Lecture notes on risk management, public policy, and the financial system

Interest rates and credit spreads

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Observing interest rates

- Interest rates and yield curves
- Bond math: spot, forward and par yield curves
- Estimating the yield curve
- Interest rate volatility

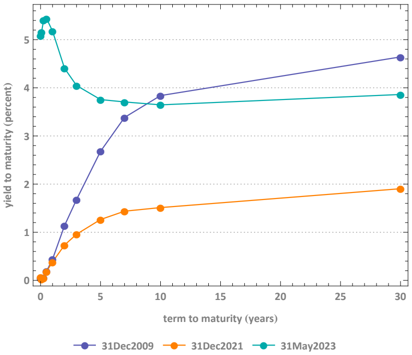
Explaining interest rates

Credit spreads and spread risk

Interest rate risk measurement

Yield curve

- Interest rates—risk-free or for a given obligor—are not a single risk factor
- Typically vary by the term of the loan or security
- **Term structure of interest rates** or **yield curve:** rates as function of maturity



U.S. on-the-run Treasury yield curve. *Source*: Bloomberg Financial LP.

Typical behavior of the term structure

- Level and shape of curve based primarily on expected future short-term rates
- But also contains liquidity, credit, interest-rate and other **risk premiums**
- Term structure generally upward-sloping due to
  - Longer-term risk-free interest rates and credit spreads generally higher than short-term
  - Particularly steep at very short end due to (→)**money premium**
- But may be downward-sloping—**yield curve inversion**—overall or in some segments, e.g.
  - Short-term rates spike, not expected to persist, e.g. emerging-market rates under foreign exchange pressure
  - High demand for safe long-term bonds, e.g. 2005 conundrum in U.S.

Compounding

- Like other asset returns, interest compounds over time

  Discrete compounding: calculated as if interest paid and added to principal at discrete intervals

  Continuous compounding: calculated as if interest paid continuously in infinitely small increments
- Some fixed income instruments pay **coupon** at discrete intervals
  - As fraction of **par value**—number of currency units—of **principal**
  - Generally annually, semiannually or quarterly
  - **Accrued interest:** owed but not yet paid out

Spot, discount and continuously compounded rates

- Consider a riskless zero-coupon security with price S, paying \$100 one year hence
- Corresponding **discount factor** or **rate** is the arithmetic return

$$r = \frac{1-S}{S} = \frac{1}{S} - 1 \quad \leftrightarrow \quad S = \frac{1}{1+r}$$

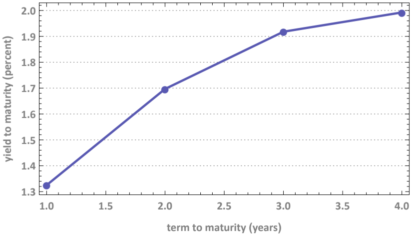
- Continuously compounded rate is $\ln\left(\frac{1}{S}\right) = -\ln(S)$
- **Example:** $S = 98.692$
  - Discount rate is $1.325 = 100\left(\frac{1}{0.98692} - 1\right)$
  - Continuously compounded rate is $1.3163 = -100\ln(0.98692)$

Equivalent ways of expressing the term structure

- Term structure can be represented in different ways, each useful in certain contexts
- **Yield curve:** yields of bonds/loans as a function of term
  - Yield to maturity calculations assume reinvestment of coupons at that yield
  - Ambiguity due to varying coupon size (→**par yield curve**), accrued interest
- **Spot** or **zero-coupon curve** and **spot** or **zero-coupon rates:**
  - Money lent now and repaid at single specific time in the future
  - No coupon or other interim payments
- **Forward curve** and **forward rates:**
  - Loans of a specified term to maturity commencing at future **settlement date**
  - E.g. rates on 3-month loans settling immediately, in 1 month, in 3 months, ...
  - Can be transformed into spot curve and v.v.

Bond prices and yield to maturity

- Simple example:
  - Shortest-term issue: a 1-year bill or zero-coupon
  - Coupon bonds maturing in 2, 3 and 4 years
- Annualized interest rates, annual pay frequency and compounding



term	coupon	price	yield
1 year	0.00	98.6923	1.32500
2 years	1.75	100.1039	1.69674
3 years	2.00	100.2368	1.91804
4 years	2.00	100.0286	1.99249

Bond price, yield, and spot rate relationships

- Quote data in the form of prices and/or yields
- Current price S_t of a t-year bond with coupon c_t per unit of par value is related to its (generally observable) yield y_t by

$$S_t = c_t \left[\frac{1}{1+y_t} + \frac{1}{(1+y_t)^2} + \cdots + \frac{1}{(1+y_t)^t} \right] + \frac{1}{(1+y_t)^t}$$

  - Converts quote in price terms to yield terms and v.v.
- Bond price also related to its (generally unobservable) spot rates $r_1, r_2, \ldots, r_t$:

$$S_t = c_t \left[\frac{1}{1+r_1} + \frac{1}{(1+r_2)^2} + \cdots + \frac{1}{(1+r_t)^t} \right] + \frac{1}{(1+r_t)^t}$$

