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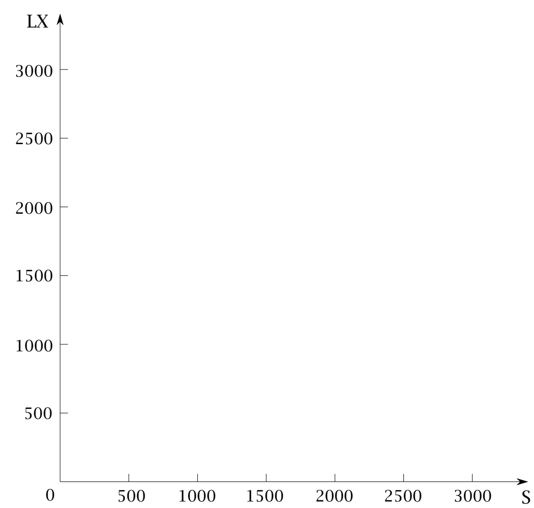
Graphical Analysis

For problems with 2 variables, we can represent each solution as a point in the plane. The Shelby Shelving model (see the readings book or pp.68-69 of the text) is repeated below for reference:

$$
\begin{array}{rrcrcll}
\text{max} & 260\,S & + & 245\,LX & & & \text{(Profit)} \\
\text{subject to:} & & & & & & \\
\text{(S assembly)} & S & & & \leq & 1900 & \\
\text{(LX assembly)} & & & LX & \leq & 1400 & \\
\text{(Stamping)} & 0.3\,S & + & 0.3\,LX & \leq & 800 & \\
\text{(Forming)} & 0.25\,S & + & 0.5\,LX & \leq & 800 & \\
\text{(Nonnegativity)} & & & S,\ LX & \geq & 0 &
\end{array}
$$

In the Shelby Shelving example, if we measure the number of S-shelves along the horizontal axis and the number of LX-shelves along the vertical axis (or vice versa), every solution can be uniquely represented as a point in the plane, see Figure 1.



S = # of Model S shelves to produce next month
LX = # of Model LX shelves to produce next month

Figure 1.

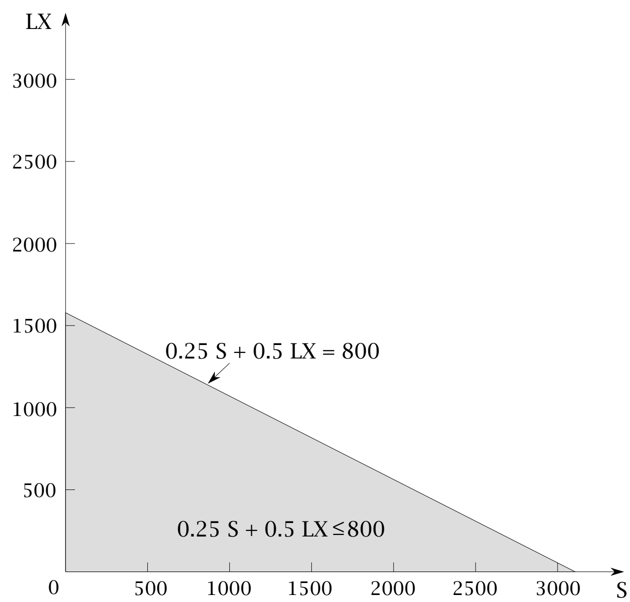
To represent the model graphically, we start by identifying the so-called "feasible region," i.e. the part of the plane which contains all feasible solutions. The feasible region is determined by considering each of the constraints sequentially. (It does not matter in what specific sequence the constraints are enumerated.)

Recall that in a linear program, each constraint is either a linear equation or a linear inequality. Before we continue, first a quick review of the graphical representation of such equations or inequalities.

Linear Equations

A linear equation corresponds, as the term *linear* indicates, with a *line*. A line is uniquely determined by two points on the line. It thus suffices to find an arbitrary pair of points on the line. Often, it is convenient to find the points of intersection with the vertical and horizontal axis.

