10: Dynamic Controllers: Basics

- minimal realizations (D&P 2.5)
- bounded-input bounded-output stability
- well-posed systems (D&P 5.1)

Realizations

consider $x(0) = 0$ and the sysem

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du,$$

the map from input u to output $y(t)$ is

$$y(t) = \int_0^t Ce^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau + Du(t)$$

- take the view point that the map $u \mapsto y(y)$ is the object of interest
- the state space system (A, B, C, D) is just a realization of this map
- the same $u \mapsto y(y)$ can be "realized" from many different choices of (A, B, C, D)

$$(A, B, C, D) \to (TAT^{-1}, TB, CT^{-1}, D)$$

Equivalent realizations

if for all u and $t \geq 0$

$$\int_0^t Ce^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau + Du(t) \quad = \quad \int_0^t C_1 e^{A_1(t-\tau)}B_1 u(\tau)\mathrm{d}\tau + D_1 u(t) \qquad (\dagger)$$

then the realizations (A, B, C, D) and (A_1, B_1, C_1, D_1) are said to be **equivalent**

- clearly it must hold that $D = D_1$ because we require $Du(0) = D_1 u(0)$
- note that $Ce^{A(t-\tau)}B$ hides the state dimension
- among all equivalent realizations, it is convenient to work with realization with smallest state dimension

Equivalence tests

Lemma

The realizations (A, B, C, D) *and* (A_1, B_1, C_1, D_1) *are equivalent, if and only if* $D = D_1$ *and*

$$Ce^{At}B = C_1 e^{A_1 t} B_1 \quad \textit{for all } t \geq 0.$$

Equivalence tests

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an alternative characterization:

Lemma

The realizations (A, B, C, D) and (A_1, B_1, C_1, D_1) are equivalent, if and only if $D = D_1$ and

$$CA^k B = C_1 A_1^k B_1 \quad \textit{for all } k \geq 0.$$

Lemma

The realizations (A, B, C, D) *and* (A_1, B_1, C_1, D_1) *are equivalent, if and only if* $D = D_1$ *and*

$$Ce^{At}B = C_1 e^{A_1 t} B_1 \quad \text{for all } t \geq 0.$$

Proof[scalar case]

- sufficiency is clear, focus on necessity
- instead of (†) consider

$$\int_0^t \left\{ Ce^{A(t-\tau)}B - C_1 e^{A_1(t-\tau)} B_1 \right\} u(\tau) \mathrm{d}\tau = 0,$$

  need to show that the term in blue is identically zero

- assume the contrapositive; for some $t_0 \geq 0$, no equality, define

$$u(t) := Ce^{A(t_0+1-t)}B - C_1 e^{A_1(t_0+1-t)} B_1 \quad \Longrightarrow \quad u(1) \neq 0$$

- thus at $t = t_0 + 1$, apply $u(t)$ above

$$\int_0^{t_0+1} \left\{ Ce^{A(t_0+1-\tau)}B - C_1 e^{A_1(t_0+1-\tau)} B_1 \right\} \mathrm{d}\tau = \int_0^{t_0+1} |u(\tau)|^2 \mathrm{d}\tau > 0.$$

Controllability and observability

Theorem

Suppose (A, B, C, D) *is a system realization. Then, if* (A, B) *is not controllable, or,* (A, C) *is not observable, then there exists a lower-order realization for the system.*

Controllability and observability

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Proof

- focus on controllability, analogous result for observability
- put system into controllable canonical form

$$TAT^{-1} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}, \quad TB = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix}, \quad \text{and} \quad CT^{-1} = [\tilde{C}_1 \ \ \tilde{C}_2],$$

then

$$Ce^{At}B = [\tilde{C}_1 \ \ \tilde{C}_2] \begin{bmatrix} e^{\tilde{A}_{11}t} & ? \\ 0 & ? \end{bmatrix} \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix} = \tilde{C}_1 e^{\tilde{A}_{11}t} \tilde{B}_t$$

- apply lemma with $A_1 = \tilde{A}_{11}, B_1 = \tilde{B}_1, C_1 = \tilde{C}_1$

Minimal realizations

Definition

A realization with the with the smallest possible state dimension is called a **minimal realization**.

