1: Vector spaces

- fields
- vector spaces
- subspaces
- range and nullspace

Beyond $\mathbb{R}^n$ and $\mathbb{C}^n$

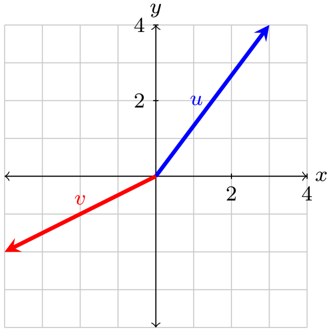
- $\mathbb{R}^n$ denotes the set of n-vectors with **real** elements
- $\mathbb{C}^n$ denotes the set of n-vectors with **complex** elements
- $x \in \mathbb{C}^n$ denotes that x "belongs to the set" of complex n-vectors, e.g.;

$$\text{let} \quad x = \begin{pmatrix} 4+3\mathbf{i} \\ -2-2\mathbf{i} \\ 1 \\ -2\mathbf{i} \end{pmatrix}, \quad \text{then} \quad x \in \mathbb{C}^4$$

- it is possible and often desirable to work with vectors whose elements are not just real or complex numbers
- a **vector space** generalizes the idea of **addition** and **scalar multiplication** to more general sets

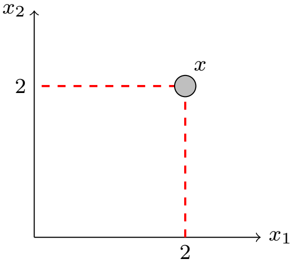
Example: $\mathbb{R}^2$

Displacement (relative to the origin)



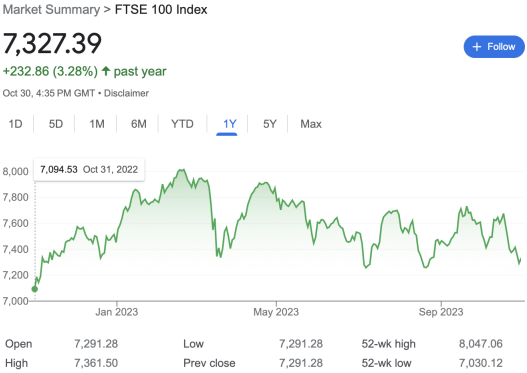
$$u = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \quad v = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$$

Position (relative to the origin)



Example $\mathbb{R}^{365}$

Time-series data



- FTSE 100 Index
- vector with 365 entries
- source: Google Finance

Fields

A field is a set F on which two operations $+$ and $\cdot$ are defined such that for all $\alpha, \beta \in F$, unique elements $\alpha + \beta$ and $\alpha \cdot \beta$ in F exist, moreover they must satisfy the following conditions for all $\alpha, \beta, \delta \in F$:

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7. **Multiplicative inverse:** if $\alpha \neq 0$, an element $\alpha^{-1} \in F$ exists with $\alpha \cdot \alpha^{-1} = 1$

Uniqueness

Proposition

Let F be a field.

1. the additive identity in F is unique
2. the additive inverse of an element of F is unique
3. the multiplicative identity in F is unique
4. the multiplicative inverse of a nonzero element of F is unique

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Intuition:

A field is the set of numbers (or "scalars") you use to measure and scale vectors.

Examples

- real numbers $\mathbb{R}$
- complex numbers $\mathbb{C}$
- rational numbers $\mathbb{Q}$

$$\frac{p}{q} \in \mathbb{Q} \quad \Longleftrightarrow \quad p, q \in \mathbb{Z}$$

Non-examples

- integers $\mathbb{Z}$ (axiom 7)

$$2 \in \mathbb{Z} \quad \text{but no } n \in \mathbb{Z} \text{ such that } 2n = 1$$

Vector spaces

- a **vector space** over a **field** F is a non-empty set V together with two binary operations that satisfy a set of axioms (to be defined)
- the elements of V are called **vectors**, and the elements of F are called **scalars**
- when $F = \mathbb{R}$ we call V a **real vector space**
- when $F = \mathbb{C}$ we call V a **complex vector space**
- vector spaces are the simplest structures that allow for the most general computations

Vector space axioms

A **vector space** over a **field** F is a non-empty set V together with two binary operations (addition and scalar multiplication) that satisfy the axioms listed below.

