

Homogeneous Transformation

■ Transformation Rules

By adding an extra dimension beyond Cartesian Coordinates, the perspective transformation can be performed with ease. This added dimension also assists with scaling and other transformations.

```
CartesianToHomogeneous[x_]:= Append[x, 1]

HomogeneousToCartesian[x_]:= Drop[x, -1]/Last[x]

PerspectiveTransformation[t_,x_]:=Module[
{OldHomogeneousCoordinates= CartesianToHomogeneous[x],
  NewHomogeneousCoordinates},
NewHomogeneousCoordinates = (
  OldHomogeneousCoordinates . t);
NewCartesianCoordinates = HomogeneousToCartesian[
  NewHomogeneousCoordinates]
];((VectorQ[x] && MatrixQ[t]) &&
(Dimensions[t]==(Length[x] + 1){1,1}))
```

■ Examples of Transformations Rules

```
x={x1,y1};
y={x1, y1, z1}
```

```
{x1, y1, z1}
```

```
CartesianToHomogeneous[x]
```

```
{x1, y1, 1}
```

```
HomogeneousToCartesian[y]
```

```
{  $\frac{x1}{z1}$ ,  $\frac{y1}{z1}$  }
```

The homogenous coordinates combine to form the transformation matrix. The matrix naming convention below will be used throughout this demonstration.

```
t={ {a,b,p},
    {c,d,q},
    {1,m,s} }

{{a, b, p}, {c, d, q}, {1, m, s}}
```

MatrixForm[t]

$$\begin{pmatrix} a & b & p \\ c & d & q \\ 1 & m & s \end{pmatrix}$$

PerspectiveTransformation[t,x]

$$\left\{ \frac{1 + a x_1 + c y_1}{s + p x_1 + q y_1}, \frac{m + b x_1 + d y_1}{s + p x_1 + q y_1} \right\}$$

■ effect of s

$$t1 = \left\{ \begin{array}{l} \{1, 0, 0\}, \\ \{0, 1, 0\}, \\ \{0, 0, .5\} \end{array} \right\}$$

$$\{\{1, 0, 0\}, \{0, 1, 0\}, \{0, 0, 0.5\}\}$$

PerspectiveTransformation[t1, x]

$$\{2. x_1, 2. y_1\}$$

■ effect of l

$$t2 = \left\{ \begin{array}{l} \{1, 0, 0\}, \\ \{0, 1, 0\}, \\ \{1, 0, 1\} \end{array} \right\}$$

$$\{\{1, 0, 0\}, \{0, 1, 0\}, \{1, 0, 1\}\}$$

PerspectiveTransformation[t2, x]

$$\{1 + x_1, y_1\}$$

■ effect of p

$$t3 = \left\{ \begin{array}{l} \{1, 0, 1\}, \\ \{0, 1, 0\}, \\ \{0, 0, 1\} \end{array} \right\}$$

$$\{\{1, 0, 1\}, \{0, 1, 0\}, \{0, 0, 1\}\}$$

PerspectiveTransformation[t3, x]

$$\left\{ \frac{x_1}{1 + x_1}, \frac{y_1}{1 + x_1} \right\}$$

Notice: **s** affects the scaling, **l** and **m** are used for translation, and **p** and **q** distort by changing the angles of intersection and line length.

