A Hilbert Type Deductive System for Sentential Logic, Completeness and Compactness.

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A (Hilbert type) *deductive system* is a system consisting of a language and of (I) a set of wffs called *axioms*, and (II) a set of *inference rules*. An inference rule consists of (i) a list of wffs called the *premises* of the rule, (ii) a wff called the rule's *conclusion*. Usually, inference rules do not have more than two premises.

A *proof* in a given deductive system is a finite sequence of wffs:

$$\alpha_1, \alpha_2, \ldots, \alpha_n$$

in which every α_i is either an axiom, or can be inferred from previous α_j's by an inference rule. The proof is a *proof of* α, if α is the last member of the proof. A wff is said to be *provable* (in the given system) if it has a proof in it. Obviously, every wff occurring in a proof is provable, since the initial segment of the proof up to, and including, the wff is itself a proof. Wffs that are provable in the system are also called *theorems* (of the system).

Axioms of Sentential Logic:

- **(i)** $\alpha \rightarrow (\beta \rightarrow \alpha)$
- **(ii)** $[\alpha \rightarrow (\beta \rightarrow \gamma)] \rightarrow [(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)]$
- **(iii)** $(\neg\alpha \rightarrow \neg\beta) \rightarrow (\beta \rightarrow \alpha)$

(Additional axioms are included if other connectives are included as primitives. If, by definition, other connectives are rewritten in terms of $\neg$ and $\rightarrow$, no additional axioms are required.)

Inference Rule: Modus ponens:
$$\frac{\alpha \rightarrow \beta, \quad \alpha}{\beta}$$

A *theorem of (sentential) logic* is any wff provable in the system just given. We use

$$\vdash \alpha$$

to say that α is a theorem.

Note: If $\vdash \alpha$ and $\vdash \alpha \rightarrow \beta$, then $\vdash \beta$. The (very easy) argument runs as follows:
Let $\beta_0, \beta_1, \ldots \beta_j, \alpha \rightarrow \beta$ and $\alpha_0, \alpha_1, \ldots, \alpha_k \ \alpha$ be, respectively, proofs of $\alpha \rightarrow \beta$ and of β. Then

$$\beta_0, \beta_1, \ldots, \beta_j, \alpha \rightarrow \beta, \alpha_0, \alpha_1, \ldots, \alpha_k, \alpha, \beta$$

is a proof of β.

Theorem 1. $\vdash \alpha \rightarrow \alpha$

Proof: The following is a proof of $\alpha\rightarrow\alpha$

$\alpha\rightarrow[(\alpha\rightarrow\alpha)\rightarrow\alpha]$, $\{\alpha\rightarrow[(\alpha\rightarrow\alpha)\rightarrow\alpha]\}\rightarrow\{[\alpha\rightarrow(\alpha\rightarrow\alpha)]\rightarrow(\alpha\rightarrow\alpha)]$,
$(\alpha\rightarrow(\alpha\rightarrow\alpha))\rightarrow(\alpha\rightarrow\alpha)$, $\alpha\rightarrow(\alpha\rightarrow\alpha)$, $\alpha\rightarrow\alpha$

The first wff is an instance of Axiom (i), the second—of Axiom (ii), the third is inferred from the first two via modus ponens, the fourth is an instance of Axiom (i) and the fifth is inferred from the third and the fourth via modus ponens.

Note: Theorem 1 says that every wff of a certain form is provable in the above deductive system. It is a theorem *about* the deductive system, not a theorem *in* it. A theorem in it would be any particular wff of the form $\alpha\rightarrow\alpha$. We can call theorems that are about the deductive system *metatheorems.* Theorems proved in a logic course are as a rule metatheorems.

Note: The proof of any wff $\alpha\rightarrow\alpha$ uses only the first two axioms and modus ponens.