Let $\alpha, \beta \in F$ and $u, v, w \in V$, then the following must hold:

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8. **Distributivity of scalar multiplication w.r.t. field addition:** $(\alpha + \beta)v = \alpha v + \beta v$

Consequences

- vector subtraction is defined as

$$v - w = v + (-1)w$$

- let $\alpha \in F$ and $v \in V$, then

$$\begin{aligned} 0v &= 0_n \\ \alpha 0_n &= 0_n \\ (-1)v &= -v \\ \alpha v &= 0_n \quad \Longrightarrow \quad \alpha = 0 \text{ or } v = 0_n \end{aligned}$$

admittedly this seems fairly tedious, however, it allows us to define very useful objects

Examples

- set of all real and complex n-vectors, $\mathbb{R}^n$, $\mathbb{C}^n$
- the set of all $m \times n$ matrices $\mathbb{F}^{m \times n}$ with $\mathbb{F} = \{\mathbb{R}, \mathbb{C}\}$
- the set of (unordered) pairs of integers
- the set of real numbers x, y, z such that $x + y + z = 1$

Subspaces

"A subspace is like Las Vegas. What happens in Vegas, stays in Vegas"

objects of interest are typically not just points in a space, but *lines*, *planes*, and *subsets*

Given a non-empty set S defined over a vector field $V(F)$, if $S \subseteq V$ and

1. $x + y \in S$ for all $x, y \in S$, and
2. $\alpha x \in S$ for all $\alpha \in F$, and $x \in S$,

then S is a **subspace**

Remarks:

- a subspace is itself a vector space
- a subspace is **closed under addition** *and* **closed under scalar multiplication**

specializing to $V = \mathbb{R}^n$, we have the following definition

Given a non-empty set $S \subseteq \mathbb{R}^n$, if

1. $x + y \in S$ for all $x, y \in \mathbb{R}^n$, and
2. $\alpha x \in S$ for all $\alpha \in \mathbb{R}$, and $x \in \mathbb{R}^n$,

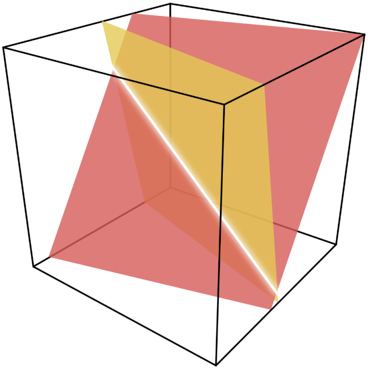
then S is a **subspace**

we will mostly be working with real vector spaces

Intersection rule

Theorem

The intersection of a collection of subspaces is a subspace.



Examples

suppose $V = \mathbb{R}^n$

- $S_1 = V$
- $S_2 = \{\mathbf{0}\}$
- given vectors $v_1, \ldots, v_l \in \mathbb{R}^n$, all linear combinations of the v_k's

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_l v_l, \quad \alpha_i \in \mathbb{R}$$

- let $n = 2$

$$S_3 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; y = 7x \right\}$$

- $\mathbb{R}^{n-1}$ (non-example)

Direct sum

Sum of two subspaces

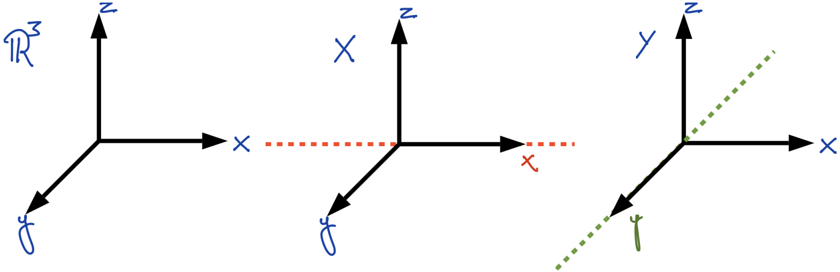
if X and Y are subspaces, then $X + Y$ is a subspace

$$X + Y = \{x + y \mid x \in X,\ y \in Y\}$$

Example

let $X = (x, 0, 0) \in \mathbb{R}^3$ with $x \in \mathbb{R}$, and $Y = (0, y, 0) \in \mathbb{F}^3$ with $y \in \mathbb{R}$, then

