3: Linear dynamical systems

- state-space representations
- transfer matrices
- matrix exponential
- eigenvalues and vectors
- diagonalizable autonomous systems

State-space system representation

system of coupled linear differential equations and an algebraic output equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0, \\ y(t) &= Cx(t) + Du(t), \end{aligned}$$

where

- $x(t) \in \mathbb{R}^n$ is the *state*
- $u(t) \in \mathbb{R}^m$ is the *input* or *control action*
- $y(t) \in \mathbb{R}^p$ is the output

Notation

- $x(t)$ is an n-dimensional vector
- $x(\cdot)$ is a signal
- will use x as shorthand for both (context will be clear)
- Compactly, the system may be written as

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \quad \text{or} \quad (A, B, C, D)$$

Higher order systems

matrix notation packs quite a punch, consider the **autonomous** system

$$x^{(k)} = A_{k-1}x^{(k-1)} + \ldots + A_1 x^{(1)} + A_0 x, \quad x(t) \in \mathbb{R}^n$$

where $x^{(l)}$ denotes the l^{th} derivative w.r.t. t

define a new variable $z = \begin{bmatrix} x \\ x^{(1)} \\ \vdots \\ x^{(k-1)} \end{bmatrix} \in \mathbb{R}^{nk}$, then

$$\dot{z}(t) = \begin{bmatrix} x^{(1)} \\ \vdots \\ x^{(k)} \end{bmatrix} = \begin{bmatrix} 0 & I & 0 & \ldots & 0 \\ 0 & 0 & I & \ldots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \ldots & I \\ A_0 & A_1 & A_2 & \ldots & A_{k-1} \end{bmatrix} z(t)$$

which is just a first-order system with enlarged state: $\dot{z} = Az$

Mechanical systems

mechanical systems with k degrees of freedom (d.o.f.)

$$M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = u(t)$$

- $q(t) \in \mathbb{R}^k$ is the **configuration** or system **coordinates**
- $M, D, K,$ are the **mass, damping**, and **stiffness** matrices
- $y(t) \in \mathbb{R}^p$ is the output

State space form

define the state as $x(t) = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}$

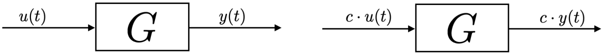
$$\dot{x}(t) = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} u(t)$$

select velocity to be the measured output

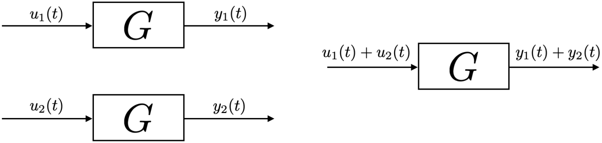
$$y(t) = \begin{bmatrix} 0 & I \end{bmatrix} x(t)$$

Linear time-invariant systems

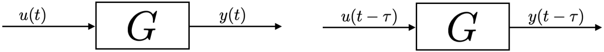
Homogeneity: scaling the input scales the output



Superposition: addition of inputs produces an addition in the output



Time-invariance: delaying the input delays the output



Operations on systems

suppose we have two systems $G_i = \left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right]$ for $i \in \{1, 2\}$

Parallel connection:

$$G_1 + G_2 = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right]$$

Series connection:

$$G_1 G_2 = \left[\begin{array}{cc|c} A_1 & B_1 C_2 & B_1 D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1 C_2 & D_1 D_2 \end{array}\right]$$

Transfer functions

let $f : \mathbb{R}_+ \to \mathbb{R}^{p\times q}$, recall the Laplace transform of a function f:

$$F(s) = \mathcal{L}\{f(t)\}, \quad \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\mathrm{d}t$$

with domain of convergence $\{s \mid \mathbf{real}(s) > \alpha\}$

- defines a linear mapping between time and frequency domain

$$\alpha f(t) + \beta g(t) \quad \longleftrightarrow \quad \alpha F(s) + \beta G(s)$$

- applied to vectors and matrices element-by-element
- convention is to use uppercase letters or "hats" to represent transformations
- inverse operation $\mathcal{L}^{-1}\{F(s)\}$

Derivative property

assuming $\dot{f}(t)$ has a Laplace transform, then it is given by $s\hat{f}(s) - f(0)$, i.e.,

$$\mathcal{L}\{\dot{f}(t)\} = s\hat{f}(s) - f(0)$$

to show this, apply definition

$$\begin{aligned}
\mathcal{L}\{\dot{f}(t)\} &= \int_0^\infty e^{-st}\dot{f}(t)\mathrm{d}t \\
&= e^{-st}f(t)\Big|_{t=0}^{t\to\infty} + s\int_0^\infty e^{-st}f(t)\mathrm{d}t \\
&= \lim_{t\to\infty} e^{-st}f(t) - e^{s\cdot 0}f(0) + s\int_0^\infty e^{-st}f(t)\mathrm{d}t
\end{aligned}$$