Take, for example, the equation $0.25S + 0.5LX = 800$. The point on the S-axis is obtained by setting $LX = 0$. Thus, $0.25S = 800$ or $S = 3200$. The point on the LX-axis is obtained by setting $S = 0$; thus, $0.5LX = 800$ or $LX = 1600$. We conclude that the equation $0.25S + 0.5LX = 800$ corresponds with the line going through the points $(S, LX) = (3200, 0)$ and $(S, LX) = (0, 1600)$.



S = # of Model S shelves to produce next month

LX = # of Model LX shelves to produce next month

Figure 2.

Linear Inequalities

To represent a linear *inequality*, we first consider the equation which arises when the inequality is replaced by an equality. (For example, the inequality $0.25S + 0.5LX \leq 800$ is replaced by the equation $0.25S + 0.5LX = 800$.) As we just reviewed, the equation corresponds with a straight line which is easily found by identifying 2 points on the line. The *inequality* corresponds with one of the two half planes bordered by this line. To identify *which* of the *two* half planes is to be chosen, we select an arbitrary point outside the line and check whether its coordinates satisfy the inequality. See Figure 2.

Definition: For a given feasible solution, we define the slack with respect to a given constraint, as the difference between the permitted value (or right hand side) and the actually used value (= left hand side of the constraint).

Example: The solution $(S = 500, LX = 500)$ is feasible. With respect to the constraint $0.25S + 0.5LX \leq 800$ its slack is $800 - 375 = 425$.

The Feasible Region

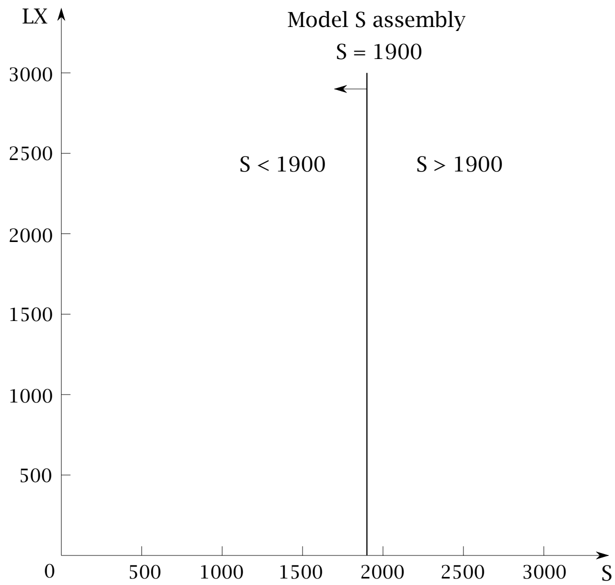
We are now ready to determine the feasible region. As mentioned, we sequentially introduce each of the constraints, identify the line or half plane corresponding with the constraint and keep track of the intersection with the feasible region identified thus far. Figure 3 shows the half plane corresponding with the S-assembly constraint $S \leq 1900$. The set of points which satisfy this constraint and the non-negativity constraints $S \geq 0$, $LX \geq 0$ is thus given by the (half open) rectangle bordered by the S-axis, and the lines $S = 0$ and $S = 1900$.

Figure 4 introduces the “LX assembly” constraint. The set of points which satisfy this constraint is given by the half plane below the horizontal line $LX = 1400$. The set of points which are feasible with respect to *both* assembly constraints and the non-negativity constraints is thus given by the shaded rectangle.

Figure 5 introduces the Stamping constraint. The half plane below the “Stamping Line” represents all points that are feasible with respect to this constraint (taken by itself). The shaded pentagon is the region of points which are feasible with respect to the first three constraints and the non-negativity constraints. Finally, Figure 6 adds the last, i.e. the Forming constraint and shows how the feasible region again reduces, now to the smaller shaded pentagon.

