8: Controllability: An operator perspective

- minimum norm control problem (D&P §4.3)
- $\mathcal{L}_2$ space (D&P §3.1–3.2 & 3.3.1)
- controllability operator (D&P §4.3)
- controllability ellipsoid (D&P §4.3)

Controllability

we have seen several tests to determine if the system $\dot{x} = Ax + Bu$ is controllable:

- rank test

$$\operatorname{rank} \mathcal{C}_{AB} = \operatorname{rank}(\begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}) \stackrel{?}{=} n$$

- if (A, B) stable, controllability Gramian test:

$$AX + XA^T + BB^T = 0, \qquad X \stackrel{?}{\succ} 0$$

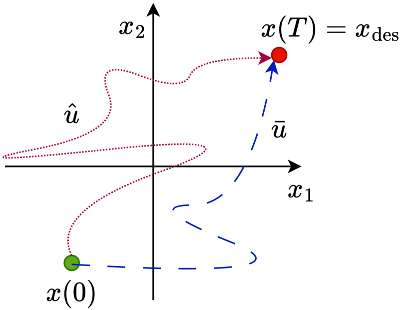
- for unstable (A, B), PBH test:

$$\textbf{rank}\,[A - \lambda I \quad B] \stackrel{?}{=} n \quad \text{for all } \lambda \in \bar{\mathbb{C}}^+$$

Note

controllability does not address **how** to get to a specific state

State transfer

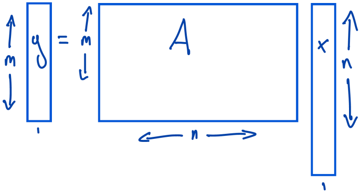


- time duration T
- desired state x_{des}
- how to select input u to get from $x(0)$ to $x(T) = x_{\mathrm{des}}$

Underdetermined systems of linear equations

consider the system of equations

$$y = Ax, \quad \text{where } A \in \mathbb{R}^{m \times n} \text{ with } m < n, \quad \operatorname{rank}(A) = m$$



- more unknowns (x) than equations (rows of A, b)
- x is *underspecified*; many (infinite) choices of x satisfy the equation
- A has a nontrivial nullspace
- all solutions are given by

$$\{x \mid Ax = y\} = \{x_p + z \mid z \in \operatorname{null}(A)\}$$

where x_p is any particular solution, i.e., $y = Ax_p$

Minimum norm solution

one choice of solution to $y = Ax$ is

$$x_{\mathrm{ln}} = A^T(AA^T)^{-1}y$$

it was shown that x_{ln} solves $y = Ax$ with the smallest $\|x\|_2$, i.e., it solves

$$\begin{array}{ll} \text{minimize} & \|x\|_2 \\ \text{subject to} & y = Ax \end{array}$$

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Minimum norm control

- **objective function:** choose a "minimal" input function u
- **constraints:** start at x_0, get to x_{des} under the system dynamics $\dot{x} = Ax + Bu$

Functions

recall our goal is to design a control action $u(t) = \phi(x(t))$ that changes the behavior of

$$\dot{x}(t) = Ax(t) + Bu(t)$$

we will restrict our attention to functions that belong to the space $L_2(\mathcal{I})$ where

$$\mathcal{I} := [a, b], \quad a \leq b$$

$L_2(\mathcal{I})$ is:

- the space of square integrable functions
- a vector space, and a Banach space, and a Hilbert space

The $L_2(\mathcal{I})$ function space

fix the interval $\mathcal{I}$ (for example $[0,1]$), then $L_2[0,1]$ is a **vector space** defined as

$$L_2([0,1]) := \{u \;:\; [0,1] \to \mathbb{C} \mid u \text{ is Lebesgue measurable and } \|u\|_{L_2[0,1]} < \infty\}$$

where

$$\|u\|_{L_2[0,1]} := \left(\int_0^1 \|u(t)\|_2^2 \, \mathrm{d}t \right)^{\frac{1}{2}}$$

associated to $L_2[0,1]$ is the inner product:

$$\langle x, y \rangle_{L_2[0,1]} := \int_0^1 x^*(t) y(t) \mathrm{d}t$$

- easy to show that $\langle \cdot, \cdot \rangle_{L_2[0,1]}$ satisfies the definition of an **inner product**
- the norm is **induced** by the inner product

$$\langle x, x \rangle_{L_2[0,1]} = \|x\|_{L_2[0,1]}^2$$

Examples

set $I = [0, \infty)$, do the following functions belong to $L_2[0, \infty)$?

