1 Review: Not to be turned in for credit (solutions will be posted soon) This is a review for you for using calculus, etc.

1. Given a constant $\lambda > 0$, let

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Compute the following:

(a) $\int_0^\infty f(u)du$ and $\int_0^\infty f(x)dx$
(b) $\int_0^\infty xf(x)dx$
(c) $\int_0^\infty x^2f(x)dx$
(d) Let $F(x) = \int_0^x f(u)du$. Graph the function $\overline{F}(x) \equiv 1 - F(x)$. Show that $\int_0^\infty xf(x)dx = \int_0^\infty \overline{F}(x)dx$.

2. Let

$$f(x) = \begin{cases} \frac{1}{4}, & \text{if } 2 < x < 4; \\ 0, & \text{otherwise.} \end{cases}$$

Graph $f(x)$ and then compute the following:

(a) $\int_0^3 f(x)dx$ and $\int_0^\infty f(x)dx$
(b) $\int_0^\infty xf(x)dx$
(c) $\int_0^\infty x^2f(x)dx$
(d) Let $F(x) = \int_0^x f(u)du$. Show that

$$F(x) = \begin{cases} 0, & \text{if } x \leq 2; \\ (x - 2)/2, & \text{if } 2 < x < 4; \\ 1, & \text{if } x \geq 4, \end{cases}$$

and that (recall that $\overline{F}(x) \equiv 1 - F(x)$)

$$\overline{F}(x) = \begin{cases} 1, & \text{if } x \leq 2; \\ (4 - x)/2, & \text{if } 2 < x < 4; \\ 0, & \text{if } x \geq 4, \end{cases}$$

Graph both $F(x)$ and $\overline{F}(x)$.

(e) Show that $\int_0^\infty xf(x)dx = \int_0^\infty \overline{F}(x)dx$.

3. Let

$$f(x, y) = \begin{cases} 2e^{-x}e^{-2y}, & \text{if } x \geq 0, \ y \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Compute $\int_0^1 \int_0^4 f(x, y)dxdy$ and $\int_0^\infty \int_0^\infty f(x, y)dxdy$. 
4. (a) Show that for any $x$,

$$
(1 + x + x^2 + \cdots + x^n) = \frac{1 - x^{n+1}}{1 - x}.
$$

(b) Use (a) to show that if $|x| < 1$, then

$$
\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x}.
$$

5. It is well known in mathematics that the sum of the integers up to $n$ (where $n$ is a positive integer) is given by the formula

$$
1 + 2 + \cdots + n = \frac{n(n + 1)}{2}.
$$

Can you (try to) prove this by induction on $n$?