Application to linear systems

recall our linear system description:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad \text{with} \quad x(0) = 0, \\ y(t) &= Cx(t) + Bu(t), \end{aligned}$$

applying the Laplace transform, gives

$$\hat{y}(s) = \hat{G}\hat{u}(s), \quad \text{where} \quad \hat{G}(s) = C(sI - A)^{-1}B + D$$

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](s) = C(sI - A)^{-1}B + D$$

the function $\hat{G} : \mathbb{C} \to \mathbb{C}^{p \times m}$ defined as

$$\hat{G}(s) = C(sI - A)^{-1}B + D$$

is called a *transfer function* or *transfer matrix*

Rational functions

the scalar function $\hat{g} : \mathbb{C} \to \mathbb{C}$ is called *rational* if

$$\hat{g}(s) = \frac{b_m s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

- when a_i, b_i are real, $\hat{g}$ is **real-rational**
- $\hat{g}$ is **proper** if $n \geq m$, and **strictly proper** if $n > m$

Transfer matrices

- transfer matrix is rational if **all** of its entries are rational
- likewise, proper, if all entries are proper

Solution to a state-space equation

given the system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0,$$
$$y(t) = Cx(t) + Bu(t),$$

we want to know:

- what form does the state equation x take? ... and y
- long-term behavior of the system:

$$\lim_{t\to\infty} x(t), \quad \lim_{t\to\infty} y(t)?$$

- can this behavior be determined directly from A, B, C, D?
- can/how do we modify the system behavior?
- is it tractable to do so?

Autonomous systems

autonomous systems have no outside force acting on them, *i.e.*, $u(t) \equiv 0$

$$\dot{x}(t) = Ax(t), \quad \text{with initial condition } x(0) = x_0$$

Scalar differential equations

$$\dot{x}(t) = \alpha x(t), \quad x(0) = x_0, \quad \text{where } \alpha \in \mathbb{R}$$

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the solution is given by

$$x(t) = e^{\alpha t} x_0$$

- **solution** is the map $x_0 \to x(\cdot)$
- solve via Laplace **or** direct integration
- behavior of x depends on the sign of α

Beyond scalar systems

the matrix exponential is defined for a **square** matrix as

$$e^M = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \cdots \frac{M^k}{k!} + \cdots$$

- the series **always** converges
- similar behavior to the scalar exponential – some important differences
- $[e^M]_{i,j} \neq e^{m_{ij}}$
- MATLAB or Python use >> `expm(M)`
- e^M is always invertible

Computing e^M

computing the matrix exponential is via it's definition

$$e^M = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \cdots \frac{M^k}{k!} + \cdots$$

is rarely a good idea

- safest choice is to use `>> expm(M)`
- in certain cases, definition is ok:

$$M = \begin{bmatrix} 5 & -3 & 2 \\ 15 & -9 & 6 \\ 10 & -6 & 4 \end{bmatrix}, \quad M^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Longrightarrow \quad e^M = I + M$$

The matrix exponential: Properties

$$e^M = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \dots \frac{M^k}{k!} + \dots$$

1. $e^0 = I$
2. $e^{M^T} = \left(e^M\right)^T$
3. $\frac{\mathrm{d}}{\mathrm{d}t} e^{Mt} = M e^{Mt}$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} e^{Mt} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(I + Mt + \frac{(Mt)^2}{2!} + \frac{(Mt)^3}{3!} + \dots + \frac{(Mt)^k}{k!} + \cdots\right) \\ &= M\left(I + Mt + \frac{(Mt)^2}{2!} + \frac{(Mt)^3}{3!} + \dots + \frac{(Mt)^{k-1}}{(k-1)!} + \cdots\right)\end{aligned}$$

4. $e^{M+N} \neq e^M e^N$ unless M and N **commute**, *i.e.*, $MN = NM$

Autonomous system solution

Theorem

The autonomous system $\dot{x}(t) = Ax(t)$ *with initial condition* $x(0) = x_0$*, where* $A \in \mathbb{R}^{n \times n}$*, has the unique solution* $x(t) = e^{At}x_0$. □

Stability

- the system is said to be **stable** iff

$$\lim_{t \to \infty} e^{At}x_0 = 0$$

  for all $x_0 \in \mathbb{R}^n$

- equivalently, there exist constants $c_1, c_2 > 0$ such that

$$\|e^{At}x_0\| \leq c_1 e^{c_2 t}\|x_0\|$$

  for all $x_0 \in \mathbb{R}^n$

when considering the system $G = (A, B, C, D)$, if $\dot{x} = Ax$ is stable, then we say G is **internally stable**