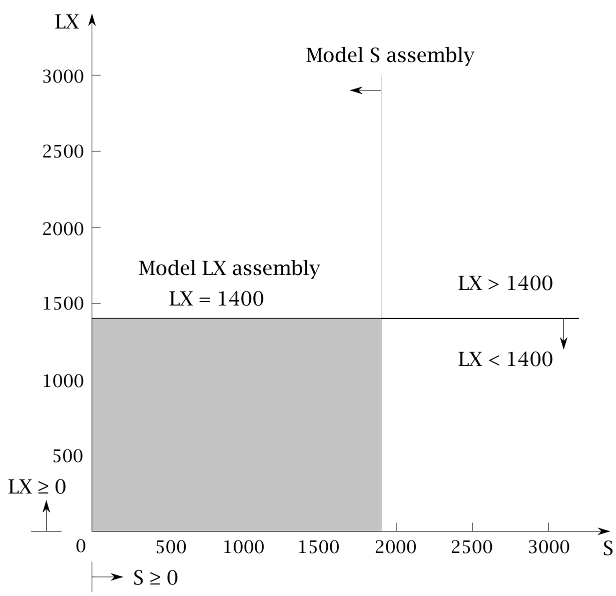
Conclusions:

1. The feasible region is a polygon, i.e., it is the intersection of several half planes and lines.
2. Each additional constraint cuts off part of the feasible region, or at best leaves the feasible region unaltered.



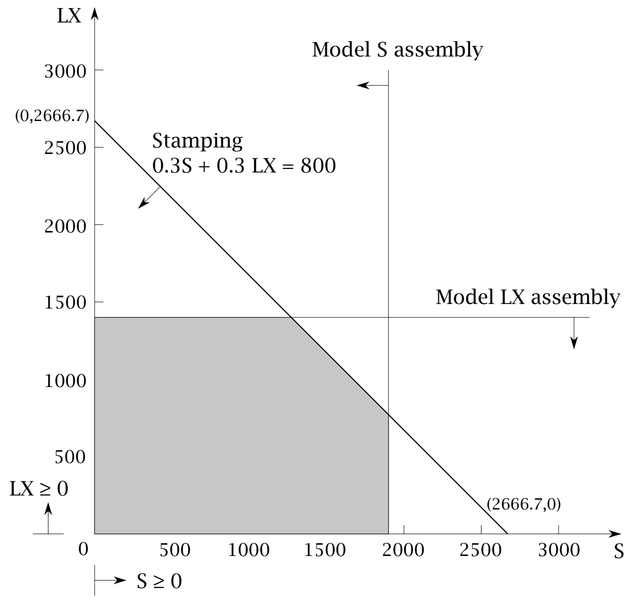
S = # of Model S shelves to produce next month
LX = # of Model LX shelves to produce next month

Figure 3.



S = # of Model S shelves to produce next month
LX = # of Model LX shelves to produce next month

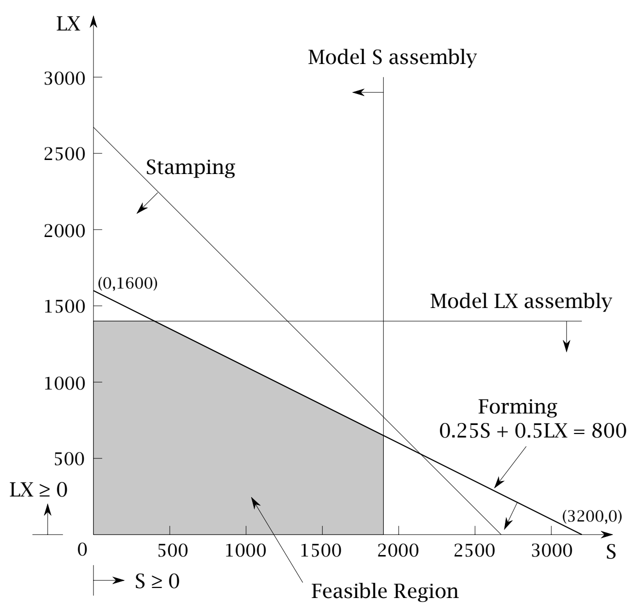
Figure 4.



S = # of Model S shelves to produce next month

LX = # of Model LX shelves to produce next month

Figure 5.



S = # of Model S shelves to produce next month

LX = # of Model LX shelves to produce next month

Figure 6.

