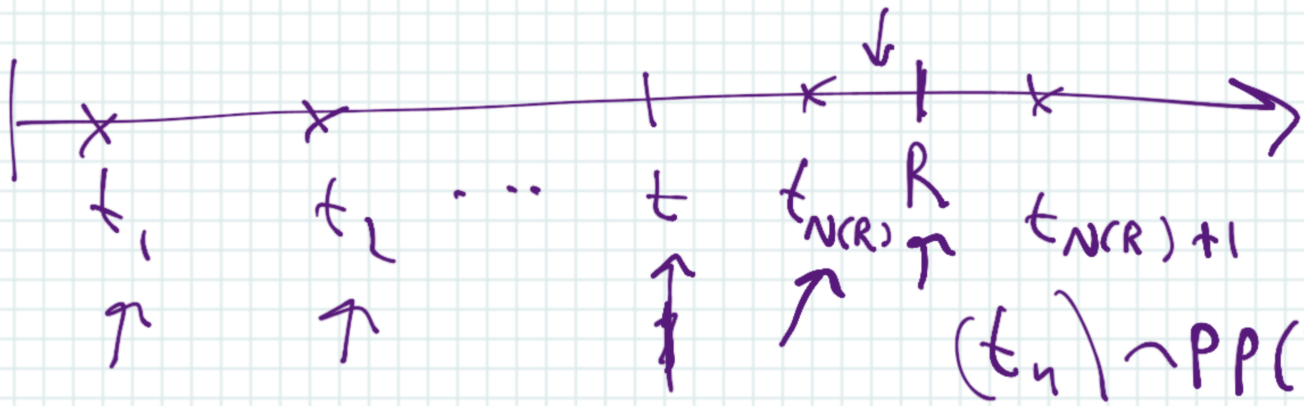
IEOR 4106 lec 13

Examples / Exercises

involving the Poisson Process

Optimization problem



$(t_n) \sim PP(\lambda)$

$\boxed{(R > \lambda^{-1})}$

Strategy to maximize the prob of choosing the last one before time R (e.g. choosing $t_{N(R)}$)

- Pick $0 < t < R$
-

Wait until time $t < R$

$\uparrow$

fixed

choose first arrival (if any) between (t, R)

"win" iff $N(R) - N(t) = 1$

Prob of winning $= p(t) = \mathbb{P}(N(R) - N(t) = 1)$

(Stationary increments) $= \mathbb{P}(N(R-t) = 1) = \boxed{e^{-\lambda(R-t)}\lambda(R-t)}$

$$P(t) = e^{-\lambda(R-t)} \lambda(R-t) \qquad 0 < t < R$$

Let's choose t to maximize $p(t)$

change variables $\boxed{x = \lambda(R-t)}$

$f(x) = x e^{-x}$

$f'(x) = 0$, $\quad e^{-x} - x e^{-x} = 0$

$e^{-x}(1-x) = 0$, $\quad x = 1$ $\Rightarrow$ $\boxed{t = R - \frac{1}{\lambda}}$

$(f'(x) > 0$

$f(x)$ is increasing on $(0, 1)$

decreasing on $(1, R\lambda)$

$(f'(x) < 0)$ So we did find the **max**

Subways : W 96 St. Station

(Downtown)

Express trains $\sim PP(3)$, $\lambda_E = 3$

independent

Local trains $\sim PP(6)$ $\lambda_L = 6$

a) Let $\{t_n\}$ $(N(t))$ be the Superposition of the 2 $(N_E(t))$ $(N_L(t))$

Given $\{N(t)=4\}$

$p = \mathbb{P}($3 are E and 1 is L$)$

$(t_4) \sim PP(\lambda)$ $\quad \lambda = \overset{3}{\lambda_E} + \overset{6}{\lambda_L}$

$= 9$

each arrival is, independently,

E, $wp = \dfrac{\lambda_E}{\lambda_E + \lambda_L} = \dfrac{3}{3+6} = \dfrac{1}{3}$

L $\quad wp = \dfrac{2}{3}$

(given that $N(t) = 4$
K = # out of 4 that are E
$\sim$ Binomial$(4, p)$ dist.
$p = \frac{1}{3}$)

