5: Eigenvalues and vectors

- definitions
- counting eigenvalues
- eigenvalue decomposition
- computing eigenvalues
  - special cases
  - power iteration algorithm

Eigenvalues and eigenvectors

Definition

Given a matrix $A \in \mathbb{C}^{n \times n}$, a non-zero vector $x \in \mathbb{C}^n$ is an eigenvector of A, and $\lambda \in \mathbb{C}$ is its corresponding eigenvalue, if

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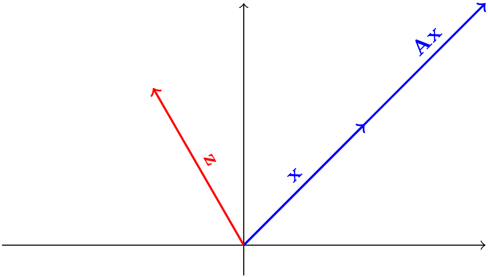

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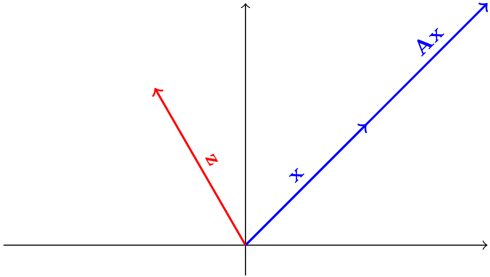

- the action of A on x only scales x
- eigenvectors are vectors whose direction does not change when acted on by A
- the eigenvalue gives the size of the scaling of the eigenvector

Basics of eigen-pairs

$$Ax = \lambda x$$

- if $Ax_i = \lambda_i x_i$, we say that x_i is the eigenvector corresponding to λ_i

- (x_i, λ_i) are eigen-pairs

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- the set of eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ is called the **spectrum** of A

- there are n eigenvalues associated to $A \in \mathbb{C}^{n\times n}$ counted with multiplicity

Characteristic polynomial

the **characteristic polynomial** of $A \in \mathbb{C}^{n \times n}$, denoted p_A, is the degree n polynomial

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the characteristic polynomial provides a method for computing eigenvalues of A

- λ is an eigenvalue of A iff $p_A(\lambda) = 0$

$$\begin{aligned} \lambda \text{ is an eigenvalue of } A \quad &\Longleftrightarrow \quad \text{there is a non-zero } x \text{ s.t. } \lambda x - Ax = 0 \\ &\Longleftrightarrow \quad \lambda I - A \text{ is singular} \\ &\Longleftrightarrow \quad \det(\lambda I - A) = 0. \end{aligned}$$

- the implication of this is that real matrices may have some complex eigenvalues

Example

consider the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ which rotates a vector 90° counter-clockwise

$$p_A(\lambda) = \det\left(\begin{bmatrix} \lambda & 1 \\ -1 & \lambda \end{bmatrix}\right) = \lambda^2 + 1$$

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- $p_A(\lambda) = 0 \Longrightarrow \lambda^2 = -1$ and so $\lambda = \pm i$
- consider each eigenvalue in turn to find eigenvectors

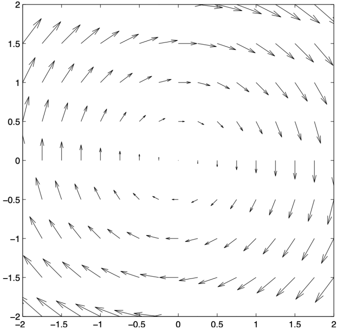
$$\lambda_1 = i : \quad A - \lambda_1 I = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \quad \Longrightarrow \quad x_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$\lambda_2 = -i : \quad A - \lambda_2 I = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \quad \Longrightarrow \quad x_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

The harmonic oscillator

complex eigenvalues have “real” impact

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t)$$



$$x(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} x(0)$$

TLDR - E6602 if interested

Eigenvalues/eigenvectors

$$p_A(\lambda) = \lambda^2 + 1 \implies \lambda = \pm i.$$

$$v_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}, \quad v_2 = \overline{v_1} = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Diagonalize

$$A = V \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} V^{-1}, \quad V = [v_1 \ v_2]$$

Matrix exponential:

$$e^{At} = V \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} V^{-1}$$

$$= V \begin{pmatrix} \cos t + i \sin t & 0 \\ 0 & \cos t - i \sin t \end{pmatrix} V^{-1}$$

$$\implies e^{At} = \cos t \, I + \sin t \, A = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