Optimal Solutions

We now identify which point in the feasible region is optimal, i.e., achieves the highest profit level. Clearly, it is in this part of the graphical representation that the objective function is used. Observe first that the constant term in the objective function has no impact whatsoever on the relative desirability of one solution over another. This constant term may thus be ignored for the purpose of identifying an optimal solution; assume therefore that the objective is given by $260S + 245LX$.

We first select an *arbitrary* target level for our objective, e.g. \$260,000, and represent the collection of points which result in this specific profit level. This set of points or "Iso-Profit Curve" is represented by the equation $260S + 245LX = 260{,}000$ which, as we have learned, corresponds with a line, through the points (1000,0) and (0,1061). See Figure 7.

We draw this Iso-Profit line and observe that it intersects with the feasible region, so that the target level of \$260,000 is achievable. Can a higher profit level be reached? Clearly, a higher profit level corresponds with a new Iso-Profit line, parallel to the first line. Thus, by pushing the Iso-profit line out in parallel we reach better and better profit values. We continue doing this until the line ceases to intersect the feasible region. Notice this occurs when the Iso-profit line goes through the "corner" solution corresponding with the point of intersection of the lines representing the S-assembly and forming constraints. See Figure 8.

To identify this corner solution, we either "read" its coordinates from the picture or solve the system of two equations in two unknowns:

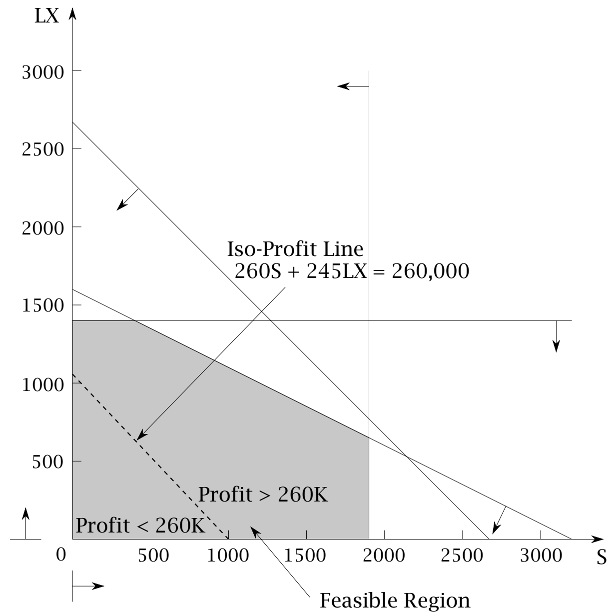
$$\begin{aligned} (S\text{-Assembly constraint}) \quad & S = 1900 \\ (\text{Forming constraint}) \quad & 0.25S + 0.5LX = 800 \end{aligned}$$

The optimal solution has $(S = 1900, LX = 650)$ and results in a Profit value of \$653,270. (This number is obtained by plugging the values of S and LX into the objective function.)

Conclusions:

1. The optimal solution in a linear program is always found in one of the corner solutions because as we push the "iso-objective line" in parallel to better and better levels, we go out of the feasible region through an extreme point.

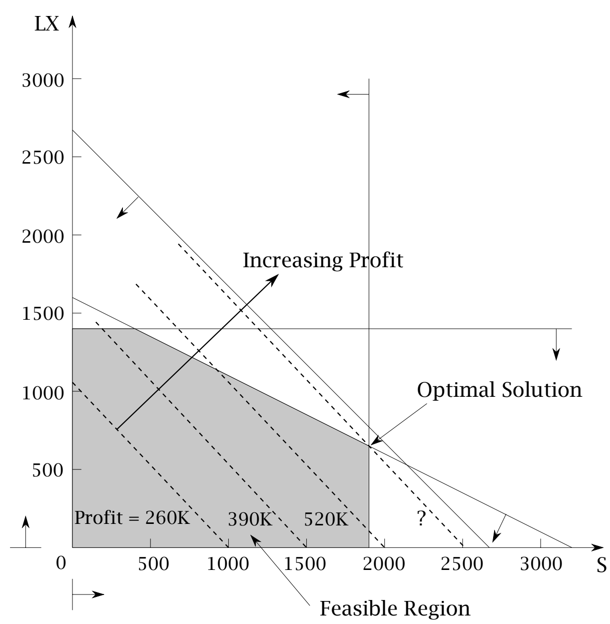
2. In some special situations, the iso-profit or iso-objective line is parallel to one of the constraint lines. For example, if the variable profit contribution of LX were 130 (instead of 245) the iso-profit line would be parallel to the Forming constraint line. In that case, moving the iso-profit line out in parallel would result in a highest achievable iso-profit line coinciding with an entire line segment, not just a single corner point. Our conclusions remain however valid even in this special case. The optimal solution can always be found in a corner. Sometimes there are multiple optimal corners, in which case all intermediate points (line segments) are optimal as well.