$P = \binom{4}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^1$

$\mid N(t)=n$

In general, out of these n pts $t_1, \dots, t_n$ from $\{t_n\}$

$K = \#$ of the n that are $E \mid N(t)=n$

$$\boxed{\begin{gathered} K \sim \text{Binomial}\left(n, \tfrac{1}{3}\right) \\ \mathbb{P}(K=k) = \binom{n}{k}\left(\tfrac{1}{3}\right)^k \left(\tfrac{2}{3}\right)^{n-k} \\ 0 \le k \le n \end{gathered}}$$

c) $\underline{t = 10}$ (hrs), n fixed $n \geq 1$

$M =$ number out of the n ($\{t_n\}$) which arrived during the first 6 hrs

Compute $E(M)$ and the distribution of M

$$\boxed{\mathbb{P}(M = k) = \mathbb{P}(N(6) = k \mid N(10) = n)}$$

$(t_1, \dots, t_n) = (V_{(1)}, \dots V_{(n)})$

order stats

$$M = \sum_{i=1}^{n} I\{V_{(i)} \le 6\} = \sum_{i=1}^{n} I\{V_i \le 6\}$$

unordered

$V_{(i)}$ are the order stats of iid unif(0, 10)

$V_1, V_2, \dots V_n$

$E(M) = np = \frac{3}{5} n$

$$\boxed{\mathbb{P}(M = k) = \binom{n}{k} p^k (1-p)^{n-k} \quad 0 \le k \le n \quad p = \frac{3}{5}}$$

Binomial(n, p)

$p = \mathbb{P}(V \le 6) = \frac{6}{10} = \frac{3}{5}$

d) $\mathbb{P}$(exactly 4 trains arrive by time 2)

$$= \mathbb{P}(N(2)=4) = e^{-2\lambda} \frac{(2\lambda)^4}{4!}$$

$$\lambda = 3+6 = 9$$

$$= e^{-18} \frac{(18)^4}{4!}$$

$t=2$
$E(V)=1$

e) for $n \geq 1$,

compute $E\left(t_1+t_2+t_3+t_4 \,\middle|\, N(t)=4\right)$

$$= E\left(\sum_{i=1}^{4} V_{(i)}\right) = E\left(\sum_{i=1}^{4} V_i\right)$$

$$= 4E(V) = 4$$

(f) would the previous answer change ?
if $(N(t))$ is replaced by $(N_E(t))$?
$\lambda=9$ $\lambda=3$

No! $V_{(i)}$ are

order stats of

n iid unif$(0,t)$

For any $PP(\lambda)$

$| N(t) = n$

3) Cars are rented out

$\sim PP(\lambda)$ $\quad \lambda = 20/\text{day}$

Each car is rented (i.i.d.)

Unif(1, 7) days

$X(t) =$ # cars out rented at time t

$(X(0) = 0)$

a) Compute $E(X(t))$ $\quad t = 1, 2, 8$

and in the limit

$t \to \infty$

$E(X(\infty))$

M/G/∞ queue

G is $\boxed{\text{unif}(1, 7)}$

"service time" distribution

service times (S_n)

$$E(X(t)) = m(t) = \lambda \int_0^t \overline{G}(u)\,du = \lambda \int_0^t \mathbb{P}(S > u)\,du$$

tail

$$\mathbb{P}(S > t) = \begin{cases} 1, & 0 \le t < 1 \\ \frac{7-t}{6}, & 1 \le t < 7 \\ 0, & t > 7 \end{cases}$$

$(t = 1, 2, 8$

$$E(X(1)) = \alpha(1) = \lambda \int_0^1 \mathbb{P}(S > u)\,du$$

$$= \lambda \int_0^1 1\,du = \lambda = 20$$

$$E(X(2)) = \alpha(2) = \lambda \int_0^1 1\,du + \lambda \int_1^2 \left(\frac{7-u}{6}\right) du$$

$$= 20\left\{1 + \frac{11}{12}\right\} = \frac{115}{3}$$

$$E(X(8)) = \rho = \lambda E(S) = (20)(4) = 80$$

$8 \geq 7$ and for <u>any</u> $t \geq 7$

$$\alpha(t) = \lambda \int_0^t \mathbb{P}(S > u)\,du = \lambda \int_0^\infty \mathbb{P}(S > u)\,du = \lambda E(S) = \rho$$

$$E(X(\infty)) = \varrho$$

$$\lim_{t\to\infty} \alpha(t) = \varrho \qquad \left(\alpha(t) = \rho \quad t \geq 7 \right|$$

b) $\mathbb{P}(X(2) = 0)$

$$\mathbb{P}(X(2) = \; e^{-\alpha(2)} \frac{\alpha(2)^0}{0!} = e^{-\alpha(2)}$$

$$\left(\left(\alpha(2) = E(X(2)) = \frac{115}{3} \right) \right. \qquad = e^{-\frac{115}{3}}$$

(More generally

$$\mathbb{P}(X(2)=k) = e^{-\lambda(2)} \frac{\lambda(2)^k}{k!}, \quad k \geq 0$$
)

c) Suppose instead of $X(0)=0$,
we have $X(0)=1$, and this 1 car
was just rented at at $t=0$.
Re-answer a, b now.