  - Derives unobserved spot rates from observed quote data

Zero-coupon bonds

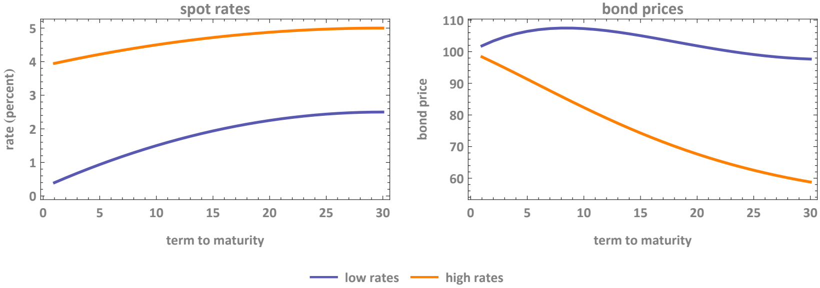
- Zero-coupon bonds: coupon $c_t = 0$
- Price, yield of t-year zero coupon bond and t-year spot rate related by

$$S_t = \frac{1}{(1+y_t)^t} = \frac{1}{(1+r_t)^t},$$

with y_t, r_t denoting t-year annually compounded rates

Impact of yield curve shifts

- Rise in interest rates goes hand in hand with fall in bond prices
- The relationship is strongest for longer-term bonds
- **Example:** rise in long-term rates from 2.5 to 5 percent and curve flattening



Left panel: spot rate curve as a function of maturity T, initially given by $100(0.025 - 0.000025(T-\tau)^2)$, to $100(0.05 - 0.0000125(T-\tau)^2), \tau = 1, \ldots, T$. Right panel: corresponding bond prices as a function of maturity. All bonds have an annual coupon of 2.25 percent.

Equivalent ways of expressing the spot curve

Discount curve and discount factors: present values corresponding to spot rates

- Discount factor corresponding to r_1 is $(1 + r_1)^{-1}$, discount factor corresponding to r_2 is $(1 + r_2)^{-2}$, etc.

Par yield curve: hypothetical bonds with coupons equal to yields

- Computed from spot curve by setting price to par and solving for coupon rate

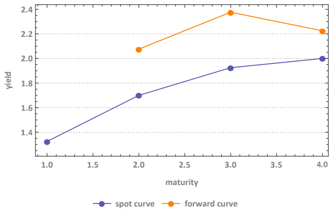
term	discount factor	par yield
1 year	0.98692	1.32500
2 years	0.96685	1.69683
3 years	0.94440	1.91831
4 years	0.92385	1.99252

Forward and spot rates: example

- Forward and spot rates linked by arbitrage
- Current forward rate $f_{t,t+1}$ from t to $t+1$ related to r_t r_{t+1} by

$$f_{t,t+1} = \frac{(1+r_{t+1})^{t+1}}{(1+r_t)^t} - 1$$

- Forward rate $f_{0,1}$ settling immediately is identical to shortest-term spot rate
- Forward higher than spot rates if yield curve positively sloped
- Forward curve falling due to **convexity** of spot curve
  - Slope of spot curve positive but declining



Assumed spot and derived forward curves, percent.

term	spot	forward
1 year	1.325	
2 years	1.700	2.0764
3 years	1.925	2.3765
4 years	2.000	2.2253

Challenges in yield curve measurement

- Derived yield curves used in pricing, valuation, risk analysis
    - Derived curves—spot curve or other forms—impose consistency
- Yield curve presents interest rates of different maturities on otherwise similar fixed-income instruments
    - But many different types of instruments
    - For most instruments, only some maturities have observable rates
    - Market quotes for different securities on same curve may employ different pay frequency, day count and other conventions
- Yield curve can be equivalently presented as spot, discount or forward curve
    - Derived forward curve may have odd appearance
    - Small changes in observable interest rates may lead to large changes in forward curve and large changes in prescribed hedges

Common techniques for estimating the yield curve

Bootstrapping: technique for obtaining spot curve sequentially

- Enables uniform representation of observable rates as spot, discount or forward curve

Spline interpolation of observable interest rates

Least-squares fitting or **parsimonious** yield curve estimation

- Relies on parameters capturing typical shapes of the yield curve

Bootstrapping the spot curve from bond prices

- Bootstrapping starts with shortest-maturity security, uses each successively longer security to capture one longer-term spot rate
- For each bond maturity, assumes we have either a price or yield (or both—and coupon rate)

Bootstrapping spot rates from prices: example

- 1-year bond has no coupon (i.e. $c_1 = 0$), so S_1 and r_1 satisfy

$$S_1 = 0.986923 = \frac{1}{1+r_1} = \frac{1}{1.01325}$$

- Next, solve for $r_2 = 0.0170$ from:

$$\begin{aligned} S_2 = 1.00104 &= c_2 \left[\frac{1}{1+r_1} + \frac{1}{(1+r_2)^2} \right] + \frac{1}{(1+r_2)^2} \\ &= \frac{0.0175}{1.01325} + \frac{0.0175+1}{(1+r_2)^2} \end{aligned}$$