- $f(t) = e^{\alpha t}$

- $f(t) = \cos t$

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Frequency domain:

Parseval's theorem tells us (for $\mathcal{I} = [0, \infty)$):

$$\|u\|^2_{L_2[0,\infty)} = \|\hat{u}\|^2_{\hat{L}_2(j\omega)} = \left(\frac{1}{2\pi} \int_0^\infty \hat{u}^*(j\omega)\hat{u}(j\omega)\ d\omega \right)$$

L_2 Summary

- $L_2(\mathcal{I})$ is a vector space
- the set of square integrable functions
- functions with bounded "energy"
- for specific choices of $\mathcal{I}$ very precise frequency domain characterizations exist
- as well as being a vector space, $L_2(\mathcal{I})$ is a normed vector space
- and an inner product space

Definition

A Hilbert space is a complete inner product space with the norm induced by its inner product.

- $\mathbb{R}^n, \mathbb{C}^n, \mathbb{C}^{m \times n}, L_2(\mathcal{I})$ are Hilbert spaces

$L_\infty(\mathcal{I})$ spaces

the L_∞ space of signals on an interval $\mathcal{I}$ is

$$L_\infty(\mathcal{I}) = \{u : \mathcal{I} \to \mathbb{C}^m \mid \|u\|_\infty < \infty\}$$

where

$$\|u\|_\infty = \operatorname*{ess\,sup}_{t\in\mathcal{I}} \|u(t)\|_\infty$$

Notes

- $\|u\|_\infty$ measures the peak of the signal
- in contrast $\|u\|_2$ captures the volume
- $\operatorname{ess\,sup}_{t\in\mathcal{I}} \|u(t)\|_\infty < 1 \iff \|u(t)\|_\infty < 1 \ \forall t$ except at a finite set of points
- not an inner product space, not a Hilbert space

Operators

a bounded linear map is called an **operator**

Linear maps

let V and Z be vector spaces, the map $F : V \to Z$ is a linear map if

$$F(\alpha v + \beta w) = \alpha F(v) + \beta F(w), \quad \text{for all } v, w \in V, \quad \alpha, \beta \in \mathbb{R}.$$

- common to write Fv instead of $F(v)$

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Bounded linear maps

if $F : V \to Z$ is a linear map, it is bounded if there exists a $c > 0$ such that

$$\|Fv\|_Z \leq c\|v\|_V, \quad \text{for all } v \in V.$$

- the space of bounded linear maps from V to Z is denoted $\mathcal{L}(V, Z)$

Adjoints

Definition

Suppose V and Z are Hilbert spaces, and $F \in \mathcal{L}(V, Z)$. The operator $F^* \in \mathcal{L}(Z, V)$ is called the adjoint of F if

$$\langle z, Fv \rangle_Z = \langle F^* z, v \rangle_V$$

for all $v \in V$ and $z \in Z$.

can be viewed as overloading of transpose to linear operators

- the adjoint of $A \in \mathbb{R}^{m \times n}$ is A^T
- extension: adjoint of a complex matrix is its conjugate transpose

useful property of adjoints:

$$\|F\| = \|F^*\| = \|F^* F\|^{\frac{1}{2}}$$

- more on operator norms later

Controllability Operator

consider the linear dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(-\infty) = 0$$

the **controllability operator** is a function

$$\Psi_c : L_2(-\infty, 0] \to \mathbb{C}^n$$

defined as

$$u \mapsto \int_{-\infty}^{0} e^{-A\tau} B u(\tau) \mathrm{d}\tau.$$

Minimum norm control

given the system $\dot{x}(t) = Ax(t) + Bu(t)$ with $x(-\infty) = 0$ construct a control law

$$u \in L_2(-\infty, 0]$$

such that $x(0) = x_{\mathrm{des}}$ such that $\|u\|_{L_2(-\infty,0]}$ is minimized

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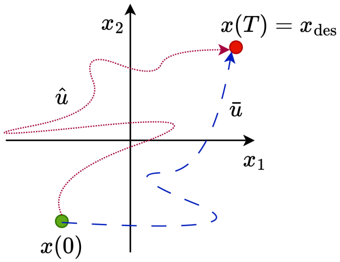

