4: Autonomous systems

- diagoanlizable system stability (recap)
- Jordan canonical form for non-diagonalizable systems
- Lyapunov theory
- Lyapunov theory for linear systems

Autonomous systems

systems with no external inputs and no output map are said to be **autonomous**

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \quad \text{where } \mathbf{dim}(A) = n \times n$$

- the solution to the set of n-coupled linear differential equations is

$$x(t) = e^{tA} x_0$$

- an autonomous system is said to be stable iff

$$\lim_{t \to \infty} x(t) = 0 \quad \text{for all } x_0 \in \mathbb{R}^n$$

- if A has n linearly independent eigenvectors, then the solution takes the form

$$x(t) = V e^{t\Lambda} V^{-1} x_0$$

  and stability can be determined by the spectrum of A:

$$\mathbf{spec}(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} = \{\lambda \in \mathbb{C} \mid \lambda I - A \text{ is singular}\}$$

Stability of diagonalizable systems

consider the solution $x(t) = Ve^{t\Lambda}V^{-1}x_0$

- e^{tA} is a diagonal matrix, specifically

$$\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}$$

- which makes it clear that

$$\lim_{t\to\infty} x(t) = 0 \quad \text{for all } x_0 \in \mathbb{R}^n \quad \text{is equivalent to} \quad \lim_{t\to\infty} e^{\lambda_i t} \to 0 \quad \text{for all } i$$

- stability reduced to checking a the sign of n eigenvalues

not all matrices are diagonalizable

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Nilpotent matrices

non-diagonalizable matrices can be approximately diagonalized

recall the definition of **matrix commutation**: $MN = NM$

- for $\sigma \in \mathbb{C}$, we have that σI and N commute
- therefore, $e^{\sigma I + N} = e^{\sigma I} e^{N}$

an $n \times n$ matrix is said to be **nilpotent** of order k if $N^{k-1} \neq 0$ and $N^k = 0$.

$$N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad N^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad N^3 = 0_{3\times 3}$$

Algebraic multiplicity

the **characteristic polynomial** of $A \in \mathbb{C}^{n \times n}$, denoted p_A, is the degree n polynomial

$$p_A(z) = \det(zI - A)$$

the characteristic polynomial provides another characterizations of the spectrum of A

- λ is an eigenvalue of A iff $p_A(\lambda) = 0$

$$\begin{aligned} \lambda \text{ is an eigenvalue of } A \quad & \iff \quad \text{there is a non-zero } x \text{ s.t. } \lambda x - Ax = 0 \\ & \iff \quad \lambda I - A \text{ is singular} \\ & \iff \quad \det(\lambda I - A) = 0. \end{aligned}$$

- p_A can always be expressed as $p_A(z) = (z - \lambda_1)(z - \lambda_2) \ldots (z - \lambda_n)$
- the **algebraic multiplicity** of an eigenvalue is the number of times it appears in the factorized expression of p_A

Jordan blocks

given $A \in \mathbb{C}^{n\times n}$, let λ be an eigenvalue of A of multiplicity l,

the **Jordan block** associated to λ is the $l \times l$ matrix

$$J = \lambda I + N = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}, \quad N \in \mathbb{C}^{l\times l} \text{ nilpotent}$$

- J has one eigenvalue, λ, of multiplicity l
- J has only one linearly independent eigenvector

for any matrix $A \in \mathbb{C}^{n\times n}$, there exists an invertible matrix T and Jordan blocks J_i such that

$$TAT^{-1} = J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_p \end{bmatrix}$$

Solution via Jordan canonical form

using similar arguments to the diagonalizable setting

$$A^k = TJ^kT^{-1}, \text{ where } J = \text{blkdiag}(J_1, \ldots, J_p)$$

and so,

$$e^{At} = Te^{Jt}T^{-1} = T \begin{bmatrix} e^{(\lambda_1 I + N)t} & & \\ & \ddots & \\ & & e^{(\lambda_p I + N)t} \end{bmatrix} T^{-1}.$$

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each exponential block has the form

$$e^{J_i t} = \begin{bmatrix} e^{\lambda_i t} & & \\ & \ddots & \\ & & e^{\lambda_i t} \end{bmatrix} \left[I + Nt + \frac{(Nt)^2}{2!} + \cdots + \frac{(Nt)^{k-1}}{(k-1)!} \right]$$

Autonomous system stability

- the system $\dot{x}(t) = Ax(t)$ with $x(0) = x_0 \in \mathbb{R}^n$, is said to be stable iff $\lim_{t\to\infty} e^{At}x_0 = 0$ for all $x_0 \in \mathbb{R}^n$.
- all systems admit a Jordan decomposition
- the solutions consist of linear combinations of functions of the form $\frac{t^k}{k!}e^{\lambda t}$
- stability is thus determined by the eigenvalues of A

Theorem

An autonomous system is stable **if and only if** *all of the eigenvalues of A have strictly negative real part. That is*

$$\mathbf{Re}(\lambda) < 0 \text{ for all } \lambda \in \mathrm{eig}(A). \qquad \square$$

- matrices whose eigenvalues have real parts strictly less than zero are said to be **Hurwitz**.