- Solve the same type of equation to get $r_3 = 0.01925$:

$$\begin{aligned} S_3 = 1.00237 &= c_2 \left[\frac{1}{1+r_1} + \frac{1}{(1+r_2)^2} + \frac{1}{(1+r_3)^3} \right] + \frac{1}{(1+r_3)^3} \\ &= 0.02 \times 1.95377 + \frac{0.02+1}{(1+r_3)^3} \end{aligned}$$

- And, finally, $r_4 = 0.02$

Cubic spline interpolation

- Spline interpolation: general technique for connecting discrete points with smooth curves
- Curves are defined piecewise as polynomials

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$$

  through groups of neighboring observations
  - Interpolated curve is continuous
  - Smoothness of interpolation governed by order n of polynomials
- Rates then estimated for maturities not observed in data
- Cubic splines frequently used for yield curves
  - Continuous first and second derivatives
  - Avoids abrupt changes in slope and curvature of estimated curve

Parsimonious yield curve estimation

- Yield curve at a point in time represented as a polynomial or other closed-form function of maturity
- Use curve fitting technique to estimate parameters
- Fitted function may not pass through observed interest rates
  - If many points on yield curve observable market prices, fitted curve may be quite different from interpolation

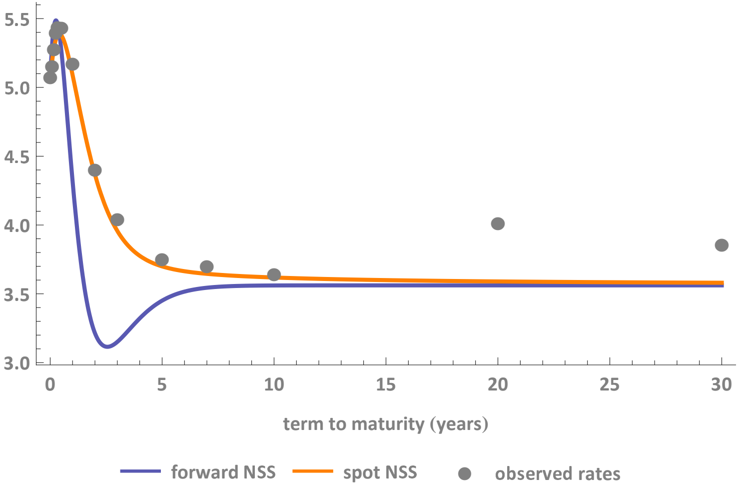
The Nelson-Siegel function

- **Nelson-Siegel** model
  - Forward curve specified as a function of time to maturity τ

$$f(\tau; \beta_0, \beta_1, \beta_2, \theta) = \beta_0 + \beta_1 e^{-\frac{\tau}{\theta}} + \beta_2 \frac{\tau}{\theta} e^{-\frac{\tau}{\theta}}$$

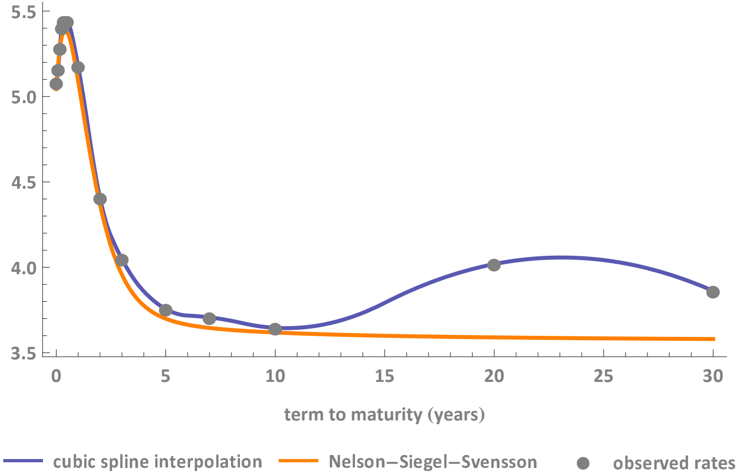
  - Parameters $\beta_0, \beta_1, \beta_2$, and θ capture level, slope and curvature of yield curve
- **Nelson-Siegel-Svensson** extension: additional term better captures hump shape and other features of yield curve

The Nelson-Siegel-Svensson yield curve



U.S. Treasury yield curve May 31, 2023.

Estimated yield curve



U.S. Treasury yield curve May 31, 2023.