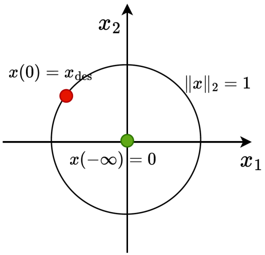

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we can express this using the controllability operator as

$$\begin{array}{ll} \text{minimize} & \|u\|_{L_2(-\infty,0]} \\ \text{subject to} & \Psi_c u = x_0 \end{array}$$

- decision variable: the **function** u
- constraint: u must move $x(-\infty) = 0$ to x_0, u belongs to $L_2(-\infty, 0]$
- w.l.o.g. $\|x_0\|_2 = 1$

A comparison

proof later, first let's use intuition

$$\begin{array}{ll} \text{minimize} & \|x\| \\ \text{subject to} & Ax = y \end{array} \qquad \begin{array}{ll} \text{minimize} & \|u\|_{L_2(-\infty,0]} \\ \text{subject to} & \Psi_c u = x_0 \end{array}$$

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Similarities

- objective functions are both norms induced by inner products
- constraints are linear
- both are **convex**

Differences

- the control problem is **infinite dimensional**
- we don't **yet** have a solution
- we do know $x^\star = x_{\text{ln}} = A^T(AA^T)^{-1}y$

The Adjoint of Ψ_c

to determine the optimal input u we need an expression for the adjoint of Ψ_c

$$\Psi_c : L_2(-\infty, 0] \to \mathbb{C}^n, \quad \text{therfore} \quad \Psi_c^* : \mathbb{C}^n \to L_2(-\infty, 0]$$

assume $\xi \in \mathbb{C}^n$ and $u \in L_2(-\infty, 0]$, apply definition of an adjoint:

$$\begin{aligned}
\langle \Psi_c^* \xi, u \rangle_{L_2(-\infty,0]} &= \langle \xi, \Psi_c u \rangle_{\mathbb{C}^n} \\
&= \xi^* \int_{-\infty}^{0} e^{-A\tau} B u(\tau) \mathrm{d}\tau \\
&= \int_{-\infty}^{0} \xi^* e^{-A\tau} B u(\tau) \mathrm{d}\tau \\
&= \left\langle B^* e^{-A^*\tau} \xi, u \right\rangle_{L_2(-\infty,0]}
\end{aligned}$$

and so we conclude that

$$\Psi_c^* : \xi \mapsto \begin{cases} B^* e^{-A^*\tau} \xi & \text{for } \tau \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

Controllability gramian

- the controllability operator, Ψ_c, is a map from $L_2(-\infty, 0] \to \mathbb{C}^n$ defined as

$$\int_{-\infty}^{0} e^{-A\tau} B u(\tau) \mathrm{d}\tau$$

- its adjoint operator Ψ_c^* is a map from $\mathbb{C}^n \to L_2(-\infty, 0$ given by

$$\begin{cases} B^* e^{-A^*\tau}\xi & \text{for } \tau \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

- it follows that $\Psi_c \Psi_c^*$ is the solution to the Lyapunov equation:

$$\Psi_c \Psi_c^* = \int_{-\infty}^{0} e^{-A\tau} B B^* e^{-A^*\tau} \mathrm{d}\tau = \int_{0}^{\infty} e^{A\tau} B B^* e^{A^*\tau} \mathrm{d}\tau = X_c$$

  where X_c solves $A X_c + X_c A^* + B B^* = 0$

- it follows that $(\Psi_c \Psi_c^*)^{-1}$ exists iff (A, B) is controllable

Solving the minimum norm control problem

Theorem

Suppose (A, B) is controllable. Then

1. $\Psi_c \Psi_c^*$ *is invertible. (Define* $X_c := \Psi_c \Psi_c^*$*)*
2. *For any point* $x_0 \in \mathbb{C}^n$*, the input* $u_{\text{opt}} = \Psi_c^* X_c^{-1} x_0$ *solves the minimum norm control problem.*
3. *The optimal input "energy" is*

$$\|u_{\text{opt}}\|_{L_2}^2 = x_0^T X_c^{-1} x_0.$$

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Comparison to minimum norm problem

minimize	$\|x\|$	minimize	$\|u\|_{L_2(-\infty,0]}$
subject to	$Ax = y$	subject to	$\Psi_c u = x_0$
	$x_{\mathrm{ln}} = A^T (AA^T)^{-1} y$		$u_{\mathrm{opt}} = \Psi_c^* (\Psi_c \Psi_c^*)^{-1} x_0$

Solving the minimum norm control problem

Proof sketch.