Proofs from Premises:

A *proof from* a set, Γ, of wffs is a sequence of wffs, in which every member is either a logical axiom (in our case, 'logic' means sentential logic) , or a member of Γ, or inferred from previous members by an inference rule–in our case: modus ponens. In this context, the members of Γ are referred to as *premises.* The proof is a proof of the last member of the sequence. A wff is *provable from* Γ if there is a proof of it from Γ. We write

$$\Gamma \vdash \alpha$$

to say that α is provable from Γ.

If Γ consists of the wffs $\alpha_1, \alpha_2, \ldots, \alpha_n$, we write: $\Gamma = \alpha_1, \alpha_2, \ldots, \alpha_n$ and rewrite the above as:

$$\alpha_1, \alpha_2, \ldots, \alpha_n \vdash \alpha.$$

We use 'Γ,α' to denote the set of premises consisting of the members of Γ and of α.

Deduction Theorem:

$$\Gamma, \alpha \vdash \beta \quad \Leftrightarrow \quad \Gamma \vdash \alpha \rightarrow \beta$$

Sketch of Proof: The implication $\Leftarrow$ is trivial, because if from Γ we can derive $\alpha\rightarrow\beta$, then, having also α as a premise, we can get β by modus ponens. The main claim is in the $\Rightarrow$ direction.

Let $\beta_0, \beta_1, \ldots, \beta_{n-1}$ be a proof of β from Γ,α.

Show by induction, that, for every $i < n$, $\alpha \rightarrow \beta_i$ is provable from Γ. There are several cases:

(i) β_i is a logical axiom or a member of Γ.

In that case β_i can be used in proving $\alpha \rightarrow \beta_i$. from Γ. Now, it is not difficult to show that, for every γ and δ:

$$\gamma \vdash \delta \rightarrow \gamma.$$

Hence from β_i we can get $\alpha \rightarrow \beta_i$.

(ii) $\beta_i = \alpha$; in this case $\alpha \rightarrow \beta_i = \alpha \rightarrow \alpha$, and, by theorem 1, $\alpha \rightarrow \alpha$ is provable in the deductive system.

(iii) β_i is inferred via modus ponens from two previous wffs. Say they are $\gamma \rightarrow \beta_i$ and γ. By the induction hypothesis, $\vdash \alpha \rightarrow (\gamma \rightarrow \beta_i)$ and $\vdash \alpha \rightarrow \gamma$. The argument can be now brought to finish by showing:

$$\alpha \rightarrow (\gamma \rightarrow \beta_i), \quad \alpha \rightarrow \gamma \vdash \alpha \rightarrow \beta_i.$$

(This is not immediate, but easier than (ii); use axiom **(ii)** and twice modus ponens.)

QED

Note: The proof of the deduction theorem carries through whenever the system in question has modus ponens as a single inference rule and includes the axioms (i) and (ii) among its theorems. Hence the deduction theorem obtains for all such systems.

Theorem 2. $\beta, \neg\beta \vdash \gamma$.
Sketch of Proof: $\neg\beta \vdash \neg\gamma \rightarrow \neg\beta$, using Axiom (iii), we get: $\neg\beta \vdash \beta \rightarrow \gamma$, and from β and $\beta \rightarrow \gamma$ we get γ, via modus ponens.

Theorem 3. $\vdash \neg\neg\beta \rightarrow \beta$.
Sketch of Proof: Let α be any axiom. By Theorem 2, $\neg\neg\beta, \neg\beta \vdash \neg\alpha$; by the deduction theorem $\neg\neg\beta \vdash \neg\beta \rightarrow \neg\alpha$; using Axiom (iii), we get: $\neg\neg\beta \vdash \alpha \rightarrow \beta$; since α is an axiom, we get: $\neg\neg\beta \vdash \beta$. Now use the deduction theorem.

Theorem 4. $\vdash \beta \rightarrow \neg\neg\beta$
Sketch of Proof: By Theorem 3 $\vdash \neg\neg\neg\beta \rightarrow \neg\beta$; using Axiom (iii), we get: $\vdash \beta \rightarrow \neg\neg\beta$.