S = # of Model S shelves to produce next month

LX = # of Model LX shelves to produce next month

Figure 7.



S = # of Model S shelves to produce next month

LX = # of Model LX shelves to produce next month

Figure 8.

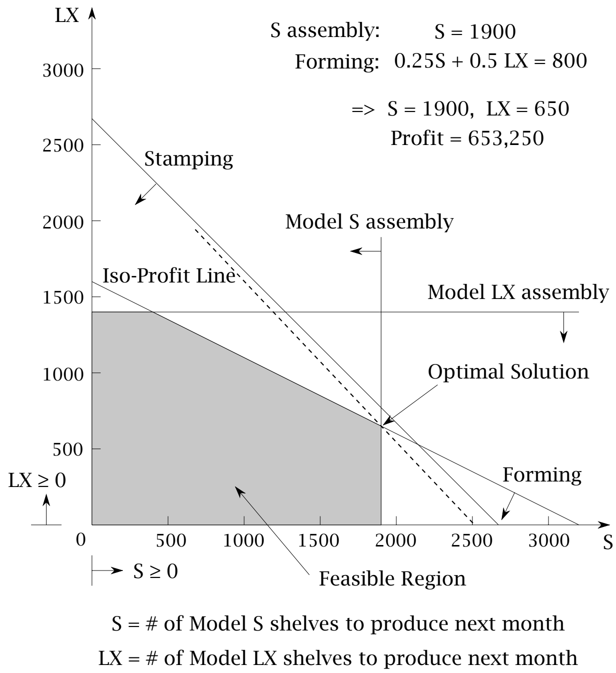


Figure 9.

Summary of 2-Variable Graphical Solution Procedure

1. Graph individual constraints; identify the feasible region
2. Graph an arbitrary iso-objective line
3. Move iso-objective line, in a parallel fashion, to identify optimal solution (corner)

   Note: The direction to move the objective depends on whether the model is a maximization or a minimization problem.

4. Identify constraints that determine optimal corner; solve simultaneous equations for optimal solution
5. Plug optimal solution into objective function to determine optimal objective function value

Sensitivity Analysis: Dual Prices

Because data is usually never known precisely, we often would like to know: How does the optimal solution change when the LP data changes, i.e., how *sensitive* is the solution to the data? Or phrased another way, how much would the management of Shelby be willing to pay to increase the capacity of the Model *S* assembly department by 1 unit, i.e., from 1900 to 1901?

Shelby Shelving Linear Program

$$
\begin{array}{rrcl}
 & \max \quad 260\,S + 245\,LX & & \text{(Profit)} \\
\text{subject to:} & & & \\
\text{(S assembly)} & S \qquad\qquad & \leq & 1900 \\
\text{(LX assembly)} & LX & \leq & 1400 \\
\text{(Stamping)} & 0.3\,S + \ 0.3\,LX & \leq & 800 \\
\text{(Forming)} & 0.25\,S + \ 0.5\,LX & \leq & 800 \\
\text{(Nonnegativity)} & S,\ LX & \geq & 0
\end{array}
$$

Optimal solution: $S = 1900$, $LX = 650$, Profit = 653,250

Would Shelby be willing to pay \$260? No, because producing 1 more Model S would require additional hours in the forming department (which is used at full capacity). Hence, producing 1 more Model S would require a cut in Model LX production.

