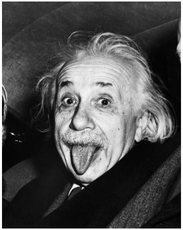
4: Matrices

- what is a matrix?
- basics
- matrix-vector operations
- applications

Interpretations of a matrix

A table of numbers

- **grey-scale image:** grid of integers from 0–255 representing pixel intensity



- **bus schedule:** each column represents arrival times for a specific route

Q102

Bus Timetable
MTA Bus Company

Astoria - Roosevelt Island Via 30Th Av / 31St St
Local Service
Effective September 4, 2022

For accessible subway stations, travel directions and other information:
Visit www.mta.info or call us at 511

Q102 Weekday — **To Astoria**

Roosvelt Island **W Loop Rd / N Loop Rd**	Long Island City **Vernon Bl / 36 Av**	Long Island City **Queens Plz S / 27 St**	Astoria **27 Av / 2 St**
5:05	5:10	5:18	5:37
6:05	6:10	6:18	6:37
6:35	6:40	6:48	7:07
7:05	7:11	7:23	7:45

a collection of vectors

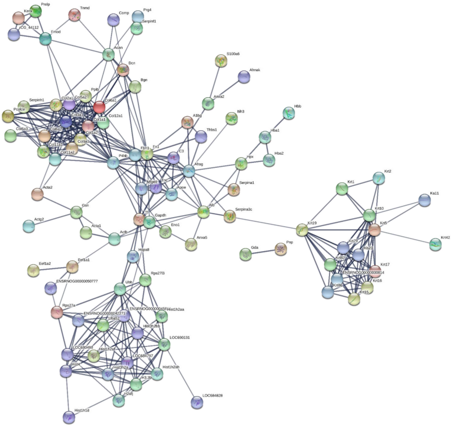
time-series data: each column-vector corresponds to data points at specific times

$$X = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}$$

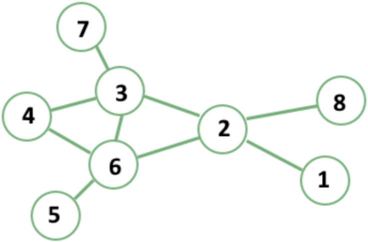
[image from Yahoo finance]

representation of interactions

- **graphs:** objects of interest denoted by vertices, interaction between objects indicated with an edge



- protein–protein interaction in an adult tendon [Choi *et al.* elife, 2020;9:e55262] many matrices can describe a graph

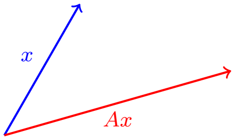


one such matrix is the **adjacency** matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

as a map $x \mapsto Ax$

- a matrix acts on a vector mapping it from one vector space to another

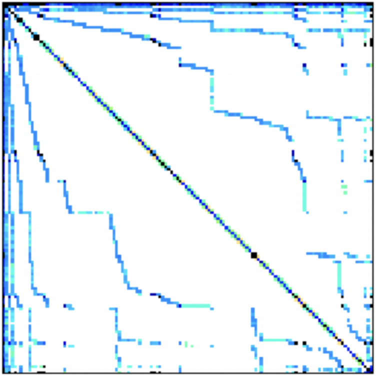


the 2×2 matrix A maps the vector $x \in \mathbb{R}^2$ to a new vector $Ax \in \mathbb{R}^2$

Basics

- $\mathbb{F}^{m \times n}$ denotes a matrix with m rows and n columns, with entries from the field $\mathbb{F}$
- $\mathbb{F}^{m \times n}$ is a vector space
- If $X \in \mathbb{F}^{m \times n}$ the x_{ik} denotes the element in row i, column k
- a matrix is
  - **square** if $m = n$
  - **tall** if $m > n$
  - **wide** if $m < n$
- a matrix with mostly zero entries is said to be **sparse**

Sparse example



- $2,987,012 \times 2,987,012$ matrix
- $\mathbf{nnz}(X) = 26,621,983$
- credit: circuit simulation, Kiran Gullapalli, Freescale Semiconductor, Inc.
- SuiteSparse Matrix Collection: `http://sparse.tamu.edu`