Theorem 5. $\alpha \rightarrow \beta \vdash \neg\beta \rightarrow \neg\alpha$.
Sketch of proof: Using Theorems 3, 4, one shows that $\alpha \rightarrow \beta \vdash \neg\neg\alpha \rightarrow \neg\neg\beta$. Then use Axiom (iii).

Theorem 6. If $\Gamma, \alpha \vdash \beta$ and $\Gamma, \alpha \vdash \neg\beta$, then $\Gamma \vdash \neg\alpha$.
Sketch of Proof: From Theorem 2, the assumptions imply: $\Gamma, \alpha \vdash \gamma$, where γ is any wff. Choose γ to be $\neg\delta$, where δ is an axiom, then $\Gamma, \alpha \vdash \neg\delta$; by the deduction theorem, $\Gamma \vdash \alpha \rightarrow \neg\delta$. Using Theorem 3, we get: $\Gamma \vdash \neg\neg\alpha \rightarrow \neg\delta$; from this (via Axiom (iii)) $\Gamma \vdash \delta \rightarrow \neg\alpha$. Since δ is an axiom, $\Gamma \vdash \neg\alpha$.

Call a premise set from which both β and $\neg\beta$ are provable, *contradictory*. Then Theorem 2 implies that every wff is provable from contradictory premises, and Theorem 6 says that if Γ, α is contradictory, then $\Gamma \vdash \neg\alpha$.

Theorem 7. $\alpha \rightarrow \beta, \neg\alpha \rightarrow \beta \vdash \beta$.

Sketch of Proof: Since, by Theorem 5, $\alpha \rightarrow \beta \vdash \neg\beta \rightarrow \neg\alpha$ and $\neg\alpha \rightarrow \beta \vdash \neg\beta \rightarrow \neg\neg\alpha$, it is sufficient to show that $\neg\beta \rightarrow \neg\alpha, \neg\beta \rightarrow \neg\neg\alpha \vdash \beta$. Now, $\neg\beta \rightarrow \neg\alpha, \neg\beta \rightarrow \neg\neg\alpha, \neg\beta$ is a contradictory premise set, since both $\neg\alpha$ and $\neg\neg\alpha$ are provable from it. Hence, $\neg\beta \rightarrow \neg\alpha, \neg\beta \rightarrow \neg\neg\alpha \vdash \neg\neg\beta$. Since $\neg\neg\beta \vdash \beta$, we get the desired conclusion.
Corollary: If $S, \alpha \vdash \beta$ and $S, \neg\alpha \vdash \beta$, then $S \vdash \beta$.
(By the deduction theorem we get: $S \vdash \alpha \rightarrow \beta$ $S \vdash \neg\alpha \rightarrow \beta$; then apply Theorem 7.)

Theorem 8. (I) $\alpha, \neg\beta \vdash \neg(\alpha \rightarrow \beta)$. **(II)** $\neg(\alpha \rightarrow \beta) \vdash \alpha$ **(III)** $\neg(\alpha \rightarrow \beta) \vdash \neg\beta$.
Proof: $\alpha, \neg\beta, \alpha \rightarrow \beta$ are contradictory premises (both β and $\neg\beta$ are provable from them); hence (I) follows via Theorem 6. To get (II), observe that $\neg\alpha, \alpha \vdash \beta$; from this, via the deduction theorem: $\neg\alpha \vdash \alpha \rightarrow \beta$; applying Theorem 5, get: $\neg(\alpha \rightarrow \beta) \vdash \neg\neg\alpha$; (II) now follows via Theorem 3. (III) follows by applying Theorem 5 to: $\beta \vdash \alpha \rightarrow \beta$.

Definition: A set S of wffs is *inconsistent* if, for some β, $S \vdash \beta$ and $S \vdash \neg\beta$. In view of Theorem 2, this is equivalent to saying that, for every γ, $S \vdash \gamma$. It is also equivalent to saying that, for some logical axiom β, $S \vdash \neg\beta$ (because if β is a logical axiom, then trivially, $S \vdash \beta$.)