Measures of interest rate volatility

- Two standard conventions in fixed-income option markets for implied volatility

  Normal volatility: standard deviation of changes in yield $\sigma_{\mathrm{n},t}$ in basis points

  Yield or **Black volatility:** standard deviation of proportional changes in yield $\sigma_{\mathrm{y},t}$ in percent

  - Divide normal vol by yield:

$$\sigma_{\mathrm{y},t} = \frac{\sigma_{\mathrm{n},t}}{y_t}$$

  - Measures yield changes—oddly—in percent rather than bps
  - But avoids—perhaps unnecessarily—negative yield scenarios
- Implied volatilities are (→)risk-neutral estimates
  - Reflect market information, but embed risk premiums embedded in option prices

Applying interest rate volatility

- **Example:** typical (as of late 2016) at-the-money short- to medium-term swaption on 10-year USD swap
  - Normal volatility of about 90bps at annual rate
  - Swap rate of 2.25 percent
  - $\Rightarrow$Yield volatility of about $\frac{90}{2.25} = 40$ percent at annual rate
  - Or $\frac{0.4}{\sqrt{256}} = 0.025$ (2.5 bps) per trading day applying ($\rightarrow$)square-root-of-time rule

Observing interest rates

Explaining interest rates

- Sources of interest rate risk
- Interest rate models
- Inflation and inflation risk

Credit spreads and spread risk

Interest rate risk measurement

Sources of fixed income return

Deterministic sources: returns with an unchanged yield curve

positive return to capital investment

Time preference: preference for consumption sooner, greater impatience$\rightarrowtail$higher interest rates

Productivity-of-waiting: more roundabout production process$\rightarrowtail$higher return

Opportunity cost: higher general productivity$\rightarrowtail$higher interest rates

Interest rate risk: yield curve may change over time, affecting security values

- Changes in interest rates random and arise for many reasons

Credit, inflation and liquidity risk: value and timing of future payments uncertain

Why is there a term structure?

- Term structure arises because
  - Borrowers issue debt with varying initial maturity and at different points in time
  - Maturity of any particular issue declines over its life
- Borrowers motivated to issue both short- and long-term debt
  - Volume of short-term debt can be varied to adjust to business fluctuations
  - Long-term debt provides greater certainty of financing for long-lived projects and core business requirements

Decomposing interest rates: inflation

- **Nominal rates** express interest and principal payments in units of money
- **Real rates** express payments in units of goods
  - E.g. gold or determined with reference to a price index
  - Measures of constant **purchasing power**
- **Inflation compensation** or **breakeven inflation** rate: difference between nominal and real rates
  - May include an inflation risk premium
- Observed nominal rate can be decomposed into estimates of unobservable real rate and inflation compensation

Decomposing interest rates: risk

Risk-free or **pure rate of interest** compensates for passing of time

Expected future interest rates: longer-term yield is based on short-term rates expected to prevail up to maturity

Term premium: additional yield compensating for bearing interest-rate risk of holding longer-term securities

- Aversion to term risk may change without change in expected future interest rates

Spreads over the risk-free rate compensate primarily for

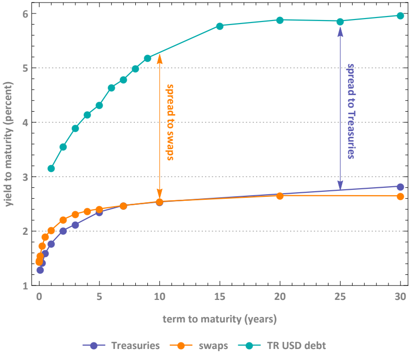
Credit risk: compensation for risk of default and credit migration losses

Liquidity risk: compensation for cost and risk of exiting, adjusting or maintaining position

Inflation risk: compensation for risk of higher-than-expected inflation

Spreads

- Risk-free curve: a base/benchmark relative to which risky yield curves measured
- Risky curve: generally expressed in terms of spreads relative to risk-free
- **Credit spread:** difference between interest rate on a risky and congruent credit risk-free security



Risk-free: U.S. on-the-run Treasury curve; risky: Republic of Turkey U.S. dollar-denominated yield curve, 16Jan2018. *Source*: Bloomberg Financial LP.

Attributing spreads to risks

- Each source of risk has an expected value and risk impact
- Wider spreads make bonds cheaper, and thus increase prospective future returns
  - Higher future return provides **risk premium** compensating for expected and unexpected losses
- Models needed to decompose spreads into various sources
- Spreads can also be measured across currencies

Deterministic sources of fixed income return

Yield of the security

Rolldown or **theta:** term to maturity shortens over time as maturity date nears

- Generally positive, since yield curves generally upward-sloping
- **Example:** a 3-year bond held for 1 year becomes a 2-year bond, typically with a lower yield and higher market value

Credit risk premium

- Credit risk generally the main driver of spreads and changes in spreads over risk-free yields
- Credit spread compensates for expected value of default—default probability times loss given default
- But also compensates for “putting up with” default risk
- Includes the market risk of changes in credit-related portion of spread

Liquidity risk premium

- Liquidity risk can have countervailing impact on spreads
- **Liquidity premium:** aversion to risk of holding longer-term securities may change
    - More likely to affect corporate and sovereign issues outside advanced market economies
    - Sovereign issues of advanced market economies may have negative liquidity risk premium
- Financial stress likely to impair liquidity conditions, induce flight to quality, lowering yields on most-liquid sovereign bonds
- **Money premium** due to provision of money services by safe very short-term debt
    - Very short end of Treasury yield curve steeper than otherwise

Fitting, explaining and predicting

- Fitted yield curves summarize the observed yield curve
- Interest rate models are intended to explain and predict future interest rates