Let $S \sim \text{unif}(1,7)$ denote the return
time of this 1 car

Let $Y(t) = \#$ additional cars out at t

$(Y(t))$ is the standard $M/G/\infty$, $Y(0) = 0$

$$X(t) = Y(t) + I\{S > t\}$$

$$\boxed{E(X(t)) = \lambda(t) + \mathbb{P}(S > t)}$$

$\mathbb{P}(S > t) = 1, \quad 0 < t < 1 \qquad = 0, \quad t \geq 7$

$= \frac{7-t}{6} \qquad 1 \leq t < 6,$

$t = 1$

$\lambda(1) + 1$

$= 21$

$t = 2$

$\lambda(2) + \frac{7-2}{6}$

$= \frac{235}{6}$

$$\mathbb{E}\, X(8) = \mathbb{E}\, Y(\infty) = \rho = 80$$

$$\left(\mathbb{P}(S > 8) = 0 \right)$$

$$\mathbb{P}(X(2) = 0)$$

$$= \mathbb{P}(Y(2) = 0, S \leq 2)$$

$$= \mathbb{P}(Y(2) = 0)\, \mathbb{P}(S \leq 2) = e^{-\alpha(2)} \left(\frac{2-1}{6} \right)$$

$$= e^{-115/6} / 6$$

"Proof of $M/G/\infty$"

Partition a Poisson rv $N(t)$ fix $t>0$

$$N(t) \xrightarrow{p(t)} X(t) = L(t)$$

$$N(t) \xrightarrow{1-p(t)} D(t) = \#\text{ departures by } t$$

$$X(t) \sim \text{Poisson}(\alpha(t))$$

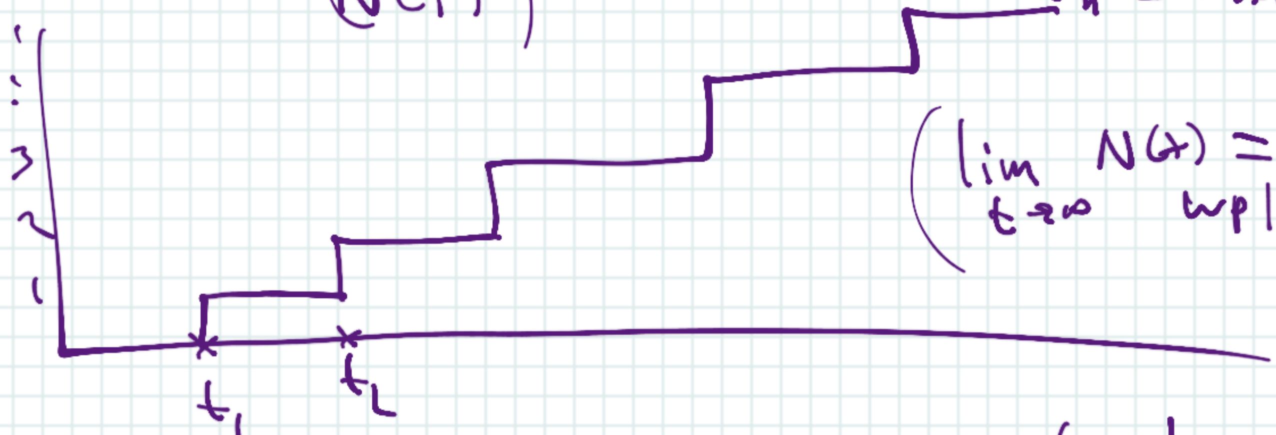
$$\alpha(t) = \lambda \int_0^t \bar{G}(u)\,du$$

What about as a stochastic process ?

$$\{X(t) : t \geq 0\} \stackrel{?}{=} \{L(t) : t \geq 0\}$$

a Counting process (Simple one $t_n < t_{n+1}$)

(N(t))



$$\lim_{t\to\infty} N(t) = \infty \quad \text{wp1}$$

L(t)

Can't be a counting process