1. already shown invertibility
2. verify that $\Psi_c u_{\mathrm{opt}} = x_0$
3. show that $\|u\| \geq \|u_{\mathrm{opt}}\|$
4. to show property 3, apply definitions

$\square$

Note

proof step 3 uses the fact that the operator

$$W := \Psi_c^* X_c^{-1} \Psi_c$$

on $L_2(-\infty, 0]$ defines an orthogonal projection

Singular Value Decomposition

given $A \in \mathbb{C}^{m \times n}$, define $p = \min\{m, n\}$, the SVD of A is given by

$$A = U\Sigma V^*$$

where

- $U \in \mathbb{C}^{m \times m}$ is unitary
- $V \in \mathbb{C}^{n \times n}$ is unitary
- $\Sigma \in \mathbb{R}^{m \times n}$ is diagonal

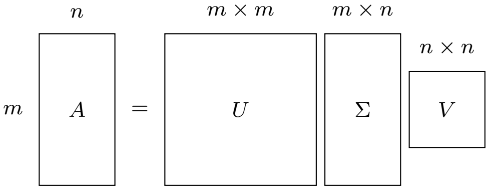
when $m \neq n$, the matrix Σ takes the form

$$\Sigma = \begin{bmatrix} \widehat{\Sigma} \\ \mathbf{0} \end{bmatrix} \quad \text{or} \quad \Sigma = \begin{bmatrix} \widetilde{\Sigma} & \mathbf{0} \end{bmatrix}$$

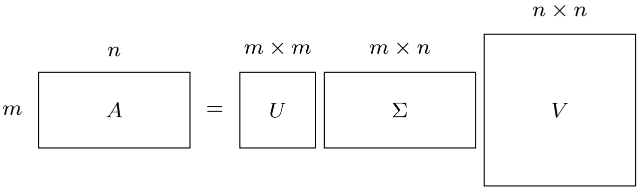
where $\widehat{\Sigma} = \mathbf{diag}(\sigma_1, \ldots, \sigma_n)$ and $\widetilde{\Sigma} = \mathbf{diag}(\sigma_1, \ldots, \sigma_m)$, and

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_p \geq 0$$

SVD for tall matrices



SVD for wide matrices



- Σ always has the same shape as A

Rank

given $A \in \mathbb{C}^{m \times n}$, the singular value decomposition is

$$A = U \Sigma V^* = \sum_{i=1}^{p} \sigma_i u_i v_i^*$$

where $U = \begin{bmatrix} u_1 & u_2 & \dots & u_m \end{bmatrix}$ and $V = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$

- σ_i is the i^{th} singular value
- u_i is the i^{th} left singular vector
- v_i is the i^{th} right singular vector

the rank of A is given by the number of non-zero singular values

Relationship to eigen-decomposition

given $A \in \mathbb{C}^{m \times n}$, the singular value decomposition is

$$A = U\Sigma V^* = \sum_{i=1}^{p} \sigma_i u_i v_i^*$$

then $A^*A = V\Sigma^* U^* U \Sigma V^* = V\Sigma^*\Sigma V^*$ and it follows that

- v_i are the eigenvectors of A^*A
- $\sigma_i = \sqrt{\lambda_i(A^*A)}$, i.e., the eigenvalues of A^*A

the same argument applied to AA^* tells us that

- u_i are the eigenvectors of AA^*
- $\sigma_i = \sqrt{\lambda_i(AA^*)}$

from this we now see that $\|A\|_2 = \sqrt{\lambda_{\text{max}}(A^*A)} = \sigma_1$