Graphically, what is the impact of this change in the "S-Assembly" constraint. Note that the Feasible Region remains unaltered, except that the S-Assembly line now moves out in parallel by a bit. This causes the corner point of intersection of the S-Assembly and Forming line to move, in that the S-coordinate increases by one to $S = 1901$ and the LX coordinate decreases by a bit.

By how much exactly? We need to find the intersection of the Forming constraint line and the shifted S-Assembly line, i.e., we need to solve the system of equations:

$$S = 1901; \quad 0.25S + 0.5LX = 800$$

$$\Longrightarrow \quad S = 1901; \quad LX = 649.5.$$

Note, the profit lines have not been affected by the change. Thus as we move the iso-profit line out in parallel, we end up in the same (albeit somewhat shifted) corner ($S = 1901$, $LX = 649.5$).

What is the net impact on the objective function? $\Delta S = +1$ so this results in an *increase* of profit by \$260. $\Delta LX = -0.5$ so this results in a *decrease* of profit by \$122.5. The net change in profit is \$137.5.

We have just calculated the dual price of the S-Assembly constraint. See Figure 10.

Dual Price (of constraint) is amount by which the objective function changes due to an increase of the Right Hand side of the constraint by one unit.

The dual price is a *marginal* quantity. However, it turns out that the same dual price value remains valid over quite a range. Consider e.g., the impact of an additional increase of the right hand side of the S-Assembly constraint, now from 1901 to 1902. The (corner) intersection point of the S-assembly and Forming constraint lines, again shifts marginally, once again by the S-coordinate increasing by one unit and the LX coordinate decreasing by +0.5 units:

$$S = 1902; \quad LX = 649.$$

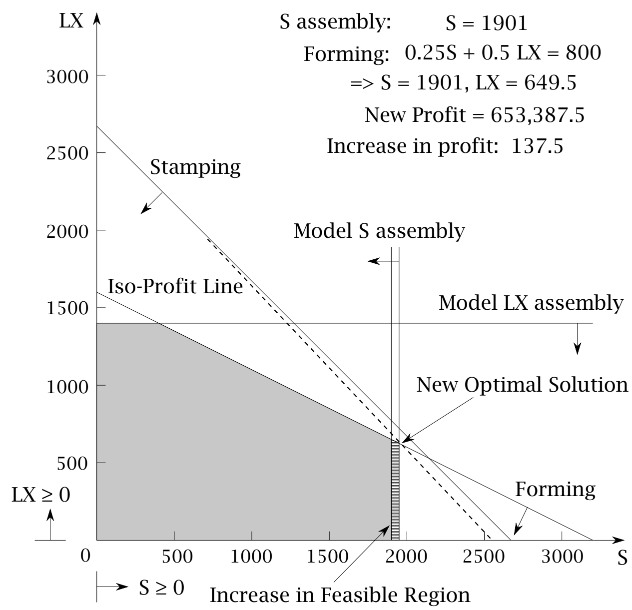


Figure 10. Dual Price: Graphical Analysis

Thus, the impact of the second increase of the right hand side value (from 1901 to 1902) is identical to that of the first increase (from 1900 to 1901). $\Delta S = +1$ results in an *increase* of profit by \$260. $\Delta LX = -0.5$ results in a *decrease* of profit by \$122.5. The dual price is the net change in profit, i.e., \$137.5.

Does the dual price of \$137.50 apply to *all* levels? No, if the (vertical) S-assembly constraint line is moved beyond the point of intersection of the Stamping and Forming constraint line, the dual price changes. This point of intersection has $S = 2133.33$ as its S-coordinate. Thus, if the S-assembly capacity is increased to a level larger than 2133, the Stamping constraint becomes binding: a capacity increase by one unit allows one to produce one additional unit of S-shelves. However, to enable this, the value of LX-shelves needs to be reduced by one (as opposed to 0.5) full unit: $\Delta S = 1$ results in an *increase* of profit of \$260. But $\Delta LX = -1$ *decreases* profit by \$245. The dual price is the net change in profit, i.e., \$260 − \$245 = \$15.