The expectations hypothesis

- Fitted yield curves summarize the observed yield curve
- **Expectations hypothesis** of the term structure of interest rates: long term interest rates are determined by the average short-term rate expected over the term of the security
    - Compares rates on obligations of the same issuer
- Equivalently: no excess returns of long-term bonds over rolling over short-term bonds

Interest rate factor models

- Model short-term interest rate as a stochastic process
  - Aligned with "memoryless" standard model of asset returns
  - Incorporates interest rate volatility through random shocks
  - Results in a term structure of rates
- **One-factor short-rate** or **Vasicek** model of short term interest changes
  - Deterministically toward a long term rate
  - In response to random shocks generated by a Wiener process

Vasicek model

- In discrete-time form, short-term interest rate follows

$$\begin{aligned}\Delta r_t &= (k\theta + \lambda - kr_t)\Delta t + \sigma z\sqrt{\Delta t} \\ &= k\left(\theta + \frac{\lambda}{k} - r_t\right)\Delta t + \sigma z\sqrt{\Delta t},\end{aligned}$$

  with z a standard normal variate
- Parameters:
  - k: speed of adjustment
  - θ: expected or equilibrium long-term rate
  - λ: term or risk premium appearing in some variants
- Short-term rate rises or falls if it is below or above θ

Decomposing nominal long-term rates

Path of short-term rates: long-term rate the sum of

- Market expectations of the average future short-term rate
- A risk or **term premium**: sum of premiums on several types of risk, may be negative

Fisher equation: simple one-period nominal interest rate R the sum of

- Real rate r
- Expected future inflation π^e
- Risk premium λ

$$1 + R = (1 + r)(1 + \pi^e)(1 + \lambda)$$

$$R \approx r + \pi^e + \lambda$$

Inflation risk premium

- Directly affects securities with payoffs defined in nominal terms
- Indirectly affects real assets by affecting macroeconomic conditions
- Inflation difficult to hedge
  - U.S. (since 1997), U.K., other countries issue **inflation-protected** or **inflation-linked bonds** that increase principal based on price index
  - **Inflation swaps** and other derivatives
  - **Gold clause** specifying gold value of principal and interest of certain U.S. Treasury bonds invalidated 1933 by Joint Resolution of Congress
- **Inflation risk premium** is positive if inflation-protected bond yield is lower than required to compensate for market's expected inflation
  - But hard to estimate since subjective or market expectations unobservable

Insurance company exposure to inflation

- Insurers may benefit from inflation
- Long-term liabilities generally defined in nominal terms
  - Generally not fully hedged against changes in interest rates
  - And substantial allocation to real assets: real estate, equities
- Permanent risk in inflation rate reduces real value of liabilities

Credit spreads and credit quality

- Credit spreads vary across several dimensions: Differences in
  - Spreads of different obligors on bonds with similar credit quality
    - Third-party guarantees, collateral, position in ($\rightarrow$)capital structure
  - Spreads of same obligor on bonds with different credit quality
    - $\rightarrow$Credit spread of senior unsecured$<$subordinated debt
  - Spreads of same obligor on bonds with different maturities
- Credit spread compensates for several sources of risk
  - Credit risk, reflecting both default probability and expected loss given default
  - Market liquidity risk: many credit-risky bonds infrequently traded or have small issuance volume

Measuring credit spreads

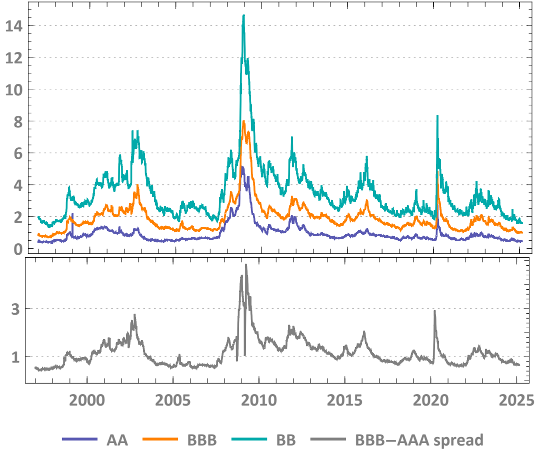
Spread to swaps or Treasuries with similar maturity. Can be implemented in different ways, e.g.

- z-**spread**: spread between zero-coupon curves
- i-**spread**: spread over interpolated risk-free curve

Option-adjusted spread (OAS): spread to a benchmark curve after accounting for the value of options embedded in the security

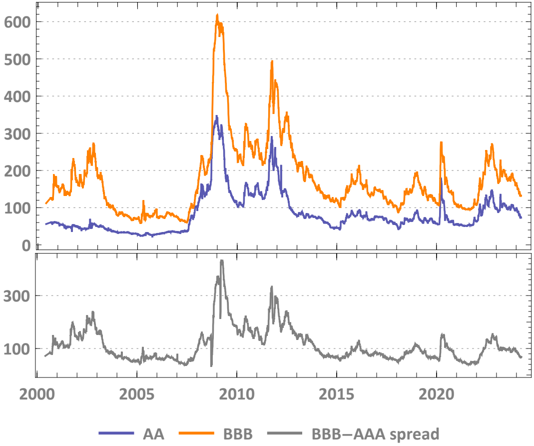
- Corporate bonds often **callable**
- Callable bond is bundled with call option on bond sold by investor to issuer

U.S. credit spreads 1996–2024



ICE BofA Merrill Lynch U.S. indexes of option-adjusted spreads (OAS) to the Treasury curve for corporate securities rated AA (C0A2), BBB (C0A4) and BB (H0A1), in percent. The BBB-AAA quality spread is the difference between the BBB and AA (C0A1) indexes. Daily, 31Dec1996 to 19Apr2024. *Source*: FRED.