Intuition

- Eigenvalues $\pm i$ give oscillatory scalar modes

$$e^{\pm it} = \cos t \pm i \sin t$$

- complex-conjugate eigenpairs combine to yield *real* rotations: imaginary parts cancel

$$V^{-1} = \frac{1}{2} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

- e^{At} is a clockwise rotation by angle t about the origin; trajectories are circles with frequency $\omega = 1$

Algebraic multiplicity

for $A \in \mathbb{C}^{n \times n}$, the fundamental theorem of algebra states that p_A can be expressed as

$$p_A(z) = (z - \lambda_1)(z - \lambda_2) \ldots (z - \lambda_n)$$

for some values $\lambda_j \in \mathbb{C}$

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for some values $\lambda_j \in \mathbb{C}$

- from p.5-4 we have seen that λ_j is an eigenvalue of A
- it's possible an eigenvalue is repeated, for example

$$\begin{bmatrix} 3 & 0 & 0 \\ 5 & 3 & 0 \\ 3 & 0 & 2 \end{bmatrix} \quad \text{has eigenvalues} \quad \lambda_1 = 3,\ \lambda_2 = 3,\ \lambda_3 = 2$$

- the **algebraic multiplicity** of an eigenvalue is the number of times it appears in the factorized expression of p_A
- eigenvalues with an algebraic multiplicity of 1 are called **simple**

Linearly independent eigenvectors

Theorem

If the $A \in \mathbb{C}^{n \times n}$ has n **simple eigenvalues**, *then the corresponding eigenvectors $x_1, \ldots, x_n$ are* **linearly independent.**

Recap

- recall, an eigenvalue is "simple" if it is not repeated
- the set of vectors $\{x_1, \ldots, x_n\}$ are linearly independent if x_i cannot be written as a linear combination of the remaining elements of the set

Geometric multiplicity

if x is an eigenvector of A, then αx is also an eigenvector ($\alpha \in \mathbb{R}$), because

$$x \in \mathbf{null}(A - \lambda I)$$

consider a specific eigenvalue λ_i

- we call E_{λ_i} the eigenspace of λ_i:

$$E_{\lambda_i} = \mathbf{null}(A - \lambda_i I)$$

- the **geometric multiplicity** of λ_i is defined as $\mathbf{dim}(E_{\lambda_i})$
- $\mathbf{dim}(E_{\lambda_i})$ describes the maximum number of linearly independent eigenvectors that can be found for λ_i

Example

Consider the 2×2 upper-triangular matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

- by inspection (see p.5-12), we have that $\lambda_1 = \lambda_2 = 0$
- 0 has an algebraic multiplicity of 2
- an eigenvector corresponding to the 0 eigenvalue must satisfy

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \text{all eigenvectors have the form } x = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

thus, the eigenvalue has geometric multiplicity 1

Calculating eigenvalues: Special cases

- **lower-triangular matrices**

  a 2×2 lower-triangular matrix takes the form $\underbrace{\begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix}}_{A}$, it follows that

$$p_A(\lambda) = \det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda - a_{11} & 0 \\ -a_{22} & \lambda - a_{22} \end{bmatrix}\right) = (\lambda - a_{11})(\lambda - a_{22})$$

  clearly, $p_A(\lambda) = 0 \quad \Longrightarrow \quad \lambda = a_{11},\ \lambda = a_{22}$

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- **diagonal matrices**

  the off-diagonal elements played no role in the above calculation

$$p_A(\lambda) \quad \text{is not a function of } a_{ij} \text{ if } i \neq j$$

  as with lower-triangular matrices, $\lambda_i - a_{ii}$

- **Hermitian matrices**

  if $A = A^*$ then each of the eigenvalues of A is **real**, to see this

$$\lambda x = Ax$$

$$\implies \qquad \lambda x^* x = x^* A x$$

$$\implies \qquad \lambda = \tfrac{x^* A x}{x^* x}$$

from where we see that

$$\lambda^* = \frac{(x^* A x)^*}{(x^* x)^*} = \frac{x^* A^* x}{x^* x}$$

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- **symmetric matrices**

  if $A = A^T$, the arguments above hold because $A = A^T = A^*$

Relationship between algebraic and geometric multiplicity

Theorem

Let A be a square matrix and let λ be an eigenvalue of A. Then

$$1 \leq (\text{geometric multiplicity of } \lambda) \leq (\text{algebraic multiplicity of } \lambda).$$

Implications

if the algebraic multiplicity of λ is 1, then

- so is the geometric multiplicity
- the corresponding eigenspace, E_λ is a line through the origin

