = 1$ and $\mathrm{S} = 0$ not allowed since totally symmetric

$\mathrm{L} = 1$ and $\mathrm{S} = 1$ allowed since symmetric $\otimes$ anti-symmetric

b)

LHS $\qquad$ RHS

$(?)\,(+)\,(+) = ? \qquad (+)\,(+)\,(-) = - \qquad \Rightarrow$ Parity (?) of pion = (-)