Transpose

let X be a **real** $m \times n$ matrix, i.e, $X \in \mathbb{R}^{m \times n}$

$$X = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{bmatrix}$$

- the **transpose** operation, denoted X^T, turns row i into column i
- $X^T \in \mathbb{R}^{n \times m}$

$$X^T = \begin{bmatrix} x_{11} & x_{21} & \dots & x_{m1} \\ x_{12} & x_{22} & \dots & x_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \dots & x_{mn} \end{bmatrix}$$

- $(X^T)^T = X$
- if $X = X^T$ then X is said to be **symmetric**
- if α is a scalar, then $(\alpha X)^T = \alpha X^T$

Conjugate transpose

the transpose operation is **almost never** used on complex matrices

let Y be a **complex** $m \times n$ matrix

$$Y = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m1} & y_{m2} & \cdots & y_{mn} \end{bmatrix}$$

- recall the conjugate of a complex number: $w = \alpha + j\beta$, then $\bar{w} = \alpha - j\beta$
- the **conjugate transpose** of Y is the $n \times m$ matrix

$$Y^* = \begin{bmatrix} \bar{y}_{11} & \bar{y}_{21} & \cdots & \bar{y}_{m1} \\ \bar{y}_{12} & \bar{y}_{22} & \cdots & \bar{y}_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{y}_{1n} & \bar{y}_{2n} & \cdots & \bar{y}_{mn} \end{bmatrix}$$

- Y is said to be **Hermitian** if $Y = Y^*$
- if β is a scalar, $(\beta Y)^* = \bar{\beta} Y^*$

Special matrices

the **Identity** matrix is a **square** and **diagonal** matrix containing only 0's and 1's

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{also denoted} \quad I = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

- sometimes it's helpful to include the dimension: I_n
- columns of I_n are the standard basis vectors $e_1, \ldots, e_n$

Zero matrix

- all elements of the zero matrix are $0's$
- does not need to be square, $0_{m \times n}$ is allowed

$$0_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Block matrices

- a matrix of matrices
- sub-matrix blocks must be of compatible dimensions

Example: 2×2 **block-matrix**

$$A = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$$

if

$$W = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad X = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, \quad Y = \begin{bmatrix} 4 & 0 & 1 \end{bmatrix}, \quad Z = \begin{bmatrix} 2 & 9 \end{bmatrix}$$

then

$$A = \left[\begin{array}{ccc|cc} 1 & 2 & 3 & 2 & 2 \\ 4 & 5 & 6 & 2 & 2 \\ \hline 4 & 0 & 1 & 2 & 9 \end{array}\right], \quad \textbf{dim}(A) = m \times n$$

- m: no. rows in W + no. rows in Y
- n: no. cols in W + no. rows in X

Operations on block matrices

addition

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} + \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} A+E & B+F \\ C+G & D+H \end{bmatrix}$$

transposition

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^T = \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix}$$

inverse

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \neq \begin{bmatrix} A^{-1} & B^{-1} \\ C^{-1} & D^{-1} \end{bmatrix}$$

(even if the matrix is square)

Matrices from vectors

often useful to construct matrices from vectors

- let $a_1, \ldots, a_k$ be n-vectors

$$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \ldots & a_k \\ | & | & & | \end{bmatrix}$$

- A is a $1 \times k$ block matrix of dimension $n \times k$

- given $b_1, \ldots, b_l \in \mathbb{R}^m$

$$B = \begin{bmatrix} \text{———} & b_1 & \text{———} \\ \text{———} & b_2 & \text{———} \\ & \vdots & \\ \text{———} & b_l & \text{———} \end{bmatrix}$$

- B is an $l \times 1$ block matrix of dimension $l \times m$

Sub-matrices

Matlab notation

consider the matrix

$$Z = \begin{pmatrix} a & b & c & d & e \\ f & g & h & i & j \\ k & l & m & m & o \\ p & q & r & s & t \\ u & v & w & x & y \end{pmatrix}$$