Theorem

If (A, B) *is controllable* **and** (A, C) *is observable, then* (A, B, C, D) *is a minimal realization.*

L_∞-norm recap

recall the definition of the $L_\infty[0,\infty)$ norm of a signal u with $u(t) \in \mathbb{C}^m$:

$$\underbrace{\|u\|_{L_\infty[0,\infty)}}_{\text{signal norm}} := \sup_{t\in[0,\infty)} \|u(t)\|_\infty = \sup_{t\in[0,\infty)} \underbrace{\max_{1\le i\le m} |u_i(t)|}_{\|u(t)\|_\infty:\ \text{vector norm}}$$

- important to draw a distinction between vectors, $u(t)$ and signals u as we will often refer to $\|u\|_{L_\infty[0,\infty)}$ as $\|u\|_\infty$
- we only consider the interval $[0,\infty)$, so we omit from notation
- we could use any ℓ_p-norm, on $u(t)$ because of equivalence of norms:

$$\|x\|_\infty \ \le\ \|x\|_2 \ \le\ \|x\|_1 \ \le\ \sqrt{m}\|x\|_2 \ \le m\|x\|_\infty$$

finite in one norm implies finite in all others

BIBO stability

Definition

The system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0,$$
$$y(t) = Cx(t) + Du(t),$$

is said to be **bounded-input, bounded-output** stable if there exists a $\eta > 0$ such that

$$\|y\|_\infty \leq \eta \|u\|_\infty \quad \text{for all} \quad u, y.$$

Note: Some textbooks us the definition: $\|y\|_\infty < \infty$ when $\|u\|_\infty < \infty$

- for linear systems, the two are equivalent

BIBO stability theorem

a linear dynamical system is full characterized by its **impulse response** matrix H

$$y(t) = \int_0^\infty H(t-\tau)u(\tau)\mathrm{d}\tau.$$

the $(i, j)^{\text{th}}$ entry of H, $h_{ij}(\cdot)$ is the impulse response from the j^{th} input to the i^{th} output.

Theorem

A continuous-time LTI system with m inputs and p outputs, and impulse response matrix $H(t)$ is BIBO stable iff

$$\max_{1\le i\le p} \sum_{j=1}^{m} \int_0^\infty |h_{ij}(t)|\mathrm{d}t < \infty.$$

BIBO proof sketch

(sufficiency)
Let u be an input signal such that $\|u\|_\infty < \infty$ and we have that

$$y(t) = \int_0^\infty H(t-\tau)u(\tau)\mathrm{d}\tau.$$

then

$$\begin{aligned}\max_{1\le i\le p} |y_i(t)| &= \max_i \left| \int_0^\infty \sum_{j=1}^m h_{ij}(t-\tau)u_j(\tau)\mathrm{d}\tau \right| \\ &\le \left[\max_i \int_0^\infty \sum_{j=1}^m |h_{ij}(t-\tau)|\mathrm{d}\tau \right] \max_j \ \sup_t |u_j(t)|.\end{aligned}$$

Thus, taking supremum over t

$$\|y\|_\infty = \sup_t \max_i |y_i(t)| \le \left[\max_i \sum_{j=1}^m \int |h_{ij}(t)|\mathrm{d}t \right] \|u\|_\infty.$$

Notes

- for state-space systems (A, B, C, D) the **impulse response matrix** is

$$H(t) = Ce^{At}B + D\delta(t).$$

- the quantity $\eta = \max_{1 \leq i \leq p} \sum_{j=1}^{m} \int_0^\infty |h_{ij}(t)| \mathrm{d}t$ is the smallest η that satisfies

$$\|y\|_\infty \leq \eta \|u\|_\infty \quad \text{for all} \quad u, y.$$

Question

how does BIBO stability relate to internal stability?

Internal Stability Implies BIBO Stability

the impulse response of (A, B, C, D) is

$$H(t) = Ce^{At}B + D\delta(t)$$

taking a Laplace transform $\mathcal{L}\{H(t)\}$ gives

$$H(s) = C(sI - A)^{-1}B + D$$

the system is BIBO stable iff the poles of $H(s)$ are in the open left half plane

equivalently A is Hurwitz

Relationship Between BIBO and Internal Stability

Theorem

- *internal stability* $\Rightarrow$ *BIBO stability.*
- *If* (A, B, C, D) *is* **controllable and observable**, *then*

$$\text{internal stability} \iff \text{BIBO stability.}$$

we won't formally prove this, instead we focus on intuition

Illustrative Example

consider the system

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix}.\end{aligned}$$

eigenvalues $\lambda = \pm 1$ so **not internally stable**

the transfer function $H(s) = C(sI - A)^{-1}B$ is:

$$\begin{aligned}H(s) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s & -1 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \frac{s-1}{(s-1)(s+1)} \\ &= \frac{1}{s+1}\end{aligned}$$

unstable pole-zero cancellation, but **stable** transfer function

Change of coordinates...

use $T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and apply $(A, B, C, D) \mapsto (TAT^{-1}, TB, CT^{-1})$ to get

$$\begin{aligned} \dot{z}(t) &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} z(t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u(t) \\ w(t) &= \begin{bmatrix} 1 & -1 \end{bmatrix} z(t) \end{aligned}$$

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$$\dot{z}(t) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} z(t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u(t)$$
$$w(t) = \begin{bmatrix} 1 & -1 \end{bmatrix} z(t)$$

the state z_1 is **unstable** and **uncontrollable**!