A decoupled system in $\mathbb{R}^2$

the system

$$\begin{aligned} \dot{x}_1(t) &= -3x_1(t), \quad x(0) = x_0 \\ \dot{x}_2(t) &= 4x_2(t) \end{aligned}$$

is **decoupled** because the value of $x_1(t)$ does not affect $x_2(t)$, and vice-versa

in matrix notation this corresponds to the autonomous system with

$$A = \begin{bmatrix} -3 & 0 \\ 0 & 4 \end{bmatrix}$$

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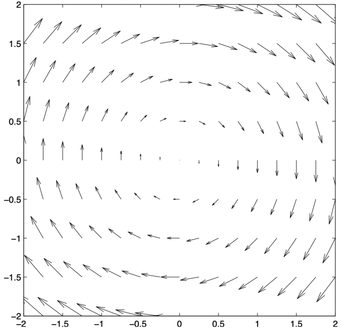
solution is given by the $x_1(t) = e^{-3t}x_1(0)$, $x_2(t) = e^{4t}x_2(0)$

- initial conditions of the form $x_0 = \begin{bmatrix} c \\ 0 \end{bmatrix}$ tend to 0 at $t \to \infty$
- system diverges for all other initial conditions

The harmonic oscillator

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t)$$



$$x(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} x(0)$$

solution via Laplace transform

1. $sI - A = \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}$ and so $(sI - A)^{-1} = \begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ \frac{-1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix}$ $\quad(\star)$

2. apply $\mathcal{L}^{-1}$ to each element:

$$\mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ \frac{-1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix}\right\} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

$(\star)$ recall the formula for 2×2 matrix inversion:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Eigenvalues and eigenvectors

for $A \in \mathbb{C}^{n \times n}$, $\lambda \in \mathbb{C}$ is an **eigenvalue** of A, denoted $\lambda \in \mathrm{eig}(A)$, if for some vector $v \in \mathbb{C}^n$ with $v \neq 0$ we have

$$Av = \lambda v$$

- eigenvectors are **not unique**, if v is an eigenvector, so is αv for any $\alpha \in \mathbb{C}$
- vectors which satisfy $w^T A = \lambda w^T$ are called **left eigenvectors** of A
- real matrices can have complex eigenvalues

Useful Eigenvalue/vector properties

- let $Ax = \lambda x$ be an eigen-pair, and k a positive integer, then $A^k x = \lambda^k x$

  show via induction:
  - (k=1): $A^1 x = Ax = \lambda x$
  - (inductive step) assume $A^k x = \lambda^k x$, show that it holds for $k+1$, i.e.,

    $$A^{k+1}x = \lambda^{k+1}x$$

    apply $A^{k+1} = AA^k$ and the assumption gives $A^{k+1}x = AA^k x$

    $$\begin{aligned} A^{k+1}Ax &= A(A^k x) \\ &= A(\lambda^k x) \\ &= \lambda^k (Ax) \\ &= \lambda^k \cdot \lambda x \\ &= \lambda^{k+1} x. \qquad \square \end{aligned}$$

Eigenvectors and dynamical systems

Given $Av = \lambda v$ and the system $\dot{x} = Ax$ with $x(0) = v$, then

$$x(t) = e^{\lambda t} v$$

this follows from the matrix exponential definition:

$$\begin{aligned} x(t) = e^{tA} v &= \left(I + tA + \frac{(tA)^2}{2!} + \dots \right) v \\ &= v + \lambda t v + \frac{(\lambda t)^2}{2!} v + \dots \qquad\qquad (\text{ from } A^k v = \lambda^k v) \\ &= e^{\lambda t} v \end{aligned}$$

- when λ is real, the motion is boring - stays on the line spanned by v
- the solution $x(t) = e^{\lambda t} v$ is called a **mode** of the system (associated to λ)

Complex eigenvalues

suppose $Av = \lambda v$ and $\lambda = \sigma + j\omega$ is complex ($\omega \neq 0$)

the trajectory $x(t) = ae^{\lambda t}v$ satisfies $\dot{x}(t) = Ax(t)$ for $a \in \mathbb{C}$

this implies that **real**$(ae^{\lambda t}v)$ is also a trajectory:

$$\begin{aligned} x(t) &= \textbf{real}(\alpha e^{\lambda t}v) \\ &= e^{\sigma t}\left[v_{\mathrm{re}}\ v_{\mathrm{im}}\right]\begin{bmatrix} \cos\omega t & \sin\omega t \\ \sin\omega t & \cos\omega t \end{bmatrix}\begin{bmatrix} \alpha \\ -\beta \end{bmatrix} \end{aligned}$$

where

$$v = v_{\mathrm{re}} + v_{\mathrm{im}}, \quad \text{and}, \quad a = \alpha + j\beta$$