- $Z_{:,1}$ is used to denote the first column of Z
- likewise, $Z_{1,:}$ denotes the first row
- $Z_{2:4,1:3}$ denotes the matrix

$$\begin{bmatrix} f & g & h \\ k & l & m \\ p & q & r \end{bmatrix}$$

Matrix-vector multiplication

Row interpretation

given $A \in \mathbb{C}^{m \times n}$ and $x \in \mathbb{C}^n$, the map $x \mapsto Ax$ produces a vector $b \in \mathbb{C}^m$

$$b = Ax$$

- the map $x \mapsto Ax$ is linear: $(\alpha \in \mathbb{C}, x, y \in \mathbb{C}^n)$

$$\begin{aligned} A(x+y) &= Ax + Ay \\ A(\alpha x) &= \alpha A x \end{aligned}$$

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- elements of b are computed via the formula

$$b_i = \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, \ldots, m$$

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \leftarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

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- **Interpretation:** A acts on x to produce b

Column interpretation let a_j denote the j^{th} column of A, then

$$b = Ax = \sum_{j=1}^{n} x_j a_j$$

b is a **linear combination** of columns of A

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$$\begin{bmatrix} \\ b \\ \\ \end{bmatrix} = \left[\begin{array}{c|c|c|c} & & & \\ a_1 & a_2 & \dots & a_n \\ & & & \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} \\ a_1 \\ \\ \end{bmatrix} + x_2 \begin{bmatrix} \\ a_2 \\ \\ \end{bmatrix} + \dots + x_n \begin{bmatrix} \\ a_n \\ \\ \end{bmatrix}$$

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- **New perspective**: x acts on A to produce b
- $\mathbf{range}(A)$ is the space **spanned** by the columns of A
- motivates the equivalent name **column space** of A

Matrix-matrix multiplication

consider the matrix product $B = AC$ where $A \in \mathbb{C}^{l \times m}$ and $C = \mathbb{C}^{m \times n}$, then

$$b_{ij} = \sum_{k=1}^{m} a_{ik} c_{kj}$$

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \longleftarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{23} \end{bmatrix}$$

matrix-matrix multiplication is just $l \cdot n$ inner products

$$b_{ij} = \langle a_{i,:}^T, x_j \rangle$$

where $a_{i,:}$ is the i^{th} row of a

Column interpretation

matrix-matrix multiplication can be expressed in terms of columns

$$\left[\begin{array}{c|c|c|c} b_1 & b_2 & \ldots & b_n \end{array}\right] = \left[\begin{array}{c|c|c|c} a_1 & a_2 & \ldots & a_m \end{array}\right] \left[\begin{array}{c|c|c|c} c_1 & c_2 & \ldots & c_n \end{array}\right]$$

algebraically, this is written as

$$b_j = Ac_j = \sum_{k=1}^{m} c_{kj} a_k, \quad j = 1, \ldots, n$$

- the columns of B are **linear combinations** of a_k with coefficients c_{kj}
- matrix-matrix multiplication is just matrix-vector multiplication repeated

Examples

- **column selector:** select column j from A

$$a_j = Ae_j$$

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- **difference:** given a vector x, return vector with elements $x_2 - x_1, x_3 - x_2, \ldots$

$$D = \begin{bmatrix} -1 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & -1 & 1 & \ldots & 0 & 0 & 0 \\ & & \ddots & \ddots & & & \\ & & & \ddots & \ddots & & \\ 0 & 0 & 0 & \ldots & -1 & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & -1 & 1 \end{bmatrix}, \quad Dx = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ \vdots \\ x_n - x_{n-1} \end{bmatrix}$$

- **permutation matrices:** create new vector $y_i = x_{\pi i}$ where π is a permutation of $1, 2, \ldots, n$

$$\text{given } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \text{want } y = \begin{bmatrix} x_3 \\ x_1 \\ x_2 \end{bmatrix}, \text{then} \quad y = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{P} x$$

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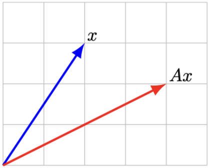
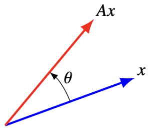
- **dilation matrices:** create a new vector that scales each component of x by a different factor

$$y = Dx, \quad \text{where} \quad D = \begin{bmatrix} \alpha_1 & \\ & \alpha_2 \end{bmatrix}$$



- **rotation matrices:** consider $x \in \mathbb{R}^2$

$$y = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} x$$



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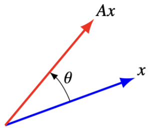

