IEOR 4106 lec 2

More on the
Gambler's Ruin Problem;
hitting probabilities
for simple Random Walks

Recall :

$(q = 1-p)$

$$P_i(N) = \begin{cases} \dfrac{1-(q/p)^i}{1-(q/p)^N} & p \neq q \quad (p \neq \tfrac{1}{2}) \\ i/N & p = q = \tfrac{1}{2} \end{cases}$$

$= \mathbb{P}($ Gambler hits N before 0 $\mid$ Gambler starts with i $)$

$1 \leq i \leq N-1$

$R_0 = i$

$R_n = \underbrace{i + \Delta_1 + \cdots + \Delta_n}_{iid}$ = total earnings after the n^{th} gamble

$n \geq 1$

$\mathbb{P}(\Delta = 1) = p, \quad \mathbb{P}(\Delta = -1) = q = 1 - p$

$P_i(N) = \mathbb{P}(R_n \text{ hits } N \text{ before } 0 \mid R_0 = i)$

$(P_0(N) = 0, \ P_N(N) = 1) \qquad 1 \leq i \leq N-1$

$$P_i(\infty) = \begin{cases} 1 - \left(q/p\right)^i & p > \frac{1}{2} \\ 0 & p \le \frac{1}{2} \end{cases}$$

$$= \lim_{N \to \infty} P_i(N)$$

$$= \mathbb{P}\left(\begin{array}{l}\text{Gambler becomes} \\ \text{infinitely rich} \\ \text{without ever} \\ \text{going broke}\end{array} \middle| \begin{array}{l}\text{Gambler} \\ \text{starts} \\ \text{with } i\end{array}\right)$$

$$1 - P_i(\infty) = \begin{cases} (q/p)^i & p > \frac{1}{2} \\ 1 & p \leq \frac{1}{2} \end{cases}$$

$$= \mathbb{P}\left(\begin{array}{c} \text{Gambler eventually goes broke} \\ \text{if allowed to play forever} \end{array} \middle| \begin{array}{c} \text{Gambler starts} \\ \text{with } i \end{array} \right)$$

Application to an Insurance risk business

each day n \$1 is always earned

and each day a \$2 claim against comes in $wp = q = 1-p$

No claim $wp = p$

net \$ per day Δ :

$\mathbb{P}(\Delta = 1) = p$ (no claim, only the \$1)

$\mathbb{P}(\Delta = -1) = q = 1-p$ (claim $= -2$ + the \$1 $-2+1 = -1$)

$R_0 = i$

$i \geq 1$, initial reserve

$$\mathbb{P}(\text{company will go out of business}) = \begin{cases} (q/p)^i = 1 - P_i(\infty) & \text{if } p > \frac{1}{2} \\ 1 & \text{if } p \leq \frac{1}{2} \end{cases}$$

getting ruined

Examples John starts with \$2,
$p = .6$

$$\mathbb{P}\left(\text{John becomes } \infty \text{ rich without going broke} \,\middle|\, R_0 = \underset{=i}{2}\right)$$