S is *consistent* if it is not inconsistent.

Note: $S \cup \{\alpha\}$ is inconsistent iff $S \vdash \neg\alpha$. The "only if" direction follows from Theorem 6. The "if" direction is trivial, because if $S \vdash \neg\alpha$, then both α and $\neg\alpha$ are provable from $S \cup \{\alpha\}$.

Note: If S is consistent then, obviously, every subset of S is consistent. On the other hand, if S is any infinite set and all its finite subsets are consistent, then S is consistent. Because, if a negation, γ, of a logical axiom, were provable from S, the proof would contain a finite number of members of S; hence some finite subset of S would be inconsistent.

Basic Semantic Notions

A *truth-value assignment* is a function, f, that assigns to every atomic sentence, A, a truth-value $f(A)$.

If α is any wff (of sentential logic), then:

$$val(f,\alpha) =_{Df} \text{the truth-value of } \alpha, \text{ under the assignment } f.$$

Note: $val(f,\alpha)$ is defined by induction on α.

Obviously, $val(f,\alpha)$ depends only on the values assigned by f to the atoms occurring in α.

Definitions and Notations:

- $S \models \beta$, if for every truth-value assignment f : if $val(f,\alpha) = \mathbf{T}$ for all $\alpha \in S$, then $val(f,\beta) = \mathbf{T}$.

- If $S = \{\alpha_1,\ldots,\alpha_n\}$, we rewrite '$S \models \beta$' as: $\alpha_1,\ldots,\alpha \models \beta$. If $S=\varnothing$ we rewrite this as $\models\alpha$. Occasionally we rewrite '$S \cup \{\alpha\}$' as: S,α.
- α and β are said to be *logically equivalent*, if $\alpha\models\beta$ and $\beta\models\alpha$. This simply means that α and β get the same truth-value under all assignments. We denote this as: $\alpha\equiv\beta$.
- A set S of wffs is *satisfiable*, if there is an assignment f, such that $va\S(f,\alpha) = \mathbf{T}$ for all $\alpha\in S$.

Obviously:

- $S \models \beta \Leftrightarrow S \cup \{\neg\beta\}$ is not satisfiable.
- $S, \alpha \models \beta \Leftrightarrow S \models \alpha\rightarrow\beta$.

The second claim, which parallels the deduction theorem, is a trivial consequence of the truth table of $\rightarrow$.

Soundness Theorem (for one wff): If $\vdash \alpha$, then $\models\alpha$.

Completeness Theorem (for one wff): If $\models\alpha$, then $\vdash \alpha$.

Soundness Theorem (for many wffs): If $S \vdash \alpha$, then $S \models\alpha$.

Completeness Theorem (for many wffs, sometimes called *Strong Completeness*): If $S \models\alpha$, then $S \vdash \alpha$.

In the case where S is finite, the many-formulas case reduces to the single-formula case, via the following equivalences:

- $\alpha_1, \ldots,\alpha_n \vdash \alpha \Leftrightarrow \vdash \alpha_1\rightarrow (\alpha_2 \rightarrow(\rightarrow \ \ldots (\alpha_n \rightarrow \alpha)\ldots)$
- $\alpha_1, \ldots,\alpha_n \models \alpha \Leftrightarrow \models \alpha_1\rightarrow (\alpha_2 \rightarrow(\rightarrow \ \ldots (\alpha_n \rightarrow \alpha)\ldots)$.

The first equivalence is, via the deduction theorem. The second follows by a straightforward argument from the definitions and the truth table of. Instead of $\alpha_1\rightarrow(\alpha_2\rightarrow(\rightarrow\ldots(\alpha_n \rightarrow \alpha)\ldots)$, we can use the logically equivalent $\alpha_1\wedge\alpha_2 \wedge \ldots\alpha_n \rightarrow \alpha$ (this would involve a proof that each wff is provable from the other).