Similarly, if the capacity of the S-assembly department is decreased, the marginal impact, or dual price, remains at the level of \$137.50 as long as the S-assembly line is not moved further to the left than beyond the point of intersection of the LX-assembly line and the Forming constraint line. This point of intersection has $S = 400$ as its S-coordinate. Thus, if the S-assembly capacity is decreased to a level lower than 400, the LX-assembly constraint becomes binding. A capacity decrease of the S-assembly department by one unit results in a decrease of the S-volume by one unit which can *not* be compensated by *any* increase in LX-volume: $\Delta S = -1$ decreases profit by \$260. $\Delta LX = 0$ increases profit by \$0. So profit decreases by \$260.

Thus, for capacity levels ≤ 400, the dual price increases to the value of \$260.

Conclusions:

1. The dual price of a constraint remains constant over a complete interval or range, but may change drastically from one interval to the next.

S-capacity	Dual Price
$0 \le S \le 400$	260
$0 \le S \le 33,35$	137.50
$2133 < S < 2666.66$	15
$666 < S$	0

The graph in Figure 11 shows how the optimal profit (in \$1000) varies as a function of the RHS of the Model S assembly constraint. The slope of the graph is the dual price of the Model S assembly constraint:

$$\text{Slope} = \frac{\text{Change in optimal profit}}{\text{Change in RHS}} = \text{Dual Price.}$$

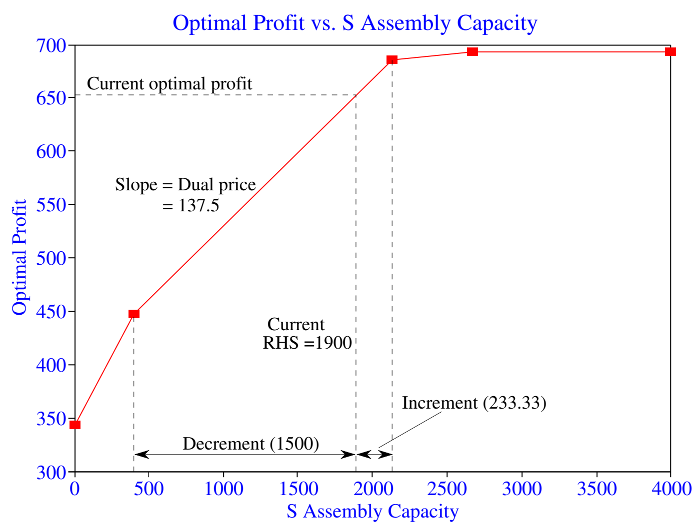


Figure 11. Optimal Objective Function versus Righthand Side

The spreadsheet optimizer's sensitivity report (see Figure 12) gives dual price information (termed *shadow prices* in the Excel report). This information is created automatically (i.e., without extra computational effort) when the LP is solved. Dual prices of nonnegativity contraints are often called *reduced costs*.

See the section "Report files and dual prices" in the reading "An Introduction to Spreadsheet Optimization Using Excel" for more information about creating reports using the Excel optimizer.

Microsoft Excel 7.0 Sensitivity Report
Worksheet: [SHELBY.XLS]Sheet1
Report Created: 1/13/96 11:00

Changing Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
C4	Production per month Model S	1900	0	260	1E+30	137.5
D4	Production per month Model LX	650	0	245	275	245

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
E15	Model S assembly Used	1900	137.5	1900	233.3333333	1500
E16	Model LX assembly Used	650	0	1400	1E+30	750
E17	Stamping (hours) Used	765	0	800	1E+30	35
E18	Forming (hours) Used	800	490	800	58.33333334	325

Figure 12. Spreadsheet Sensitivity Report

Righthand Side Ranges

The sensitivity report also gives righthand side ranges specified as "allowable increase" and "allowable decrease:" The sensitivity report indicates that the dual price for Model S assembly, 137.5, is valid for RHS ranging from

$$1900 - 1500 \quad \text{to} \quad 1900 + 233.33.$$

i.e., for Model S assembly capacity from

$$400 \quad \text{to} \quad 2133.33$$

as we verified graphically above. In other words, the equation

$$\text{Change in profit} = \text{Dual Price} \times \text{Change in RHS.}$$

is only valid for "Changes in RHS" from -1500 to $+233.33$.