$$= 1 - (q/p)^i = 1 - \left(\frac{.4}{.6}\right)^2$$

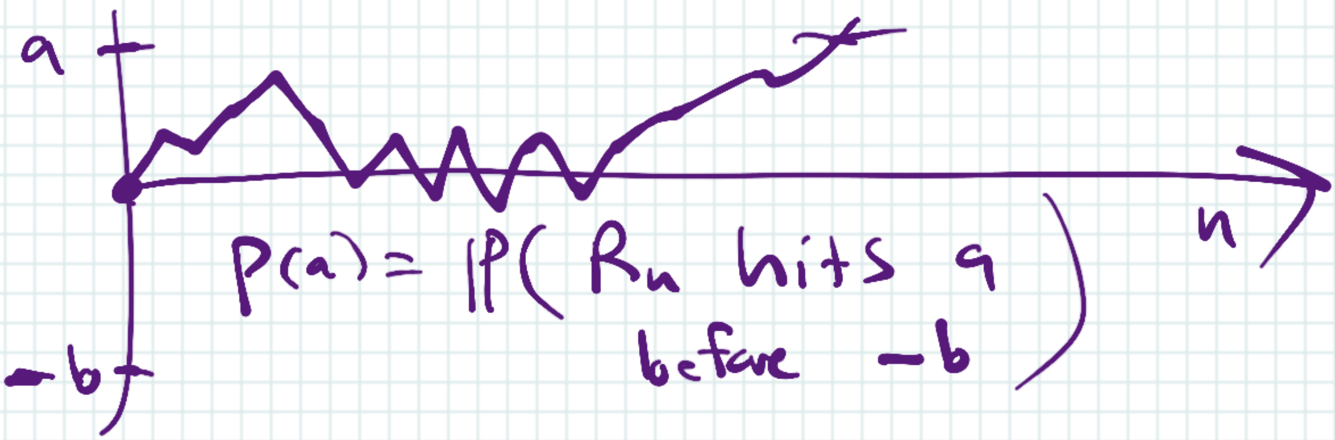
$$= 1 - \left(\frac{2}{3}\right)^2 = 5/9$$

$$\frac{4}{9} = \mathbb{P}(\text{John will go broke} \mid$$

hitting probabilities for simple random walks

$R_0 = 0$, $R_n = \sum_{i=1}^{n} \Delta_i$, $n \geq 1$,

$a > 0$, $b > 0$ integers



$p(a) = \mathbb{P}(R_n \text{ hits } a \text{ before } -b)$

Same as Gambler's Ruin Problem with $i = b$

$N = a + b$

$R_0 = b = i$

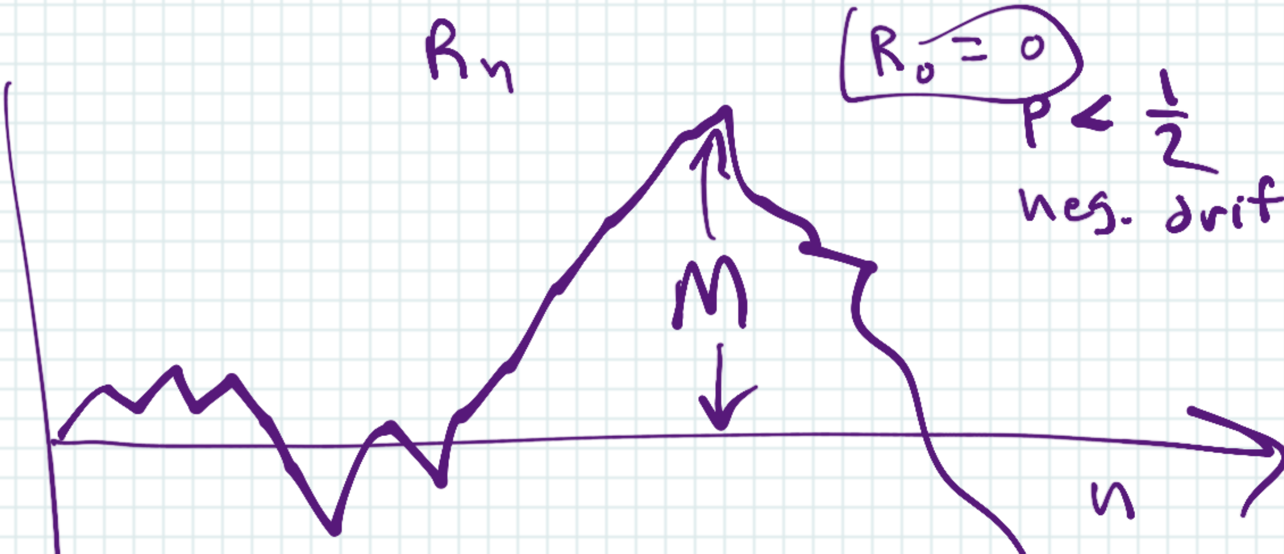
$N = a + b$

$$\boxed{P_i(N) = \frac{1 - \left(q/p\right)^{b}}{1 - \left(q/p\right)^{a+b}} \qquad p \neq \frac{1}{2}}$$