The soundness theorem is easily proved, by showing that if $\alpha_1, \ldots,\alpha_n$ is a proof of α from S, then, for every truth-value assignment, f: if $va\S(f,\beta) = \mathbf{T}$, for all $\beta\in S$, then $va\S(f,\alpha_i) = \mathbf{T}$, for i=1,…n. This is shown by (strong) induction on i, using the following easy claims:

1. Every logical axiom of sentential logic gets the value **T** under any truth-value assignment.
2. If $va\S(f,\alpha) = va\S(f,\alpha\rightarrow\beta) = \mathbf{T}$, then $va\S(f,\beta) = \mathbf{T}$.

<u>Sketch of the proof of the completeness theorem:</u>

The claim

$$S \models\alpha \Rightarrow S \vdash \alpha$$

is equivalent (via the observations above) to:

$$S \cup \{\neg\alpha\} \text{ is not satisfiable} \Rightarrow S \cup \{\neg\alpha\} \text{ is inconsistent,}$$

which is equivalent to:

$$S \cup \{\neg\alpha\} \text{ is consistent} \Rightarrow S \cup \{\neg\alpha\} \text{ is satisficable.}$$

Therefore the (strong) completeness theorem is equivalent to:

(∗) *For every set S of wffs, S is consistent* $\Rightarrow$ *S is satisfiable.*

The following proof is for the case of a countable language.

Let α_0 , α_1, α_2,…,α_n … be an enumeration of all the wffs. Given a consistent set *S*, define by induction, on *n*, a sequence of sets S_0 , S_1, … S_n, …:

$$S_0 = S$$
$$S_{n+1} = S_n \cup \{\alpha_n\}, \quad \text{if } S_n \cup \{\alpha_n\} \text{ is consistent}$$
$$S_{n+1} = S_n \cup \{\alpha_n\}, \quad \text{if } S_n \cup \{\neg\alpha_n\} \text{ is inconsistent.}$$

Obviously: $S_0 \subseteq S_1 \subseteq \ldots \subseteq S_n \subseteq S_{n+1} \subseteq \ldots$:

Claim: If a set *X* of wffs is consistent, then either $X \cup \{\alpha\}$, or $X \cup \{\neg\alpha\}$ is consistent. Otherwise, we have both: $X \vdash \neg\alpha$ and $X \vdash \neg\neg\alpha$, implying that *X* is inconsistent.

Hence all S_n's are consistent. Let $S^* = \bigcup_n S_n$. Every finite subset of S^* is a subset of some S_n, and therefore is consistent. Therefore S^* is consistent. By the construction, for each wff α, exactly one of α and $\neg\alpha$ is in S^*. Define *f* as follows:

For each atomic wff, *A*,

$$f(A) = \begin{cases} \mathbf{T} & \text{if } A \in S^* \\ \mathbf{F} & \text{if } \neg A \in S^* \end{cases}$$

Main Claim: For every wff α , if $\alpha \in S^*$ then *va§*(*f*,α) = **T**, and if $\neg\alpha \in S^*$ then *va§*(*f*,α)=**F.**

This is proved by induction on α. It holds for an atomic α, by the definition of *f*.

Assume it holds for α; show: that it holds for $\neg\alpha$. If $\neg\alpha \in S^*$, then by the induction hypothesis, *va§*(*f*,α)=**F**, hence, *va§*(*f*,$\neg\alpha$)=**T**. If $\neg(\neg\alpha) \in S^*$, then, since by Theorem 3, $\neg\neg\alpha \vdash \alpha$, $\alpha \in S^*$; hence *va§*(*f*,α)=**T** and *va§*(*f*,$\neg\alpha$)=**F.**

Assume it holds for α and β; show: it holds for $\alpha \rightarrow \beta$.

If $\alpha\rightarrow\beta \in S^*$, then either $\neg\alpha\in S^*$ or $\beta\in S^*$. For otherwise both $\alpha\in S^*$ and $\neg\beta\in S^*$; and, by Theorem 8 (I) $\alpha,\neg\beta \vdash \neg(\alpha\rightarrow\beta)$, implying that S^* is inconsistent. Hence, by the hypothesis for α and β, either *va§*(f,α)=**F** or *va§*(f,β)=**T**, implying that *va§*$(f,\alpha\rightarrow\beta)$=**T**.