Observer-based controllers

recall, that for the linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0 \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

the controller

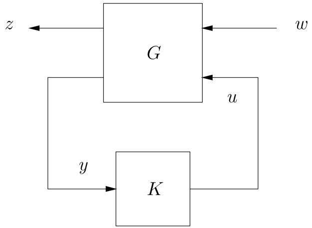
$$\begin{aligned} \dot{\hat{x}}(t) &= (A + LC + BF + LDF)\hat{x}(t) - Ly(t) \\ u(t) &= F\hat{x}(t) \end{aligned}$$

produces a closed-loop system with eigenvalues determined by $A + BF$ and $A + LC$

it was **assumed** that the interconnection between system and controller "makes sense"

a **well-posed** system is one where the plant-controller interconnection is **well defined**

General feedback arrangement



the plant G is described by the state-space system:

$$\dot{x}(t) = Ax(t) + \begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} w(t) \\ u(t) \end{bmatrix}$$

$$\begin{bmatrix} z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} x(t) + \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} w(t) \\ u(t) \end{bmatrix}$$

and controller K by

$$\begin{aligned} \dot{x}_K(t) &= A_K x_K(t) + B_K y(t) \\ u(t) &= C_K x_K(t) + D_K y(t) \end{aligned}$$

Well-posedness

it will be assumed that:

- (A, B, C) and (A_K, B_K, C_K) are **stabilizable**, *i.e.,*

$$A + BK \quad \text{and} \quad A_K + B_K F \quad \text{can be made stable}$$

- (A, B, C) and (A_K, B_K, C_K) are **detectable**, *i.e.,*

$$A + LC \quad \text{and} \quad A_K + HC_K \quad \text{can be made stable}$$

Definition

The feedback interconnection G, K is said to be **well-posed** if solutions exist for

$$x(t), \quad x_K(t), \quad y(t), \quad u(t),$$

for all initial conditions $x(0)$, $x_K(0)$ and inputs $w(t)$.

Note: all "physical" systems are well-posed

Matrix inversion lemma

let A be the square matrix $\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ with A_1, A_4 square

- assume A_1 nonsingular and define $\Delta := A_4 - A_3 A_1^{-1} A_2$. Then,

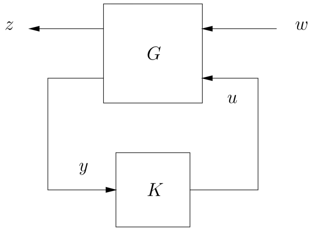
$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}^{-1} = \begin{bmatrix} A_1^{-1} + A_1^{-1} A_2 \Delta^{-1} A_3 A_1^{-1} & -A_1^{-1} A_2 \Delta^{-1} \\ -\Delta^{-1} A_3 A_1^{-1} & \Delta^{-1} \end{bmatrix}$$

- if A_4 is nonsingular, define $\hat{\Delta} := A_1 - A_2 A_4^{-1} A_3$. Then,

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}^{-1} = \begin{bmatrix} \hat{\Delta}^{-1} & -\hat{\Delta}^{-1} A_2 A_4^{-1} \\ -A_4^{-1} A_3 \hat{\Delta}^{-1} & A_4^{-1} + A_4^{-1} A_3 \hat{\Delta}^{-1} A_2 A_4^{-1} \end{bmatrix}$$

Theorem

The feedback interconnection



is well posed if $(I - D_{22}D_K)^{-1}$ *exists.*

Proof (sketch)

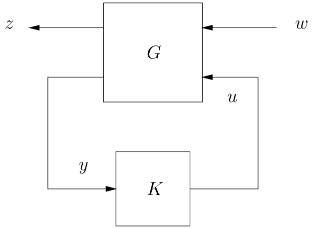
Express u and y as

$$\begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x_K(t) \end{bmatrix} + \begin{bmatrix} 0 \\ D_{21} \end{bmatrix} w(t)$$

The L.H.S. matrix is invertible iff $(I - D_{22}D_K)^{-1}$ is invertible (c.f., MIL)

Theorem

The feedback interconnection



is well posed if $(I - D_{22}D_K)^{-1}$ *exists.*

Note: if $D_{22} = 0$ or $D_K = 0$ then the closed-loop system is always well-posed

Closed-loop stability

Definition

The closed-loop system G, K is **internally stable** if it is well posed, and for all initial conditions $x(0)$ and $x_K(0)$, the limits

$$x(t), x_K(t) \to 0 \text{ as } t \to \infty \text{ hold}$$

when $w = 0$.

internal stability requires:

1. $(I - D_{22}D_K)^{-1}$ to be invertible
2. the eigenvalues of

$$\begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_2 \\ C_K & 0 \end{bmatrix}$$

   to have negative real parts

Assumptions

for the remainder of the course, unless otherwise stated, we will assume that

- all state space descriptions are minimal realizations
- all feedback connections are well-posed