$\boxed{R_0 = 0}$

$p < \frac{1}{2}$

neg. drift



what is the probability dist.

of $M = \max_{n \geq 0} R_n$? $-\infty$

$\mathbb{P}(M \geq a) = ?$

$(p < \frac{1}{2})$

the event $\{M \geq a\}$

$= \{R_n$ ever hits a for some $n\}$

$\mathbb{P}(M \geq a) = \mathbb{P}(R_n$ ever hits a for some $n)$

$$= \lim_{b \to \infty} P(a) = \left(\frac{p}{q}\right)^a, \quad a \geq 0$$

(requires some algebraic manipulation to take the limit)

$\Rightarrow M$ has a geometric

distr. with mass at 0

$$\mathbb{P}(M=a) = \mathbb{P}(M \geq a) - \mathbb{P}(M \geq a+1)$$

$$= \boxed{\left(\frac{p}{2}\right)^a \left(1 - \frac{p}{2}\right) \quad a \geq 0}$$

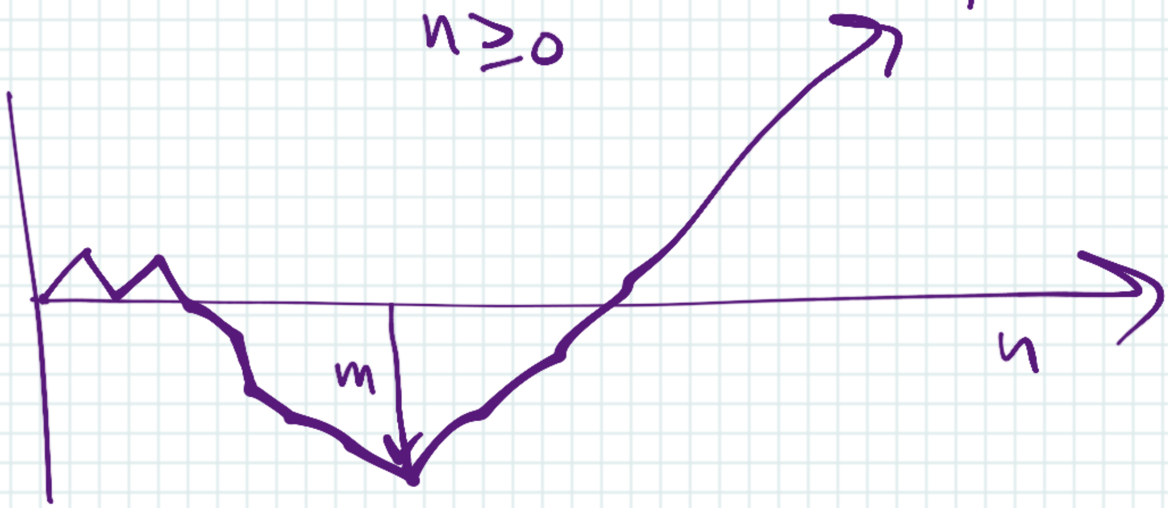
$$\mathbb{P}(M=0) = 1 - \frac{p}{2} > 0$$

If $p > \frac{1}{2}$

we can consider $\mathbb{P}(m \leq -b) = (q/p)^b$

$$m = \min_{n \geq 0} R_n$$

$+\infty$



$p = \frac{1}{2}$ (Symmetric case)

$$P_i(N) = \frac{i}{N} = \frac{b}{a+b} \xrightarrow[b \to \infty]{} 1$$

$\left(\begin{array}{l} i = b \\ N = a+b \end{array} \right)$

$$\mathbb{P}(M \geq a) = 1$$

for **all** $a \geq 0$

$= \mathbb{P}(R_n \text{ even hits } a)$

$= 1$ for <u>all</u> $a \geq 0$

$\left(\mathbb{P}(m \leq -b) = 1 \right.$ symmetry also $\left. \right)$

$$\mathbb{P}(M = \infty) = 1 = \mathbb{P}(m = -\infty)$$

when $p = \frac{1}{2}$

$\Rightarrow$ for a symmetric RW R_n hits every state $i \in \mathbb{Z}$ infinitely often