Eigenvalue decomposition

let $A \in \mathbb{C}^{n \times n}$ be a matrix with **linearly independent** eigenvectors, then we can write

$$A = X \Lambda X^{-1}$$

where

- Λ is a diagonal matrix whose entries are the eigenvalues of A
- column x_j of X corresponds to the j^{th} eigenvector of A
- X is invertible by assumption c.f., p. 5-9

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- Λ is a diagonal matrix whose entries are the eigenvalues of A
- column x_j of X corresponds to the j^{th} eigenvector of A
- X is invertible by assumption c.f., p. 5-9
- rearranging the eigenvalue decomposition gives

$$AX = X\Lambda$$

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \left[\begin{array}{c|c|c} & & \\ x_1 & \dots & x_n \\ & & \end{array}\right] = \left[\begin{array}{c|c|c} & & \\ x_1 & \dots & x_n \\ & & \end{array}\right] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

this decomposition makes it clear that $Ax_j = \lambda_j x_j$ (c.f., p.4-19)

Diagonalization

yet another rearrangement of $A = X\Lambda X^{-1}$ gives

$$X^{-1}AX = \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

which motivates the phrase "A is diagonalizable"

- for A to be diagonalizable, we require linearly independent eigenvectors
- an eigenvalue whose algebraic multiplicity exceeds its geometric multiplicity is called **defective**
- a matrix with one or more defective eigenvalues is said to be **defective**

Theorem

Only non-defective matrices can be diagonalized.

consequences of diagonalization

if A can be diagonalized, we can compute A^n **without** resorting to the definition:

$$A^n = \underbrace{AA \ldots A}_{n \text{ times}}$$

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- first consider $n = 2$, i.e., A^2

  $$A^2 = (X\Lambda X^{-1})X\Lambda X^{-1} = X\Lambda^2 X^{-1}$$

  where

  $$\Lambda^2 = \begin{bmatrix} \lambda_1^2 & & & \\ & \lambda_2^2 & & \\ & & \ddots & \\ & & & \lambda_n^2 \end{bmatrix}$$

- by induction, for arbitrary n, $A^n = X\Lambda^n X^{-1}$

- in addition, for diagonalizable matrices, the eigenvectors for A and A^n are the same

  $$Ax = \lambda x \quad \iff \quad A^2 x = \lambda^2 x$$

  $\Longleftarrow$ not true for non-diagonalizable

Basis of $\mathbf{range}(A)$

recall the definition of $\mathbf{range}(A)$ where $A \in \mathbb{R}^{n\times n}$:

$$\mathbf{range}(A) = \{Ax \in \mathbb{R}^n \mid x \in \mathbb{R}^n\} \subseteq \mathbb{R}^n$$

if A is diagonalizable, then

- A may have eigenvalues equal to zero
- the set of eigenvectors corresponding to the non-zero eigenvalues forms a basis for $\mathbf{range}(A)$

Non-Uniqueness

it is important to note that diagonalization is **not** unique

- permuting the order of the eigen-pairs will change X and Λ
- eigenvectors are not unique $\Longrightarrow X$ is not unique

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$$\begin{bmatrix} \\ \quad A \quad \\ \\ \end{bmatrix} = \left[\begin{array}{c|c|c} & & \\ x_1 & \dots & x_n \\ & & \end{array}\right] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \left[\begin{array}{c|c|c} & & \\ x_1 & \dots & x_n \\ & & \end{array}\right]^{-1}$$

$$\begin{bmatrix} \\ \quad A \quad \\ \\ \end{bmatrix} = \left[\begin{array}{c|c|c} & & \\ {\color{red}c}x_1 & \dots & {\color{red}c}x_n \\ & & \end{array}\right] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \left[\begin{array}{c|c|c} & & \\ {\color{red}c}x_1 & \dots & {\color{red}c}x_n \\ & & \end{array}\right]^{-1}$$

Relationship between diagonalization and invertibilty

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consider the four matrices A, B, C, D:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

- A is invertible and diagonalizable
- B diagonalizable but not invertible
- C is invertible but not diagonalizable
- D is neither invertible nor diagonalizable

Symmetric eigendecomposition

let A be a symmetric real matrix, then A can be diagonalized by Q

$$Q^T A Q = \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}, \quad \text{with } Q^T Q = I$$

- the columns q_j of Q form an orthonormal set (Q is said tobe **orthogonal**)
- q_j are eigenvectors of A
- λ_i's are real

Computing eigenvalues

- computation via p_A is numerically ill-conditioned
- solving $Ax = \lambda x$ for x and λ is a non-linear problem
- we must resort to iterative algorithms
- **power iterations** are one such algorithm
- not a good algorithm in general, however the idea is a good one