European credit spreads 2000–2024



Bloomberg Barclays Euro Corporate Bond Index option-adjusted spread (OAS) to the treasury curve for securities rated AA and BBB, in basis points, daily, 31May2000 to 01Apr2024. The BBB-AAA quality spread is the difference, in percent, between the BBB and AAA indexes. *Source*: Bloomberg LP.

Credit derivatives

Credit default swaps (CDS): OTC contract, one counterparty pays regular premium and is made whole by the other if default of specific bond issuer occurs

- Economically similar to guarantee or insurance on firm's debt, but without requirement of direct economic exposure
- Partial standardization: **single-name CDS** governed by master agreements
- Pre-crisis, large volumes of CDS on residential and commercial real-estate securitizations

Credit default swap indexes (CDX)

- Protection on a set of **reference entities**
- **Examples**: CDX.IG.NA series covering North America, iTraxx series covering Europe, Asia

Credit default swap derivatives based on underlying CDS or CDX

- Standard tranches ($\rightarrow$structured credit products)
- OTC swaptions on CDS and CDX

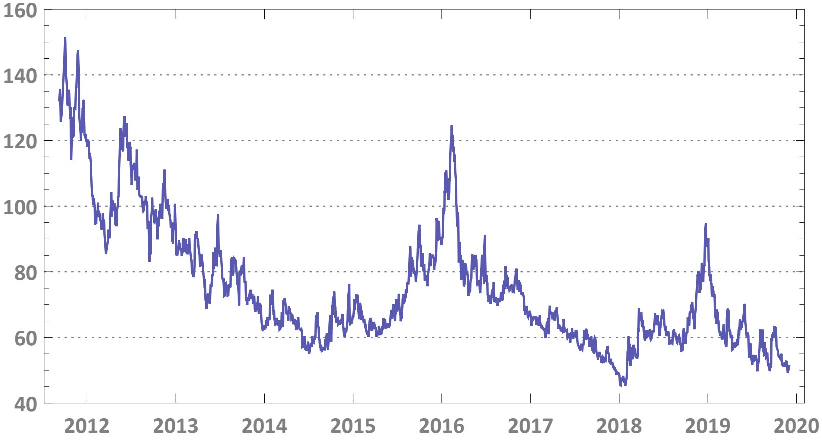
Pricing of credit derivatives

CDS spread: market-clearing premium in basis points paid for default protection

- Investment-grade CDS quoted in terms of credit spread
- High-yield CDS quoted in **points up front:** spread is standardized to 100 or 500 basis points, an upfront fee equates present value to market-based credit spread

CDS basis: difference between the CDS and bond spread

U.S. credit default swap index 2011–2019



On-the-run 5-year Markit CDX North American Investment Grade index, in basis points. Daily, 09Sep2011 to 02Jan2019. *Source*: Bloomberg Financial LP.

Credit spread risk

- **Credit spread risk**: risk of loss from change in credit spreads
  - Spread risk is a *market* risk, albeit credit-related
  - **Spread 01**: price/value impact of change in spreads, market risk of credit exposures
- Both the spread and spread risk a function of obligor credit quality
  - As well as of bond tenor, liquidity and other factors

Some special types of credit spread risk

Basis risk Risk that two similar, but not identical, securities diverge or converge in price.
Example: **Bond-CDS basis**, difference between credit spreads implied by bonds and CDS, influenced by availability of funding

Convexity risk Risk of loss from under-hedging a security with a highly non-linear payoff profile.
Example: **Structured credit products**, equity tranche may have highly non-linear response to change in default correlation (May 2005)

Citigroup CDS basis 2007–2010

Difference between the *z*-spread over Libor of Citigroup Inc. bonds and the premium on Citigroup 10-year senior unsecured CDS. The bond spread is blended from spreads on two senior unsecured issues: the 4.7% maturing May 29, 2015 (CUSIP 172967CY5) and the 5.85% maturing Aug. 2, 2016 (CUSIP 172967DQ1). *Source*: Bloomberg Financial LP.