$$X + Y = (x, y, 0) \in \mathbb{R}^3$$


$\mathbb{R}^3$
z
x
y
X
z
x
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Y
z
x
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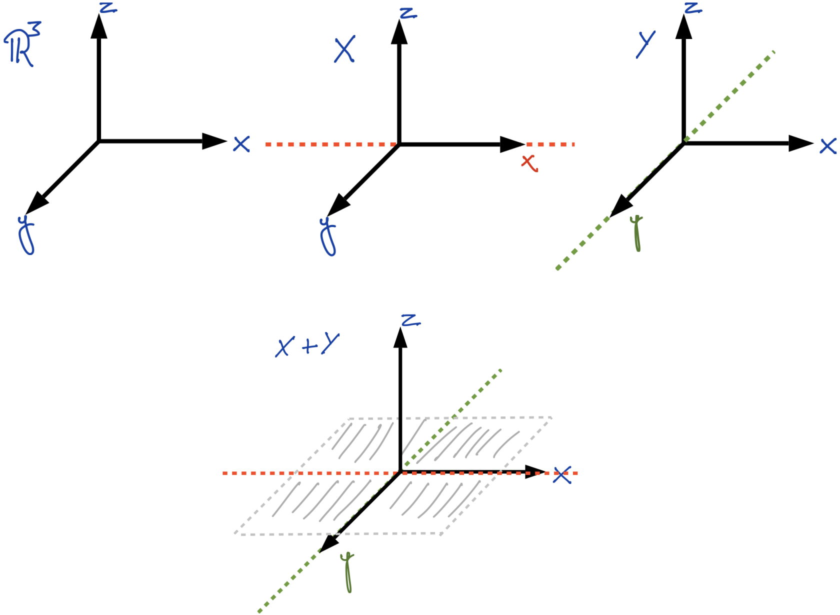

$\mathbb{R}^3$
X
Y
X + Y
x
y
z

Direct sum

assume we have 3 sets A, B, V such that

- V is a vector space over F
- A and B are subspaces of V

V is the **direct sum** of A and B, written as

$$V = A \oplus B,$$

if

1. every element of V can be written as a sum of elements from A and B:

$$V = A + B = \{a + b \mid a \in A, b \in B\}$$

2. the representation is unique, i.e., $A \cap B = \emptyset$

Nullspace

given a matrix $A \in \mathbb{R}^{m\times n}$, we define the nullspace of A as

$$\textbf{null}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\} \subseteq \mathbb{R}^n$$

- sometimes written as $\mathcal{N}(A)$, often drop the "$\in \mathbb{R}^n$" from the notation
- set of vectors "mapped to" the origin, by A, i.e., $x \mapsto Ax$
- **null**(A) is a **subspace** of $\mathbb{R}^n$

later we will see how to characterize the "size" of the nullspcace and find representations of it

Example

consider the matrix $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

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$\textbf{null}(A)$ is a subspace, so

- if $x \in \textbf{null}(A)$ then so is αx for any α
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Consequences of being in the nullspace

- if $y = Ax$ and $w \in \mathbf{null}(A)$, then $y = A(x + w)$
- if $y = Ax$ and $y = Av$, then $v = x + z$ with $z \in \textbf{null}(A)$

One-to-One mapping

a set with only one element is a **singleton**, if a subspace, S, is a singleton, then $S = \{\mathbf{0}\}$

A is said to be **one-to-one** if $\textbf{null}(A) = \{\mathbf{0}\}$

equivalently, this means:

- for $y = Ax$, different x's map to different y's (no ambiguity)
- x can be uniquely determined from A (lossless)

Interpretations

- let $y = Ax$, y denotes a measurement of x and assume $z \in \mathbf{null}(A)$
  - z is undetectable by the sensor
  - sensor cannot distinguish between x and $x + z$

null(A) characterizes the ambiguity in x from the measurement $y = Ax$

Range

given a matrix $A \in \mathbb{R}^{m \times n}$, we define the range of A as

$$\textbf{range}(A) = \{Ax \in \mathbb{R}^n \mid x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$$

- sometimes written as $\mathcal{R}(A)$, often drop the "$\in \mathbb{R}^n$" from the notation
- set of vectors "hit" by Ax
- **range**(A) is a **subspace** of $\mathbb{R}^m$

Onto matrices

- if **range**$(A) = \mathbb{R}^m$, A is said to be onto
- can solve $Ax = y$ for x, given any y

Interpretations

- given $v \in \mathbf{range}(A)$ and $w \notin \mathbf{range}(A)$, consider $y = Ax$
  - $y = v$ is a possible measurement from the sensor
  - $y = w$ is an impossible measurement – implies broken sensor or incorrect model

range(A) characterizes possible outputs

Further topics

other mathematical constructs which share some structure with a vector space:

- groups
- rings
- algebras
- modules