Dual Price (continued)

In the Shelby Shelving model, how much would they be willing to pay to increase the capacity of the Model LX assembly department by 1 unit, i.e., from 1400 to 1401?

$$\begin{array}{rrcrcl}
\max & 260\,S & + & 245\,LX & - & 385{,}000 \\
\text{subject to:} & & & & & \\
(S \text{ assembly}) & S & & & \le & 1900 \\
(LX \text{ assembly}) & & & LX & \le & 1400 \\
(\text{Stamping}) & 0.3\,S & + & 0.3\,LX & \le & 800 \\
(\text{Forming}) & 0.25\,S & + & 0.5\,LX & \le & 800 \\
(\text{Nonnegativity}) & & & S,\ LX & \ge & 0
\end{array}$$

Optimal solution: $S = 1900$, $LX = 650$, Net Profit = \$268,250.

They would not be willing to pay *anything*. Why? The capacity is 1400, but they are only producing 650 Model LX shelves. There are already 750 units of unused capacity (i.e., *slack*), so an additional unit of capacity is worth 0. So the dual price of the Model LX assembly constraint is 0.

The answer report gives the slack (i.e., unused capacity) for each constraint. A constraint is *binding*, or *tight* if the slack is zero (i.e., all of the capacity is used). The results from the sensitivity and answer reports are summarized next.

$$\max \quad 260\,S + 245\,LX - 385{,}000$$

subject to:

	Constraint	Slack	Dual Price
(S assem.)	$S \le 1900$	0	137.5
(LX assem.)	$LX \le 1400$	750	0
(Stamping)	$0.3\,S + 0.3\,LX \le 800$	35	0
(Forming)	$0.25\,S + 0.5\,LX \le 800$	0	490
(S nonneg.)	$S \ge 0$	1900	0
(LX nonneg.)	$LX \ge 0$	650	0

Optimal solution: $S = 1900$, $LX = 650$, Net Profit = \$268,250.

In general,

$$\text{Slack} > 0 \quad \Longrightarrow \quad \text{Dual Price} = 0$$

and

$$\text{Dual Price} > 0 \quad \Longrightarrow \quad \text{Slack} = 0$$

It is possible to have a dual price equal to 0 and a slack equal to 0.

Objective Coefficient Ranges

Changing Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
\$C\$4	Production per month Model S	1900	0	260	1E+30	137.5
\$D\$4	Production per month Model LX	650	0	245	275	245

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
\$E\$15	Model S assembly Used	1900	137.5	1900	233.3333333	1500
\$E\$16	Model LX assembly Used	650	0	1400	1E+30	750
\$E\$17	Stamping (hours) Used	765	0	800	1E+30	35
\$E\$18	Forming (hours) Used	800	490	800	58.33333334	325

Figure 13.

The "Changing Cells" section of the sensitivity report shown in Figure 13 also contains objective coefficient ranges. For example, the optimal production plan will not change if the profit contribution of model LX increases by 275 or decreases by 245 from the current value of 245. (The optimal profit will change, but the optimal production plan remains at $S = 1900$ and $LX = 650$.)

This robustness phenomenon can again be understood by inspecting the graphical representation. As the profit margin of LX increases, the iso-profit lines become flatter. As long as the slope doesn't change too much, we end up in the exact same corner. Only if the slope decreases beyond the point where the iso-profit lines are parallel to the Forming constraint line, do we end up in a different corner ($S = 400, LX = 1400$), see Figure 8. Let p denote the profit margin for LX. The switch between optimal solutions thus occurs when:

$$p/260 = 0.5/0.25 = \text{slope of Forming line} = 2,$$

hence when $p = 520$, i.e., when p increases by 275 from its current value of 240.

Further, the optimal production plan will not change if the profit contribution of model S increases by any amount. Why? At a production level of $S = 1900$, Shelby is already producing as many model S shelves as possible. This explains the "Allowable Increase" numbers in Figure 13, for C4 and D4.