Estimating default probability from credit spreads

- Credit spread expresses **risk-neutral hazard rate** and **default probability**
- Simplest case:
  - 1-year risk-free with yield r and credit-risky bonds with yield $r + z$ both exist
  - Recovery rate zero
- These present values should be equal:

$$\frac{(1-\tilde{\pi}_1)\times 1+\tilde{\pi}_1\times 0}{1+r}=\frac{1}{1+r+z}$$

- Spread z approximately equal to 1-year default probability $\tilde{\pi}_1$:

$$\boxed{\tilde{\pi}_1 \approx z}$$

Default probability and the hazard rate

- Credit risk models typically in continuous-time setting
- With r the risk-free continuously-compounded rate
  - z is the spread and $r + z$ is the continuously-compounded risky yield
  - Risk-neutral 1-year hazard rate $\tilde{\lambda}$:

$$\tilde{\lambda} = z$$

  - Via $e^a - 1 \approx a$
  - Approximation reasonably accurate for coupon securities, longer maturities

Credit spreads and recovery rates

- With recovery rate $R > 0$, credit spread approximately equal to expected loss (EL)

$$z \approx \tilde{\pi}_t(1 - R)$$

- If $R > 0$, risk-neutral default probability *higher*:

$$\boxed{\tilde{\pi}_1 \approx \frac{z}{1 - R}}$$

  - If you observe a given spread, then the lower the LGD, the higher must be the default probability

- **Example** (all in percent):

Credit spread	z	2.00
1-yr. risk-neutral default probability ($R = 0$)	$\tilde{\pi}_1 \approx z$	2.00
1-yr. risk-neutral default probability ($R = 0.40$)	$\tilde{\pi}_1 \approx \frac{z}{1-R}$	3.33

Bond and derivatives exposure to interest rates

- Interest rate risk can be measured as exposure to
  Parallel shifts of entire yield curve
  Yield to maturity: closed-form measures in some cases
  Key rates: impact of changes in rates for specific maturities, e.g. 3-month or 2-year rates
  - Useful for hedging, bond portfolio management
- Complexities introduced by need to measure impact of changes in *rates* on *price* or *value*

Scenario analysis

- Analysis in which yield curve assumed to change in specific ways
- Can be carried out using expression for price S_t of a bond in terms of spot rates
  - Scenario result is price change $\tilde{S}_t - S_t$ resulting from specified changes in spot rates
- Some commonly-encountered scenario analyses include

Parallel shift: measure price change if all spot rates rise, say, 25 basis points

Curve steepening: longer-term risk-free rates rise, but short-term rates and credit spreads unchanged

Credit spread widening: credit spreads increase, but risk-free rates unchanged

- Change in *price* is the same for equal change in risk-free rate or spread

Roll-down return: bond "ages" by, say, 1 year, but rates and spreads unchanged

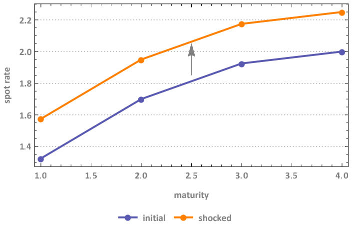
Scenario analysis: parallel shift

- Price change $\tilde{S}_t - S_t$ if all spot rates rise 25 basis points:

$$\tilde{S}_t = c_t \left[\frac{1}{1 + r_1 + 0.0025} + \cdots + \frac{1}{(1 + r_t + 0.0025)^t} \right] + \frac{1}{(1 + r_t + 0.0025)^t}$$

- All bond prices fall, but longer-term bond prices fall most
- In the **example:**

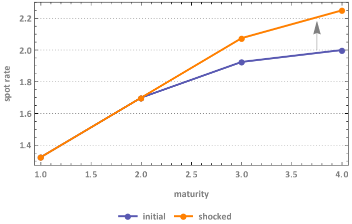
term	initial S_t	shocked $\tilde{S}_t$	$\tilde{S}_t - S_t$
1 year	98.692	98.449	-0.246
2 years	100.104	99.618	-0.486
3 years	100.237	99.517	-0.718
4 years	100.029	99.082	-0.946



Scenario analysis: curve steepening

- **Example:** 1- and 2-year spot rates unchanged, 3- and 4-year spot rates rise by 15 and 25 bps (0.0015 and 0.0025 percent)
- Short-term bond prices unchanged, but longer-term bond prices fall

term	initial S_t	shocked $\tilde{S}_t$	$\tilde{S}_t - S_t$
1 year	98.692	98.692	0.000
2 years	100.104	100.104	0.000
3 years	100.237	99.813	-0.423
4 years	100.029	99.102	-0.926



Scenario analysis: roll-down

- Assume spot curve unchanged 1 year hence
- Short-term bonds will return exactly their initial yield to maturity
- If yield curve upward-sloping, all bonds will experience positive return
  - As maturities shorten, bonds' cash flows discounted at lower spot rates
- In the **example:**

term	initial	1 year hence	Δ price
Initially 1 year	98.692	100.000	1.325
Initially 2 years	100.104	100.419	0.315
Initially 3 years	100.237	100.592	0.355
Initially 4 years	100.029	100.237	0.208

Rate sensitivity using duration and convexity

- Approximate measures of impact of small changes in yield y_t (or of small parallel shifts) on bond price S_t:

  Modified duration: denoted $\text{mdur}_t \equiv -\frac{1}{S_t}\frac{dS_t}{dy_t}$

  - Percent change in bond price as yield or level of curve rises 1 percent
  - Negative relation between price and yield→convention of multiplying by -1