■ Applications

```

OldPoints={ {0,0}, {0,1}, {1,0}}
           {{0, 0}, {0, 1}, {1, 0}}

t={ {-1,0,0},{1,1,0},{.6,.8,1/2}}
   {{-1, 0, 0}, {1, 1, 0}, {0.6, 0.8, 1/2}}

   {{1, -.2, 3}, {.2, 1, 4}, {.6, .8, 1/2}}

   {{1, -0.2, 3}, {0.2, 1, 4}, {0.6, 0.8, 1/2}}

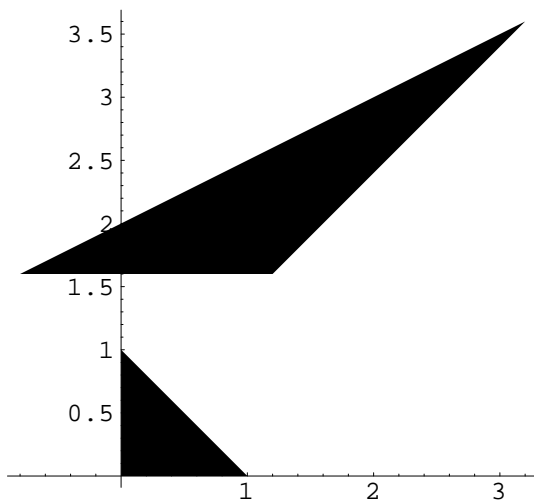
NewPoints= Map[
  PerspectiveTransformation[ t, #]&, OldPoints]
           {{1.2, 1.6}, {3.2, 3.6}, {-0.8, 1.6}}

figures = Map[
  Polygon, {OldPoints, NewPoints} ] //Graphics

- Graphics -

Show[figures, AspectRatio -> Automatic, Axes -> True]

```



- Graphics -

The straight lines remain straight even though the angles of intersection, line length, scaling, and translation have changed.

■ Part-2

■ Conclusions

An important theorem to be proven is that a straight line remains a straight line, even after perspective transformation.

Let us start with two generic points on an arbitrary straight line and use the index i .

$$\mathbf{x} = x_0 + x_1 i ; \mathbf{y} = y_0 + y_1 i ; \text{locus} = \{\mathbf{x}, \mathbf{y}\}$$

$$\{x_0 + i x_1, y_0 + i y_1\}$$

Since the affine transformation preserves linearity, it suffices to check the transformation with the perspective variables, as follows:

$$\mathbf{t} = \{ \{1, 0, p\}, \{0, 1, q\}, \{0, 0, 1\} \}$$

$$\{\{1, 0, p\}, \{0, 1, q\}, \{0, 0, 1\}\}$$

$$\text{NewLoculs} = \text{PerspectiveTransformation}[\mathbf{t}, \text{locus}]$$

$$\left\{ \frac{x_0 + i x_1}{1 + p(x_0 + i x_1) + q(y_0 + i y_1)}, \frac{y_0 + i y_1}{1 + p(x_0 + i x_1) + q(y_0 + i y_1)} \right\}$$

The initial transformed point is recovered by setting $i=0$:

$$\text{NewLoculs0} = \{\mathbf{xNew0}, \mathbf{yNew0}\} = \text{NewLoculs} /. \{i \rightarrow 0\}$$

$$\left\{ \frac{x_0}{1 + p x_0 + q y_0}, \frac{y_0}{1 + p x_0 + q y_0} \right\}$$

The marching quantity is:

$$\text{march} = \text{Together}[\text{NewLoculs} - \text{NewLoculs0}]$$

$$\left\{ \frac{i x_1 + i q x_1 y_0 - i q x_0 y_1}{(1 + p x_0 + q y_0)(1 + p x_0 + i p x_1 + q y_0 + i q y_1)}, \frac{-(i p x_1 y_0) + i y_1 + i p x_0 y_1}{(1 + p x_0 + q y_0)(1 + p x_0 + i p x_1 + q y_0 + i q y_1)} \right\}$$

...its common denominator:

$$\text{factor} = \text{march}[[1]]/\text{Denominator}$$
$$(1 + p x_0 + q y_0) (1 + p x_0 + i p x_1 + q y_0 + i q y_1)$$

Now, a non-linear function of i is defined :

$$j = i/\text{factor}$$
$$\frac{i}{(1 + p x_0 + q y_0) (1 + p x_0 + i p x_1 + q y_0 + i q y_1)}$$

To recover the new scaling parameters, use:

$$\{\text{xFactor}, \text{yFactor}\} = \text{Simplify}[\text{march}/j]$$
$$\{x_1 + q x_1 y_0 - q x_0 y_1, -(p x_1 y_0) + y_1 + p x_0 y_1\}$$

Verify whether the new loculi are linearly marched with index j .

$$\text{Simplify}[\text{(NewLoculs - NewLoculs0) - j \{xFactor, yFactor\}] == \{0, 0\}}$$

True