Nonlinear autonomous systems

consider the *time-invariant* dynamical system

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0,$$

where

- $f : \mathbb{R}^n \to \mathbb{R}^n$ is the *vector field*
- $x^\star$ is an *equilibrium point* if $f(x^\star) = 0$

Stability

when dealing with nonlinear systems, we refer to the stability of an equilibrium point, not the system.

- the eq. point $x^\star$ is *globally asymptotically stable* (G.A.S.) if for every trajectory $x(t)$

$$x(t) \to x^\star \text{ as } t \to \infty$$

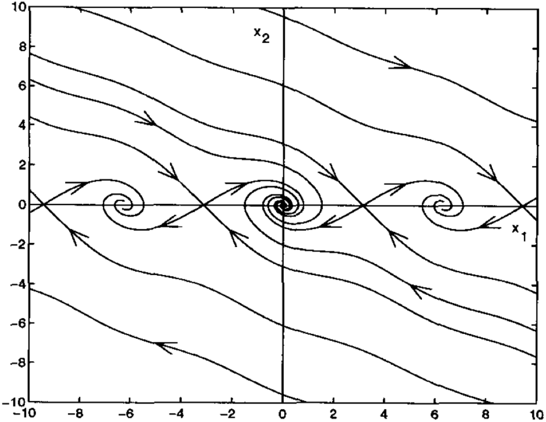
- the eq. point $x^\star$ is *locally asymptotically stable* (L.A.S.) if there exists an $R > 0$ such that

$$\|x(0) - x^\star\| \leq R \quad \Rightarrow \quad x(t) \to x^\star \text{ as } t \to \infty$$

- convenient to change coordinates so that $x^\star = 0$
- linear systems are a special case, $f(x) = Ax$
  - for linear systems G.A.S. $\iff$ L.A.S. $\iff$ stability
  - "stability" is determined by $\mathbf{Re}[\lambda_i(A)]$
- there are lots of other variations on stability
- for nonlinear f, establishing stability is a not straight forward
  - finding an eq. point can be a challenge
  - solving for $x(t)$ almost never possible
  - simulation is is inconclusive
  - most of the theory that exists is not constructive

Pendulum

$$ml\ddot{\theta} = -mg\sin\theta - kl\sin\dot{\theta} \quad \Longrightarrow \quad \begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 \end{array}$$



$$x^\star = \begin{bmatrix} n\pi \\ 0 \end{bmatrix} \quad \text{for } n = 0, \pm 1, \pm 2, \ldots$$

Positive definite functions

A function $V : \mathbb{R}^n \to \mathbb{R}$ is **positive definite** if

- $V(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$

Examples

- $V(x, y) = x^2 + y^2$
- if $z \in \mathbb{R}^n$ and P is symmetric matrix with positive eigenvalues, then

$$V(z) = z^T P z = \sum_{i=1}^{n} \sum_{j=1}^{n} P_{ij} z_i z_j > 0 \quad \text{for all } z \neq 0$$

Lyapunov theory

draw conclusions about the stability of $x^\star$ without having to calculate $x(t)$

Theorem

Consider the system $\dot{x}(t) = f(x(t))$ with $x(0) = x_0$. Without loss of generality, we assume $x^\star = 0$ and that $x^\star \in \mathcal{D} \subseteq \mathbb{R}^n$. If there exists a continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$ such that

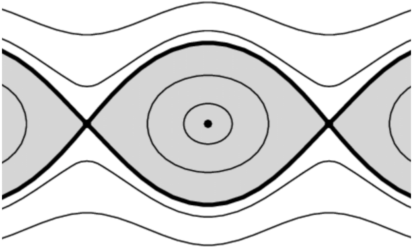
$$V(x) \text{ is positive definite on } \mathcal{D},$$
$$\dot{V}(x(t)) \text{ is negative definite on } \mathcal{D},$$

then $x^\star$ is locally asymptotically stable.