If $\neg(\alpha\rightarrow\beta) \in S^*$, then, by Theorem 8 (II) and (III), $\alpha, \neg\beta \in S^*$. Hence, *va§*$(f,\alpha)$=**T** and *va§*$(f,\beta)$=**F**, implying that *va§*$(f,\alpha\rightarrow\beta)$=**F.**

This shows that *f* satisfies S^*; therefore it satisfies *S*.

QED

Finitary Version of the Proof, for a Finite *S*:

If *S* is finite, the proof can be achieved without resort to an infinite S^*. Let $S¢$ consist of all wffs that are components of members of *S*. Note that $S \subseteq S¢$, since every wff is, by definition, a component of itself (a so-called improper component). Let $S^{\#}$ be obtained from $S¢$ by adding to it all negations of members of $S¢$ that are not themselves negations. Then $S^{\#}$ is a finite set of wffs such that:

- $S \subseteq S^{\#}$
- If $\alpha\in S^{\#}$, then every component of α is in $S^{\#}$.
- If $\alpha\in S^{\#}$, then either $\neg\alpha\in S^{\#}$, or $\alpha = \neg\beta$, for some $\beta\in S^{\#}$.

Let $\alpha_1, \ldots,\alpha_n$ be an enumeration of all members of $S^{\#}$. We now repeat the construction of S^*, going only through the members of $S^{\#}$. After *n* steps we get a consistent set $S^* \subseteq S^{\#}$, such that for all $\alpha \in S^{\#}$, if $\neg\alpha\in S^{\#}$, either $\alpha\in S^*$, or $\neg\alpha\in S^{\#}$. (note that if $\neg\alpha\notin S$, then $\alpha = \neg\beta$, and in this case either α or β is in S^*). We now define a truth-value assignment *f* for all atomic wffs in $S^{\#}$, exactly in the same way as before. For atoms not in $S^{\#}$, *f* can be defined arbitrarily. We prove, by induction on α, that, for all $\alpha \in S^{\#}$, $\alpha\in S^* \Rightarrow$ *va§*(f,α) = **T** and $\neg\alpha \in S^* \Rightarrow$ *va§*(f,α)=**F.** The proof is exactly as before.

Effective Version for the One-Formula Case:

The completeness theorem implies that every tautology is provable. But it does not provide an effective way of finding the proof, short of enumerating all finite sequences of wffs, checking each whether it is a proof of the given tautology, until we hit on a proof. The following proof provides a construction of the proof.

Theorem 9: Given a truth-value assignment, *f*, define for each atom *A*:

$$A^f = \begin{cases} A & \text{if } f(A) = \mathbf{F} \\ \neg A & \text{if } f(A) = \mathbf{F} \end{cases}$$

Then, for each wff, α, for every *f*, if all the atoms in α are among, $\mathbf{A}_1, \ldots \mathbf{A}_n$, then:

1. $\mathbf{A}_1^f, \ldots \mathbf{A}_n^f \vdash \alpha$ if *va§*(f,α)=**T**
2. $\mathbf{A}_1^f, \ldots \mathbf{A}_n^f \vdash \neg\alpha$ if *va§*(f,α)=**F**

This is proved by induction on α. The claim is trivial if α is atomic. The induction steps are exactly as in the proof of the main claim in the above proof of the completeness theorem. Note that this induction yields, for each $\mathbf{A}_1^f, \ldots \mathbf{A}_n^f$, an effective construction of the proof of α, or of $\neg\alpha$, from the premises $\mathbf{A}_1^f, \ldots \mathbf{A}_n^f$.
Assume now that α is a tautology whose atoms are among $\mathbf{A}_1, \ldots \mathbf{A}_n$. We prove, by induction on k, that for every $k \leq n$:

For every f $\mathbf{A}_1^f, \ldots \mathbf{A}_{n-k}^f \vdash \alpha$

For $k=n$ this yields $\vdash \alpha$.