Power iterations

problem data

$A \in \mathbb{C}^{n \times n}$ with linearly independent eigenvectors, eigenvalues are ordered such that

$$|\lambda_1| > |\lambda_2| \geq \ldots \geq |\lambda_n|$$

with corresponding eigenvectors $v_1, \ldots, v_n$

desired output

an approximation of the dominant eigenvector v_1, and if A is Hermitian, the dominant eigenvalue λ_1

Only the dominant eigenvector?

applications where only the dominant eigen-pair are required:

- Google's PageRank algorithm
- Stability analysis of a dynamical system

$$x_{t+1} = Ax_t, \quad t = 0, 1, 2, \ldots$$

- steady state distribution of a Markov chain
- Principal Component Analysis (PCA) – seek direction of maximum variance

Power iteration method

input: any $x^{(0)}$ such that $\|x^{(0)}\| = 1$
`for` $k = 1, 2, \ldots$

1. $w = Ax^{(k-1)}$
2. $x^{(k)} = \frac{w}{\|w\|}$
3. $\lambda^{(k)} = (x^{(k)})^T A x^{(k)}$ `//skip if` $A \neq A^*$

return $\lambda_1 = \lambda^{(k)}$, $v_1 = x^{(k)}$

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Notes

- a real implementation needs a stopping criteria
- this code will only converge under certain conditions
- convergence depends on the ratio $r = \frac{|\lambda_2|}{|\lambda_1|}$
- forms the basis of more sophisticated algorithms that are widely used
- matrix-vector multiply is most expensive operation $\Longrightarrow$ most efficient on sparse matrices

Idea behind the algorithm

1. let x be any n-vector, then we can express x as

$$x = c_1 v_1 + c_2 v_2 + \dots c_n v_n$$

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$$Ax \quad = \quad A\sum_{i=1}^{n} c_i v_i \quad = \quad \sum_{i=1}^{n} c_i A v_i \quad = \quad \sum_{i=1}^{n} c_i \lambda_i v_i$$

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3. apply A repeatedly c.f., p.119

$$A^k x = \sum_{i=1}^{n} c_i \lambda_i^k v_i$$

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$$A^k x = \sum_{i=1}^{n} c_i \lambda_i^k v_i$$

4. by assumption $|\lambda_1^k| > |\lambda_i^k|$, so for each element in the sequence

$$x, \; Ax, \; A^2x, \; A^3x, \; \ldots$$

   expect the $\lambda_1^k v_1$ term to dominate: $A^k x = c_1 \lambda_1^k v_1 +$ smaller terms

$$A^k x = \lambda_1^k \left(c_1 v_1 + \sum_{j=2}^{n} c_j \left(\frac{\lambda_j}{\lambda_1}\right)^k v_j \right)$$

Rayleigh quotient

line 3 of the power iteration algorithm had the term $\lambda^{(k)} = (x^{(k)})^T A x^{(k)}$

the **Rayleigh quotient** of a vector $x \in \mathbb{R}^n$ and Hermitian matrix $A \in \mathbb{C}^{n \times n}$ is the scalar

$$r(x, A) = \frac{x^T A x}{x^T x}$$

- if x is an eigenvector of A, then $r(x, A) = \lambda$
- $r(x, A) \in [\lambda_1, \lambda_n]$

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Interpretation

given x, what scalar, α, acts most like an eigenvalue for x in the sene of minimizing