Further properties of the SVD

```
>> [U,S,V]=svd(A)
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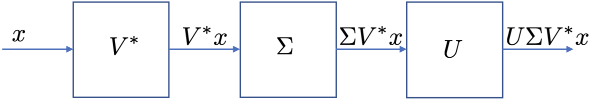
$$A = U\Sigma V^* = \sum_{i=1}^{p} \sigma_i u_i v_i^*$$

let $\mathrm{rank}(A) = r$, then

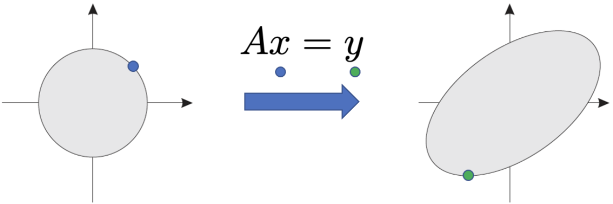
- $\{u_1, u_2, \ldots, u_r\}$ form an **orthonormal basis** for $\mathrm{range}(A)$.
- $\{v_1, v_2, \ldots, v_r\}$ form an **orthonormal basis** for $\mathrm{null}(A)^\perp$.

SVD as a linear map



rotate $\to$ scale $\to$ rotate

Geometric interpretation



$A \in \mathbb{R}^{m \times n}$ maps the unit sphere in $\mathbb{R}^n$ to an ellipsoid in $\mathbb{C}^m$

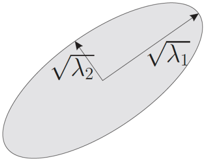
Ellipsoids

let W be a real $n \times n$ matrix and

- $W = W^T$ and $W \succ 0$ (positive definite)
- define $W = MM^T$ $(W \succ 0 \Rightarrow M \succ 0)$

W defines an ellipsoid:

$$\mathcal{E} = \left\{x \in \mathbb{R}^n \mid x^* W^{-1} x \leq 1\right\}$$



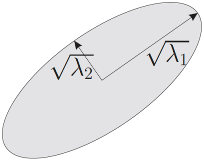
- semi-axis lengths determined by eigenvalues of W
- orientation determined by eigenvectors of W

Ellipsoid representations

let U be a Hilbert space and $M : U \to \mathbb{R}^n$ and $\mathrm{range}(M) = \mathbb{R}^n$

define $Z = MM^*$, then the following sets define the same ellipsoid:

1. $\mathcal{E}_1 = \left\{x \in \mathbb{R}^n \mid x^* Z^{-1} x \leq 1\right\}$
2. $\mathcal{E}_2 = \left\{Z^{\frac{1}{2}} y \mid y \in \mathbb{R}^n, \|y\|_2 \leq 1\right\}$
3. $\mathcal{E}_3 = \{Mu \mid u \in U, \|u\|_2 \leq 1\}$



$\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}_3$

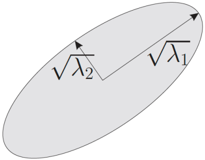
Controllability Ellipsoid

the set of states reachable with an input $u \in L_2(-\infty, 0]$ with $\|u\|_{L_2(-\infty,0]} \leq 1$ is

$$\mathcal{E}_c = \left\{ \Psi_c u \mid \|u\|_{L_2(-\infty,0]} \leq 1 \right\}$$

an alternative representation using matrices is given by

$$\mathcal{E}_c = \left\{ \xi \in \mathbb{R}^n \mid \xi^* X_c^{-1} \xi \leq 1 \right\}, \quad \text{where} \quad X_c = \Psi_c \Psi_c^*$$



where λ_i is an eigenvalue of X_c

the energy required to drive the state to $x(0) = x_\mathrm{d} \in \mathbb{R}^n$ is

$$\|u_\mathrm{opt}\|^2 = \langle \Psi_c^* X_c^{-1} x_\mathrm{d}, \Psi_c^* X_c^{-1} x_\mathrm{d} \rangle = x_\mathrm{d}^* X_c^{-1} x_\mathrm{d}$$

Computation

When A is stable, the **controllability gramian** $X_c \in \mathbb{R}^{n \times n}$ is the unique solution to

$$AX_c + X_c A^* + BB^* = 0.$$

Note: This is a Lyapunov equation.

Properties

- the matrix $X_c \succeq 0$ (because $X_c = \Psi_c \Psi_c^*$)
- if (A, B) is controllable, then $X_c \succ 0$

Controllability Summary

- if A is stable, **the controllability gramian**

$$X_c = \int_0^\infty e^{A\tau} BB^* e^{A^*\tau} \mathrm{d}\tau$$

  is real symmetric, and $X_c \succeq 0$

- $X_c = \Psi_c \Psi_c^*$
- X_c is the unique solution to $AX_c + X_c A^* + BB^* = 0$
- `>>Xc= lyap(A,B*B')`
- eigenvalues of X_c provide information how controllable the system is
- if any $\lambda_i = 0$, then (A, B) is not controllable
- Ψ_c^* and X_c combine to provide the optimal minimum norm control input

Interpretation via singular values

- eigenvalues of $\Psi_c \Psi_c^{-1}$ are the squares of the singular values of Ψ_c
- use the singular values Ψ_c instead of using $\mathrm{rank}(C_{AB})$ to determine $\mathrm{range}(\Psi_c)$