E6602: Modern control
Logistics & Intro

- Basic course information
- Motivating example
- Modeling recap

Logistics

Instructor	James Anderson	`james.anderson@columbia.edu`
TAs	TBD	
Location	702 Hamilton Hall	
When	Class	Tue, Thurs 10:10–11:25am
	James' office hours	Tue 2:45–4:00pm (709 CEPSR)
	TA office hours	TBD
Courseworks	All course material (slides/homework/project material) and announcements come through Courseworks	
Homework	Gradescope/Courseworks	

Assessment

- Homework: 6(ish) homework assignments (40%)
- Midterm: Date tba, closed-book, in-person (20%)
- Individual project: Several options: case-study, advanced topic (35%)
- Participation: Attendance (5%)

Homework

- Homework is to be submitted by 6pm on the due date.
  - ≤ 24 hours late, max 75%
  - ≤ 48 hours late, max 50%
  - > 48 hours late won't be graded.
- You may collaborate on homework, the work you turn in **must be your own**
- Any sources beyond the lecture notes must be cited. Failure to do so constitutes plagiarism
- Project work must be typeset in LaTeX

ED Discussion

- Email is not an efficient means of communication
- Put `[E6602: xyz]` in the subject to ensure (quick?) response
- Ideally, post to **ED Discussion** message board (access through Courseworks)
- Even better, attempt to answer a question
- We will monitor responses, so don't worry about being wrong
- Discretionary bonus 5% available for active participation on ED Discussion

Setting the Scene

E6602

- **Electrical Engineering:** Part of the Signals, Information and Data sequence
- **Mechanical Engineering:** Part of the Robotics and Control MS Concentration

Prerequisites

- Necessary: Undergraduate signals and systems course, comfort with linear algebra and multi-variable calculus. Introductory control class.
- Optional: Any optimization course.

Related classes (this semester)

- E4650 Convex Optimization for Electrical Engineering (Mon 4:10–6:40)
- E6616 Convex Optimization (Fri 10:10–12:40)
- E4601 Digital Control Systems (Wed 4:10–6:40)
- E6907 Special Topics: Model Predictive Control (Tu 7–9:30pm)

Topics

- **Background**

  linear algebra, convex optimization, linear matrix inequalities (LMIs), linear time invariant dynamical systems

- **Linear system analysis**

  stability, controllability, observability, Leunberger observer, $\mathcal{H}_2$ and $\mathcal{H}_\infty$ system norms, systems analysis via LMIs

- **Optimal and robust control**

  $\mathcal{H}_2$ control, $\mathcal{H}_\infty$ state feedback control, $\mathcal{H}_\infty$ output feedback control, uncertain systems

- **Advanced topics (depending on time)**

  system identification, distributed control, model predictive control

Why should I take this class?

Academic curiosity

- Did you ever wonder why a system is controllable if

$$\operatorname{rank}\left([B,\ AB,\ A^2B\ \ldots\ A^{n-1}B]\right) = n?$$

- Or, can we give more than a yes/no answer to controllability?

Practicality

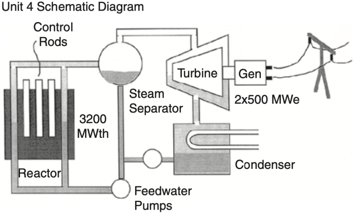
- With the exception of LQR control, most control techniques PID, pole-placement, etc., fail for systems with multiple inputs and outputs.
- The control methods you know about deal with robustness and uncertainty in a hand-wavey manner. We will fix this, everything will be made precise. This is because...

"We are permitted to do things with automatic controls that cannot be done manually and that, if done improperly, can have dire consequences for property, the environment, and human life. Most, but not all, of these dangerous applications involve open-loop unstable plants with divergence rates violent enough to elude manual control."

