IEOR 4106 lec 18

Renewal Reward Thm

more on the <u>inspection Paradox</u>

Renewal processes

Last lecture

$$A(t) = t_{N(t)+1} - t$$ forwards (excess) rec. time

$$B(t) = t - t_{N(t)}$$ backwards (age) rec. time

$$S(t) = A(t) + B(t)$$ spread

$$= t_{N(t)+1} - t_{N(t)}$$

$F(x) = \mathbb{P}(X \leq t)$

$\lambda \stackrel{def}{=} \frac{1}{E(X)}$

Stochastic processes in continuous-time, $t \geq 0$

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t A(s)\,ds = \frac{E(X^2)}{2E(X)} = \frac{\lambda E(X^2)}{2}$$

wp1

$$\frac{1}{t}\int_0^t A(s) \approx \frac{1}{t}\sum_{j=1}^{N(t)} \frac{X_j^2}{2}$$

$$\left(\frac{N(t)}{t}\right)\left(\frac{1}{N(t)}\sum_{j=1}^{N(t)} \frac{X_j^2}{2}\right)$$

(essence of the proof)

$$\frac{N(t)}{t} \xrightarrow[t\to\infty]{\text{ERT}} \lambda = \frac{1}{E(X)} \quad \text{wp1}$$

$$\frac{1}{N(t)}\sum_{j=1}^{N(t)} \frac{X_j^2}{2} \xrightarrow[t\to\infty]{\text{SLLN}} \frac{E(X^2)}{2} \quad \text{wp1}$$

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t B(s)\,ds = \frac{E(X^2)}{2E(X)} \quad \text{wp1} \quad \text{also,}$$

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t S(s)\,ds = \frac{E(X^2)}{E(X)} \geq E(X)$$

inspection paradox

"The average size of the interarrival time you land in over time is larger than the $E(X)$"

X denoting a copy of the original iid (X_n) interarrival times

$$F_e(x) = \lambda \int_0^x \bar{F}(y)\,dy$$ equilibrium dist. of F, $x \geq 0$

$$= \boxed{\lim_{t\to\infty} \frac{1}{t}\int_0^t I\{A(s) \leq x\}\,ds}$$ $x \geq 0$

$$F_e'(x) = \lambda \bar{F}(x) = f_e(x)$$ density function of the equilibrium dist.

Renewal Reward Theorem

Taxi drops off passengers at times $\{t_n : n \geq 1\}$ that form a renewal process

with interarrival times

$$X_n = t_n - t_{n-1}, \quad n \geq 1$$

$$0 < E(X) < \infty$$

$$\lambda = \frac{1}{E(X)}$$

R_n = \$ earned at time t_n (You pay driver at the end of your ride)

$$R(t) = \sum_{j=1}^{N(t)} R_j = \text{Total \$ earned by time } t$$

we assume that the vectors $\{(X_j, R_j) : j \geq 1\}$ are iid

$$\frac{R(t)}{t} \xrightarrow[t \to \infty]{} \text{long-run rate that the taxi driver earns \$}$$

$$\parallel$$

$$\frac{1}{t}\sum_{j=1}^{N(t)} R_j = \underbrace{\left(\frac{N(t)}{t}\right)}_{\substack{\text{ERT} \\ \downarrow \\ \lambda = \frac{1}{E(X)}}} \underbrace{\left(\frac{1}{N(t)}\right)\sum_{j=1}^{N(t)} R_j}_{\substack{\text{SLLN} \\ \swarrow \\ E(R)}} \longrightarrow \frac{E(R)}{E(X)}$$

"Expected value over a cycle, divided by an expected cycle length"

Renewal Reward Thm: Suppose R_j is earned during X_j (any which way)

$0 < E(X) < \infty$, $E|R| < \infty$

$R(t)$ = total \$ earned by time t

$$\lim_{t \to \infty} \frac{R(t)}{t} = \frac{E(R)}{E(X)} \quad \text{wp1}$$

Proof:

if the $R_j \geq 0$ then

$$\sum_{j=1}^{N(t)} R_j \leq R(t) \leq \sum_{j=1}^{N(t)+1} R_j$$

$$\boxed{R(t) \simeq \sum_{j=1}^{N(t)} R_j}$$

$$\frac{R(t)}{t} \to \frac{E(R)}{E(X)} \quad \text{wp1}$$

General case

$$\boxed{R_j = R_j^+ - R_j^-}$$

$R_j^+ = \max\{0, R_j\}$

$R_j^- = -\min\{0, R_j\}$

$E R_j^+ < \infty$

$E R_j^- < \infty$

if $E|R| < \infty$

Recall that ERT also has that $\dfrac{E(N(t))}{t} \xrightarrow[t\to\infty]{} \lambda$

$$\left(\frac{N(t)}{t} \xrightarrow[\text{wp1}]{} \lambda \right)$$

It
Also holds that

$$\frac{E(R(t))}{t} \xrightarrow[t \to \infty]{} \frac{E(R)}{E(X)}$$

(Proof is beyond the scope of this course)