- V is called a **Lyapunov function**
- replace "definite" with "semidefinite" to obtain the weaker notion that $x^\star$ is locally stable

Example: Frictionless pendulum

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\tfrac{g}{l}\sin x_1\end{aligned}$$



candidate Lyapunov function: $V(x) = \left(\frac{g}{l}\right)(1-\cos x_1) + \frac{1}{2}x_2^2$, $\mathcal{D} = -2\pi < x_1 < 2\pi$

$$\begin{aligned}\dot{V}(x) &= \left(\frac{g}{l}\right)\dot{x}_1 \sin x_1 + x_2\dot{x}_2 \\ &= \left(\frac{g}{l}\right)x_2 \sin x_1 - x_2\left(\frac{g}{l}\right)\sin x_1 = 0\end{aligned}$$

Lyapunov theory

a few takeaways:

- Lyapunov methods provide a powerful tool for stability analysis, but...
- where does V come from?
  - for mechanical systems, total energy $=$ potential $+$ kinetic
  - everything else?!
- if the proposed Lyapunov function fails the two tests, the eq. point may still be stable
- testing positivity of a function is usually intractable

Lyapunov theory for linear systems

define the **Lyapunov equation** as

$$A^T P + PA + Q = 0$$

where $A \in \mathbb{R}^{n \times n}$ and $P, Q \in \mathbb{S}^n$.

- linear in the matrix variable P

associated to the Lyapunov equation are

- the linear dynamical system $\dot{x}(t) = Ax(t)$
- a **quadratic** Lyapunov function candidate $V(x) = x^T P x$

apply the chain rule to obtain the derivative:

$$\dot{V}(x(t)) = -x^T Q x$$

Solution to the Lyapunov equation

when A is **stable**, there is an integral solution to the Lyapunov equation

Theorem

Suppose A and Q are square matrices and that A is Hurwitz. Then

$$P = \int_0^\infty e^{tA^T} Q e^{tA} \mathrm{d}t$$

is the unique solution to $A^T P + PA + Q = 0$.

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Proof: Substitute in the expression for P to get

$$\begin{aligned} A^T P + PA &= \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t} \left\{ e^{tA^T} Q e^{tA} \right\} \mathrm{d}t \\ &= e^{tA^T} Q e^{tA} \Big|_0^\infty \\ &= -Q. \end{aligned}$$

Uniqueness follows from defining $\Pi(P) := A^T P + PA$ and showing that $\mathrm{null}(\Pi) = 0$ $\square$

Theorem

Suppose $Q \succ 0$. Then A is Hurwitz if and only if there exists a solution $P \succ 0$ to the Lyapunov equation $A^T P + PA + Q = 0$.

Proof:
(only if direction)

- we established the unique solution $P = \int_0^\infty e^{tA^T} Q e^{tA} \mathrm{d}t$
- when $Q \succ 0$ it is clear that $P \succ 0$ (Use $Q = Q^{\frac{1}{2}} Q^{\frac{1}{2}}$)

(converse)

- Suppose $P \succ 0$ solves the Lyapunov equation and that $Av = \lambda v$
- It then follows that $\lambda^* v^* P v + \lambda v^* P v + v^* Q v = 0$, and since $P \succ 0$,

$$2\textbf{real}(\lambda) = -\frac{v^* Q v}{v^* P v} < 0,$$

it follows that A is Hurwitz $\square$

Cost-to-go interpretation

assuming A is stable and P solves $A^TP + PA + Q = 0$, then

$$\begin{aligned} V(z) &= z^T P z \\ &= z^T \left(\int_0^\infty e^{tA^T} Q e^{tA} \mathrm{d}t \right) z \\ &= \int_0^\infty x(t)^T Q x(t) \mathrm{d}t \end{aligned}$$

where $\dot{x}(t) = Ax(t)$ and $x(0) = z$.

$V(z)$ is the cost-to-go from z when considering an integral quadratic cost function

Notes

- the proof was *if and only if*
  - linear systems are internally stable if and only if there is a quadratic Lyapunov function
  - the choice of Q is arbitrary
- an alternative proof based on a Lyapunov argument was hinted at earlier
- we will see later that in some cases $Q \succeq 0$ will suffice
- to solve a Lyapunov equation in Matlab, use `>>lyap(A',eye(n))`
  - note the transpose

Summary: stability tests for autonomous systems

Two tests for internal stability

1. check eigenvalues of A
2. solve the Lyapunov equation for P (A, Q given)
3. solve the linear matrix inequalities for P (next week)

Which method to use?

- each method is equivalent (in theory)
- computing eigenvalues is often ill-conditioned (numerically problematic)
- LMIs offer greater flexibility - focus of this course