Gunter Stein, Bode Lecture, IEEE Conference on Decision and Control 1989

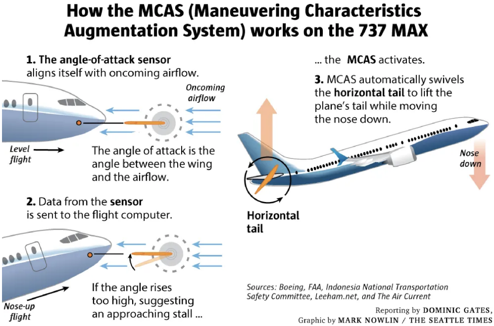
Open-loop (in)stability

Chernobyl No. 4 Reactor, 1986

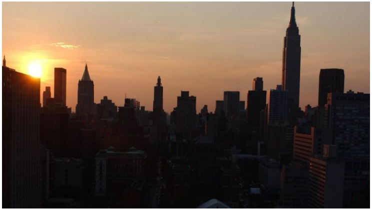


System built from unstable components are only locally stable with bounded inputs.

Boeing 737 MAX MCAS flight stabilizer



Complexity is increasing...



Northeast blackout 2013

- Estimated cost: **$4.2-8.2 billion** (Anderson Economic Group estimate)
  - $4.2 billion lost income to workers/investors
  - $100 million extra to governmental agencies
  - $1–2 billion affected utilities
  - $380–940 million lost/spoiled goods
- 55 million people affected
- Critical infrastructure: water supply, telecoms network, transportation, emergency response

Linear dynamical systems

Continuous-time, linear, time-invariant (LTI) dynamical system has the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t),$$

where

- $t \in \mathbb{R}$ denotes time
- $x(t) \in \mathbb{R}^n$ is the *state vector*
- $u(t) \in \mathbb{R}^m$ is the *input* or *control action*
- $y(t) \in \mathbb{R}^p$ is the output

Notation

- $x(t)$ is an n-dimensional vector
- $x(\cdot)$ is a signal
- x: ambiguous, will be clear from context
- Compactly, the system may be written as

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \quad \text{or} \quad (A, B, C, D)$$

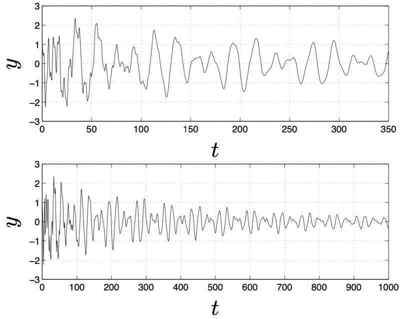
Toy example

- Consider a system with no inputs and a read out map

$$\dot{x} = Ax, \quad y = Cx$$

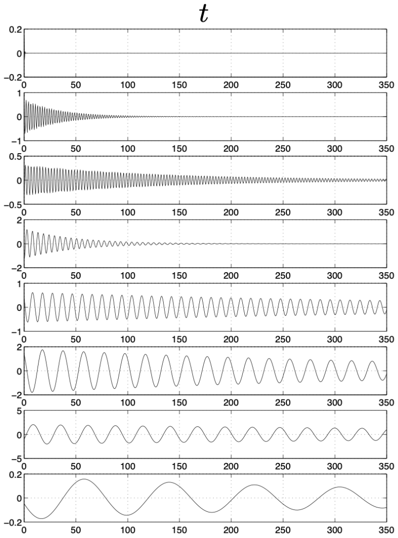
  with $n = 16$ and $p = 1$, *i.e.*, 16-states and a single output

- "model" of a lightly damped mechanical system - details not important



[Example and images from Lall & Boyd]

The output can be decomposed into modal components



Input design

add two inputs and two outputs:

$$\dot{x} = Ax + Bu, \quad y = Cy, \quad x(0) = x_0$$

withe $B \in \mathbb{R}^{16\times 2}$, $C \in \mathbb{R}^{2\times 16}$ (same A)

Objective:

- find $u : \mathbb{R}_+ \to \mathbb{R}^2$ so that $y(t) \to y_{\mathrm{des}}$
- let's select $y_{\mathrm{des}} = (1, -2)$

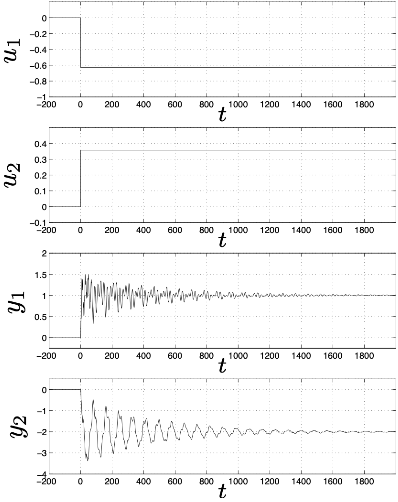
Steady state approach: consider u, x, y constant:

$$Ax + Bu_{\mathrm{static}} = 0, \quad y = y_{\mathrm{des}} = Cx$$

solve for u gives

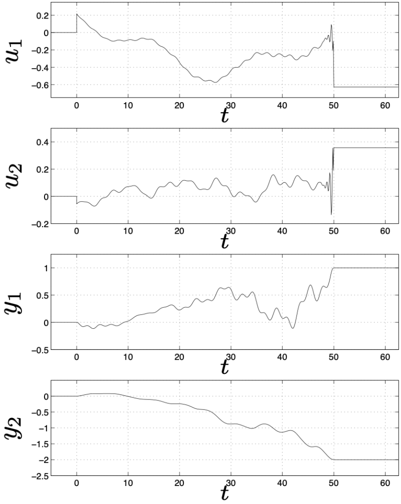
$$u_{\mathrm{static}} = (-CA^{-1}B)^{-1} y_{\mathrm{des}} = \begin{bmatrix} -0.63 \\ 0.36 \end{bmatrix}$$

applying u_{static}:

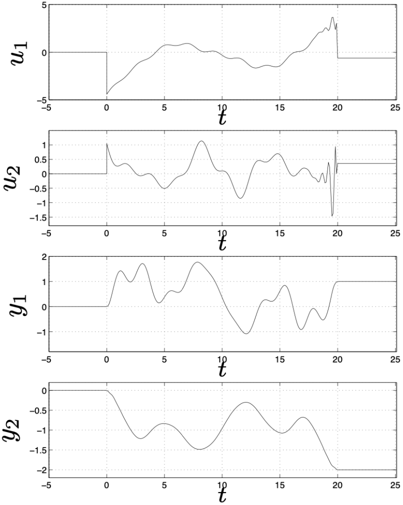


$y(t)$ converges exactly to y_{des} in ~ 1600s.

using techniques from this class we can do **much** better:



larger inputs can do even better:



we'll study

- How to design such inputs (control synthesis)
- The tradeoff between the input magnitude and rate of convergence

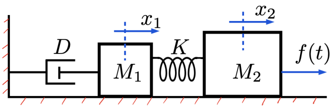
State Space Modeling

- Our starting point is a model of the form

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

- What this models and where the model comes from is important but not covered
  - from data (system identification)
  - from laws of physics

Modeling recap



Assume zero initial condition and the system is in steady state

Mass 1:

$$M_1 \frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} + D \frac{\mathrm{d}x_1}{\mathrm{d}t} + K(x_1 - x_2) = 0 \tag{1}$$

Mass 2:

$$M_2 \frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} + K(x_2 - x_1) = f(t) \tag{2}$$

Define the "state": let $v_i := \frac{\mathrm{d}x_i}{\mathrm{d}t}$ and set

$$x := \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \text{position 1} \\ \text{velocity 1} \\ \text{position 2} \\ \text{velocity 2} \end{bmatrix} \quad \Longrightarrow \quad \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \\ \dot{v}_2 \end{bmatrix}$$

Write $\dot{x}$ as a linear function of x and the input:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ & & & \\ 0 & 0 & 0 & 1 \\ & & & \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \\ 0 \\ \\ \end{bmatrix} u$$

Rearrange (1):

$$\dot{v}_1 = -\frac{D}{M_1} v_1 - \frac{K}{M_1} x_1 + \frac{K}{M_1} x_2$$

Rearrange (2):

$$\dot{v}_2 = \frac{K}{M_2} x_1 - \frac{K}{M_2} x_2 + \frac{1}{M_2} f(t)$$

Write $\dot{x}$ as a linear function of x and the input:

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \\ \dot{v}_2 \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -K/M_1 & -D/M1 & K/M_1 & 0 \\ 0 & 0 & 0 & 1 \\ K/M_2 & 0 & -K/M_2 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/M_2 \end{bmatrix}}_{B} f(t)$$

Rearrange (1):

$$\dot{v}_1 = -\frac{D}{M_1} v_1 - \frac{K}{M_1} x_1 + \frac{K}{M_1} x_2$$

Rearrange (2):

$$\dot{v}_2 = \frac{K}{M_2} x_1 - \frac{K}{M_2} x_2 + \frac{1}{M_2} f(t)$$