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- **dilate then rotate:** consider $x \in \mathbb{R}^2$

$$y = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} x$$

amplifies x_1 by a factor of 4, shrinks x_2 by a factor of $\frac{1}{2}$ and rotates by θ radians counter-clockwise

Convolution

discrete-time convolution between vectors can viewed as matrix-vector operation

- recall the continuous-time definition of confolution for two functions f and g

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\mathrm{d}\tau$$

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- when x and y are vectors of length n and m respectively

$$z = x * y \quad \text{then} \quad z_k = \sum_{i,j \in S} x_i y_j$$

  where S is the set of integers such that $i + j = k + 1$

Discrete-time convolution between vectors

given $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, $z = x * y$ is a vector of length $n + m - 1$

$$z_k = \sum_{\substack{i \text{ and } j \text{ s.t.} \\ i+j=k+1}} x_i y_j$$

Example $n = 4, m = 3$

$$\begin{aligned} z_1 &= x_1 y_1 \\ z_2 &= x_1 y_2 + x_2 y_1 \\ z_3 &= x_1 y_3 + x_2 y_2 + x_3 y_1 \\ z_4 &= x_2 y_3 + x_3 y_2 + x_4 y_1 \\ z_5 &= x_3 y_3 + x_4 y_2 \\ z_6 &= x_4 y_3 \end{aligned} \quad \Longleftrightarrow \quad \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \underbrace{\begin{pmatrix} x_1 & 0 & 0 \\ x_2 & x_1 & 0 \\ x_3 & x_2 & x_1 \\ x_4 & x_3 & x_2 \\ 0 & x_4 & x_3 \\ 0 & 0 & x_4 \end{pmatrix}}_{T} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

- can be equivalently expressed with a vector x and matrix y
- matrices with the structure T are called **Toeplitz** matrices
- this formulation strips away leading and trailng zeros

Toeplitz matrices

the general form of a Toeplitz matrix is

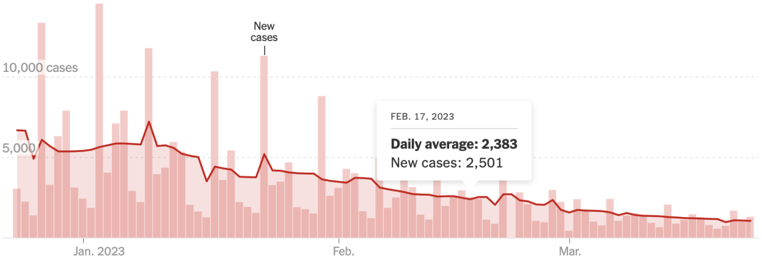
$$\begin{pmatrix} a & b & c & d & e \\ f & a & b & c & d \\ g & f & a & b & c \\ h & g & f & a & b \\ i & h & g & f & a \end{pmatrix}$$

- as seen in the convolution example, they need not be square
- autocorrelation, cross-correlation, and k-moving average can all be expressed using Toeplitz matrices
- a lower-triangular/banded structure often corresponds to causality

Moving average

k-period moving average

- let x be a time-series signal of length n
- let $a = \underbrace{(\frac{1}{7}, \ldots, \frac{1}{7})}_{7 \text{ times}}$, then $a * x$ is the moving average
- New York Times Covid data: reported cases



Stock data

- Tesla stock over 5 years
- daily data and 10-day moving average

Evaluating polynomials

- a univariate polynomial, $\mathbb{R}[t]$, with $t \in \mathbb{R}$ of degree $n-1$ is given by

$$p(t) = x_0 + x_1 t + x_2 t^2 + \ldots + x_{n-1} t^{n-1}$$

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- the **Vandermonde matrix** evaluates $p(t)$ at various values of t

$$\begin{bmatrix} p_1(t_1) \\ p_2(t_2) \\ \vdots \\ p_m(t_m) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & t_1 & t_1^2 & \ldots & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \ldots & t_2^{n-1} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 1 & t_m & t_m^2 & \ldots & t_m^{n-1} \end{bmatrix}}_{V} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

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- note that Vx is linear in the coefficients $x_0, \ldots, x_{n-1}$

- polynomial interpolation can be framed as solving $Vx = p$

- columns of V are **linearly independent**

Further topics

- Projection onto a line
- 2D convolution
- Discrete Fourier Transform (special case of a Vandermonde matrix)