  Convexity: denoted $\text{conv}_t \equiv \frac{1}{S_t}\frac{d^2S_t}{dy_t^2}$

  - Change in duration as yield or level of curve changes

- Effect on price/value of a Δy increase in yield in bps is Taylor-approximated by

$$\boxed{\frac{\Delta S_t}{S_t} \approx -\,\text{mdur}_t\,\Delta y + \frac{1}{2}\,\text{conv}_t\,\Delta y^2,}$$

  with mdur_t the delta, conv_t the gamma of a bond

Calculating duration

- **Effective duration:** approximate $\frac{dS_t}{dy_t}$ by shifting curve up and down by 1 basis point:

$$\frac{dS_t}{dy_t} \approx \frac{S_t(\text{shifted up}) - S_t(\text{shifted down})}{2 \times 0.0001}$$

  - Divide by initial bond price (and express as a positive number)

term	initial	+1bp shock	−1bp shock	duration
1 year	98.692	98.683	98.702	0.987
2 years	100.104	100.084	100.123	1.950
3 years	100.237	100.208	100.266	2.886
4 years	100.029	99.991	100.067	3.807

- Why use effective duration?
  - Often easier to implement
  - Needed for securities with cash flows that depend on interest rates, e.g. mortgage-backed securities (prepayments), bonds with embedded options
- **Effective convexity** estimated analogously

Duration and convexity of U.S. 10-year note

Screen capture of Bloomberg Quick Yield Analysis for U.S. on-the-run 10-year note on 17Jan2017.

Example: rate sensitivity of U.S. 10-year note

- 2 percent coupon, expiration 15Nov26 (term 9 year 10 months)
- Price quoted in dollars, 32nds and 64ths of a dollar per $100 of par value:

$$97\text{-}01+ \equiv 97 + \frac{1}{32} + \frac{1}{64} = 97.0468750$$

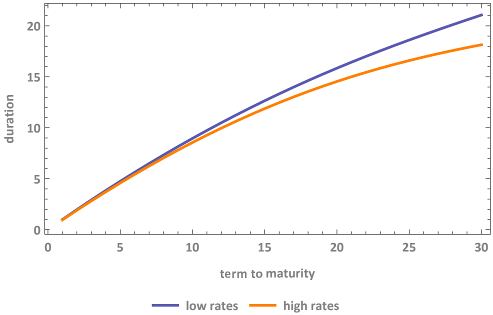
  - This quote includes 01 32nds; the + indicates an additional 64th
- Bloomberg estimate of modified duration $\text{mdur}_t = 8.8160$
  - If yield falls by 1 bp, price rises by 0.088160 percent

$$97.0468750 \times (1 + 0.088160) = 97.1324$$

- Bloomberg estimate of convexity $\text{conv}_t = 0.8713$
  - **positive convexity:** duration declines slightly as yield rises, attenuates price decline
  - Note: Bloomberg divides conventional convexity measure by 100
  - Makes it easier to work with percent rather than decimal changes
- Some types of bonds, e.g. mortgage-backed securities, display large-magnitude **negative convexity**
  - Behaves like negative option gamma, increases risk

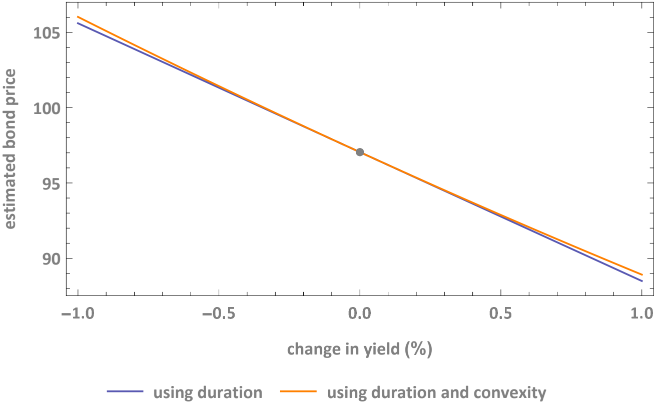
Duration and the level of the yield curve

- Duration—and therefore sensitivity to changes in rates—is higher when rates generally are lower
- The effect of the level of rates on duration is greatest for the longest-duration bonds
- Long-term bonds are exceptionally vulnerable when rates are low→SVB in 2022



Duration as a function of maturity T. All bonds have an annual coupon of 2.25 percent. Low spot rates given by $100(0.025 - 0.000025(T - \tau)^2)$; high spot rates given by $100(0.05 - 0.0000125(T - \tau)^2), \tau = 1, \ldots, T$.

Example: response of U.S. 10-year note to shocks



- Impact of 25 bps increase in rates
  - Duration: $-\,\text{mdur}_t\,\Delta y = -8.816 \cdot 0.0025 = -0.0220$ (-2.20 percent)
  - Convexity: $\frac{1}{2}\,\text{conv}_t\,\Delta y^2 = 87.13 \cdot 0.0025^2 = 0.00027$ (0.027 percent)
  - Total: $-8.816 \cdot 0.0025 + 87.13 \cdot 0.0025^2 = -0.0217$ (-2.17 percent)