For $k=0$, this is follows immediately from Theorem 9 and the assumption that α is a tautology. Assume that it is true for k. Then, for every f the above holds. Therefore, for every f, both of the following hold:

- $\mathbf{A}_1^f, \ldots \mathbf{A}_{n-k-1}^f, \ \mathbf{A}_{n-k} \vdash \alpha$
- $\mathbf{A}_1^f, \ldots \mathbf{A}_{n-k-1}^f, \ \neg\mathbf{A}_{n-k} \vdash \alpha$.

Using Theorem 7 we get: $\mathbf{A}_1^f, \ldots \mathbf{A}_{n-k-1}^f \vdash \alpha$.

Note that, since the proof of Theorem 7 yields a construction of the proof in question, we get a construction of the proof of α.

QED

Compactness Theorem: If every finite subset of a set S of wffs is satisfiable, then S is satisfiable.

Proof (for a countable set): If every finite subset of S is satisfiable, then, by soundness, every finite subset of S is consistent. Therefore S is consistent. By the completeness theorem S is satisfiable.

Note: The one-formula version of the completeness theorem, together with the compactness theorem implies the many-formula version of the completeness theorem. For assume that S is consistent. Then, every finite subset $S\text{¢} \subseteq S$ is consistent, which is equivalent to the consistency of $\{\alpha\}$, where α is the conjunction of the members of $S\text{¢}$. By the one-formula completeness theorem α is satisfiable. Hence every finite subset of S is satisfiable. By the compactness theorem, S is satisfiable.

Note: The compactness theorem is stated in purely semantic terms, without any reference to a deductive system. There is also a purely semantic proof of it; cf. Enderton's text book.

<u>An Application of Sentential Compactness:</u>

A (undirected) graph is a structure $G = (V, E, g)$ where V is a non-empty set (of objects called 'vertices'), E is a set (of objects called 'edges') and g is a function that associates with every $e \in E$ an unordered pair $\{x,y\}$ of members of V; we say that e connects x and y. (In a directed graph the values of g are ordered pairs (x,y).)

Call two vertices *neighbors* if some edge connects them.

A coloring of the graph G is a mapping, h, that associates with every vertex, x, an object $h(x)$ called *the color of* x, such that neighboring vertices always have different colors. A graph is *k-colorable*, if it has a coloring that uses $\leq k$ colors.

A *subgraph* of G is any graph $G\text{¢} = (V\text{¢}, E\text{¢}, g\text{¢})$ such that $V\text{¢} \subseteq V$, $E\text{¢} \subseteq E$ and $g\text{¢}$ is the restriction of g to $E\text{¢}$.

Theorem: For each k, if every finite subgraph of G is k-colorable, G is k-colorable.
Proof: Without loss of generality, let the colors be $0,\ldots, k-1$. For every vertex, x, of G and for every $i< k$, let $A_{x,i}$ be an atom of sentential logic. (Think of $A_{x,i}$ as saying that vertex x is colored with color i.) Consider the set, S, of all wffs that fall under one of the following cases:

1. $A_{x,0} \vee A_{x,1} \vee \ldots \vee A_{x,k-1}$, for all $x \in V$ (it says that the vertex is colored by some color).
2. $A_{x,i} \rightarrow \neg A_{x,j}$, for all $x \in V$, $i< j< k$ (it says that the vertex is not colored by more than one color).
3. $A_{x,i} \rightarrow \neg A_{y,i}$, for all x, y that are neighbors and all $i<k$ (it says that the two vertices have different colors).

It is not difficult to check that G has a k-coloring iff S is satisfiable. If every finite subgraph of G has a k-coloring, then every finite subset of S is satisfiable. By the compactness theorem, S is satisfiable. Hence G is k-colorable.

qed