Eigenspaces

for $A \in \mathbb{C}^{n\times n}$, $\lambda \in \mathbb{C}$ is an **eigenvalue** of A, denoted $\lambda \in \text{eig}(A)$, if for some vector $v \in \mathbb{C}^n$ with $v \neq 0$ we have

$$Av = \lambda v$$

- the **eigenspace** associated with the eigenvalue λ_k is the **subspace**

$$\mathcal{E}_k = \text{null}(\lambda_k I - A) \quad (\iff \quad \{v \mid \lambda_k v = Av\})$$

- suppose that a set of eigenvectors satisfies

$$\text{span}\{z_1, \dots, z_n\} = \mathbb{C}^n \quad \left(\iff \quad \bigcup_i \mathcal{E}_i = \mathbb{C}^n\right),$$

  then $V = \begin{bmatrix} z_1, \dots, z_n \end{bmatrix}$ is **invertible**

Diagonalization

let $X = \left[\begin{array}{ccc} x_1, \ldots, x_n \end{array}\right]$ be an invertible matrix built from the eigenvectors of A such that $\operatorname{span}(x_1, \ldots, x_n) = \mathbb{C}^n$

then $AX = X\Lambda$ with $\Lambda = \left[\begin{array}{ccc} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{array}\right]$

as X is invertible, we get $X^{-1}AX = \Lambda$

- we say X diagonalizes A
- in general, the transformation $W = P^{-1}ZP$ is called a **similarity transform**
- unfortunately, not all matrices can be diagonalized:

$$A = \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right], \quad \lambda_1 = \lambda_2 = 0, \quad x_1 = \left[\begin{array}{c} -1 \\ 0 \end{array}\right], x_2 = \left[\begin{array}{c} 1 \\ 0 \end{array}\right]$$

- having distinct eigenvalues is **sufficient**, but not necessary

Stability of diagonalizable systems

$\dot{x}(t) = Ax(t)$, with $x(0) = x_0$ has the unique solution $x(t) = e^{At}x_0$

suppose A is diagonalizable *i.e.*, there exists an X such that $A = X\Lambda X^{-1}$, then

$$A^k = X\Lambda X^{-1}X\Lambda X^{-1}\ldots X\Lambda X^{-1} = X\Lambda^k X^{-1}$$

together with the definition of the matrix exponential

$$\begin{aligned}
e^{At} &= \left(XX^{-1} + (X\Lambda X^{-1})t + \frac{(X\Lambda^2 X^{-1})t^2}{2!} + \cdots\right) \\
&= X\left(I + \Lambda t + \frac{(\Lambda t)^2}{2} + \cdots\right)X^{-1} \\
&= Xe^{\Lambda t}X^{-1},
\end{aligned}$$

the matrix $e^{\Lambda t}$ is a diagonal matrix with entries $e^{\lambda_i t}$

Summary

specifically, $e^{\Lambda t}$ has the form

$$\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}$$

- by definition, $Xe^{\Lambda t}X^{-1}$ and e^{At} have the same eigenvalues
- multiplication on the left and right does not change the sign of λ
- the solution $x(t)$ is a linear combination of the functions $e^{\lambda_i t}$
- we conclude that the system is stable iff

$$\lim_{t\to\infty} e^{\lambda_i t} \to 0 \quad \text{for all i}$$

which occurs iff $\mathbf{Re}\,\lambda_i < 0$ for all i

what about non-diagonalizable systems?

Modal form

- when $x(0) = v$ and $Av = \lambda v$ then

$$x(t) = e^{\lambda t} v \qquad (\dagger)$$

- when A has linearly independent eigenvectors then

$$x(t) = X e^{\Lambda t} X^{-1} x_0$$

- we can extend $(\dagger)$ to arbitrary initial conditions with independent eigenvectors by expressing x_0 as a linear combination of the eigenvectors:

$$x_0 = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \quad \text{where } Av_i = \lambda_i v_i$$

  and c_i are some (scalars) to be determined, it then follows that

$$x(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t} v_i$$

Modal form and left eigenvectors

it is straight forward to show that

$$c_i = w_i^T x(0) \quad \text{where} \quad w_i^T A = \lambda_i w_i^T$$

giving

$$x(t) = \sum_{i=1}^{n} e^{\lambda_i t} (w_i^T x(0)) v_i.$$

Hint

- relationship between W^T and V
- $x(t) = e^{At} x_0 = V e^{\lambda t} V^{-1} x_0$