Applications

$A(s) = X_1 - s, \quad 0 \le s < X_1$

$r(t) \stackrel{\text{def}}{=} A(t)$ = rate at time t at which \$ is earned

$$\int_0^t A(s)\,ds = \int_0^t r(s)\,ds$$

$$R = R_1 = \int_0^{t_1} r(s)\,ds = \int_0^{X_1} r(s)\,ds = \int_0^{X_1} (X_1 - s)\,ds = \int_0^{X_1} s\,ds = \frac{X_1^2}{2}$$

$$R_j = \frac{X_j^2}{2}$$

$$\boxed{\frac{E(R)}{E(X)} = \frac{E(X^2)}{2E(X)}}$$

$r(t) = B(t)$ $\qquad B(s) = s, \quad 0 \le s < X_1$

$$R_1 = \int_0^{X_1} s\,ds = \frac{X_1^2}{2}$$

Same $\lim_{t \to \infty} \frac{R(t)}{t} = \frac{E X^2}{2E(X)}$

$r(s) = S(s)$, $\qquad S(s) = X_1 \quad 0 \le s < X_1$

$$R_1 = \int_0^{X_1} X_1\,ds = X_1 \int_0^{X_1} ds = X_1 \cdot X_1 = X_1^2$$

$$\frac{E(R)}{E(X)} = \frac{E(X^2)}{E(X)}$$

Car Replacement problem with "T" policy

New cars cost $\$C_1$ and have (iid) lifetimes $\{V_j : j \geq 1\}$ with a cont. dist. $F(x) = \mathbb{P}(V \leq x)$

$f(t)$ density

Cost $\$C_2$ if car breaks down (dies)

For fixed time $T > 0$

we do the following to save $

Keep car until either it dies or to time T, which every happens first.

If car is still alive at T, give it away (free) and buy a new car. If car breaks down before T, pay $\$C_2$ and buy new car at $\$C_1$

$R(t)$ = total \$ spent by t

$$\lim_{t \to \infty} \frac{R(t)}{t} = ?$$

$$X_j = \min\{V_j, T\}$$

$t_j = X_1 + \cdots + X_j$

time at which a new car is bought,

$\{t_j\}$ is a renewal process

$$R_j = C_1 + C_2 I\{V_j \leq T\}$$

$$E(R) = C_1 + C_2 \mathbb{P}(V \leq T)$$

$$= \boxed{C_1 + C_2 F(T)}$$

$$E(X) = E(\underbrace{\min(V_j, T)}_{Y \geq 0}) = \int_0^\infty \mathbb{P}(Y > y)\,dy$$

$$= \boxed{\int_0^T \bar{F}(y)\,dy}$$

Also can be computed as

$$E(X) = \int_0^T x f(x)\,dx + T\,\bar{F}(T)$$

$$\frac{E(B)}{E(X)} = \frac{C_1 + C_2 F(T)}{\int_0^T \bar{F}(y)\,dy} = g(T)$$

try to find the value of T that minimizes this cost $g(T)$ (set $g'(T)=0$, solve) for example: suppose

$V \sim \text{unif}(0, 10)$

$F(T) = \mathbb{P}(V \leq T) = \frac{T}{10}, \quad 0 < T < 10$

$c_1 = 3, \; c_2 = \frac{1}{2}$

$$\frac{c_1 + c_2 \frac{T}{10}}{\int_0^T \left(1 - \frac{t}{10}\right) dt} = \frac{3 + \frac{T}{2}}{T - T^2/20} = g(T)$$

$g'(T) = 0 \iff T^2 + 120T - 1200 = 0$

$\boxed{T = 9.25}$

Train dispatching Problem

Passengers arrive to a train platform as a renewal process $\{S_n\}$

interarrival times of passengers $T_n = S_n - S_{n-1}$ $(S_0 \equiv 0)$

$T_1 = S_1$ $\left(\nu \stackrel{\text{def}}{=} \frac{1}{E(T)}\right)$ arrival rate of passengers

Train departs when the N^{th} passenger arrives

Train company incurs a "waiting cost" of

\$ nc /unit time that n passengers are waiting

+ \$ K for dispatching (cost)

iid cycle lengths distributed as

$$X = T_1 + \cdots + T_N = S_N \qquad E(T) = \frac{1}{\mu}$$

$$E(X) = N\,E(T) = \frac{N}{\mu} \qquad \left(\lambda = \frac{1}{E(X)} \right)$$

train dep. rate

$$R = (0)cT_1 + 1cT_2 + 2cT_3 + \cdots (N-1)cT_N \quad + K$$

$$E(R) = c\,E(T)\left(1+2+\cdots+N-1\right) \quad + K$$

$$= c\,\underbrace{E(T)}_{\frac{1}{\mu}}\;\frac{(N-1)(N)}{2} + K$$

$$\frac{R(t)}{t} \longrightarrow \frac{E(R)}{E(X)}$$

$$= g(N) = \frac{c(N-1)}{2} + \frac{KN}{N}$$

minimize cost:

$g'(N) = 0$

$$c/2 - \frac{KN}{N^2} = 0 \iff N = \sqrt{\frac{2KN}{c}}$$

$$g''(N) = \frac{2KN}{N^3} > 0$$, yes, a minimum.

If N is not an integer, choose the smallest one above, and largest one below and see which is better

$$\left(\begin{array}{l} K=6,\ n=1,\ c=2 \\ N=\sqrt{6} \approx 2.45 \\ 2,\ 3 \qquad g(2)=g(3) \end{array} \right)$$

Back to Inspection Paradox:

for a renewal process:

For each t

$$\mathbb{P}(S(t) > x) \geq \mathbb{P}(X > x) \quad \text{for all } x \geq 0$$

"Stochastically larger"

$$\Rightarrow E(S(t)) \geq E(X) \quad \text{for all } t$$