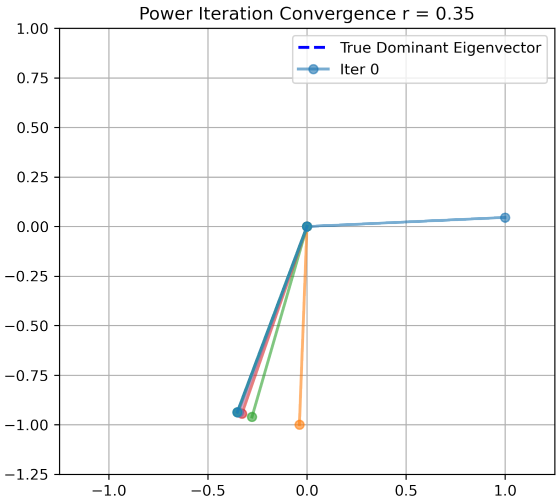
$$\|Ax - \alpha x\|_2?$$

examining the derivative of $r(x)$ w.r.t. x, we have

$$\nabla_x r(x) = \frac{2}{x^T x}(Ax - r(x, A)x)$$

Example: $r = 0.35$

$$\textbf{spec}(A) = \{0.35, 1\}$$



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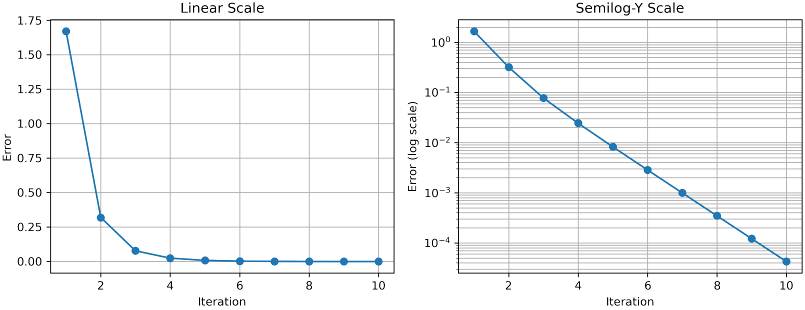

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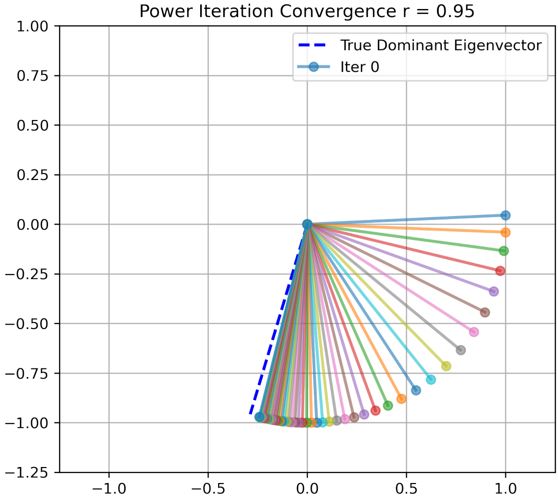

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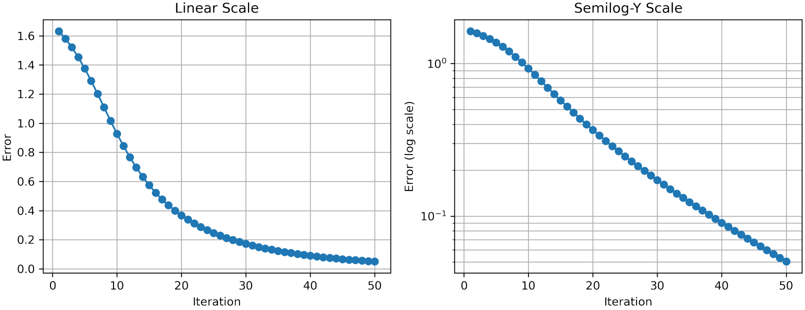

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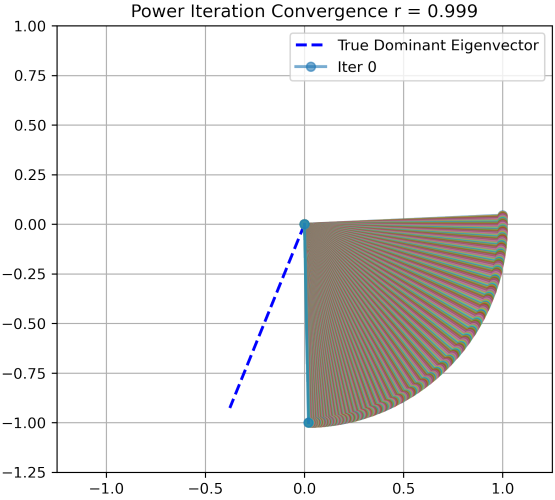

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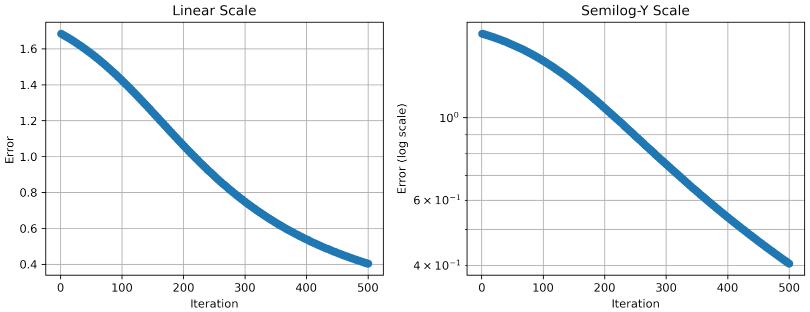

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