("recurrent case"

Let

$$T_{00} = \min\{n \geq 1 : R_n = 0 \mid R_0 = 0\}$$

return time back to state 0

We now know a return will always occur;

$$\mathbb{P}(T_{00} < \infty) = 1\ ;$$

but what about $E(T_{00})$? average # steps required

Recall that rvs can satisfy $\mathbb{P}(X < \infty) = 1$, but $E(X) = \infty$.

$$\boxed{E(X) = \sum n \, \mathbb{P}(X = n)}$$

if, for example,

$$\mathbb{P}(X = n) = \frac{c}{n^2} \quad \left(c = \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{-1} \right)$$

$$\mathbb{P}(X < \infty) = \sum_{n=1}^{\infty} \mathbb{P}(X = n) = 1$$

but

$$E(X) = \sum_{n=1}^{\infty} \frac{c}{n} = \infty$$

Computing Expected Values by integrating the tail, $\mathbb{P}(X > x)$, $x \geq 0$, for non-negative r.v.s

$$\boxed{E(X) = \int_0^\infty \mathbb{P}(X > x)\,dx}$$

$$= \int_0^\infty x f(x)\,dx$$

density function $f(x)$

CDF $F(x) = \mathbb{P}(X \le x)$

tail $\bar{F}(x) = 1 - F(x)$

$= \mathbb{P}(X > x)$

(density $f(x) = F'(x)$

if exists; "continuous case")

Proof:

$$\mathbb{P}(X>x) = E(I\{X>x\})$$

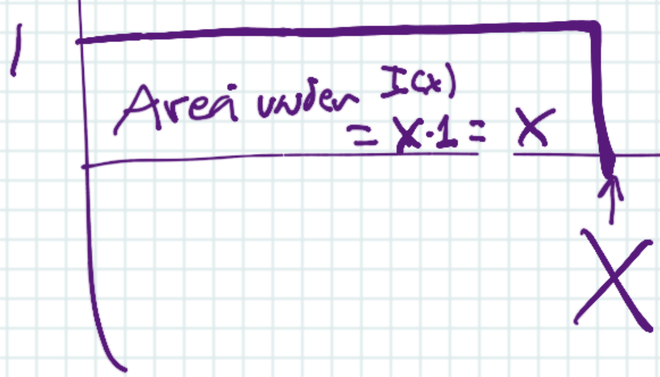
indicator of event $\{X>x\}$

$I(x)$

$$I(x) = \boxed{I\{X>x\}} \quad x \ge 0$$

$$\overset{\text{def}}{=} \begin{cases} 1 & \text{if } X>x \\ 0 & \text{if } X \le x \end{cases}$$

Graph $I(x)$, $x \ge 0$



$$E(I(x)) = \mathbb{P}(X>x)$$

$$\boxed{\int_0^\infty I(x)\,dx = X}$$

$$\Rightarrow E(X) = E\left(\int_0^\infty I(x)\,dx\right)$$

$$= \int_0^\infty E(I(x))\,dx$$

(allowed by "Tonelli's Thm" special case of Fubini's Thm)

$$= \int_0^\infty \mathbb{P}(X > x)\,dx$$

Example X is exponential with rate λ

$F(x) = \mathbb{P}(X \le x) = 1 - e^{-\lambda x}, \; x \geq 0$

$f(x) = F'(x) = \lambda e^{-\lambda x}$

$E(X) = \int_0^\infty x f(x)\,dx = \int_0^\infty \lambda x e^{-\lambda x}\,dx$

$\left(\begin{array}{c}\text{integration}\\ \text{by parts}\end{array}\right) = \frac{1}{\lambda}$

$\int_0^\infty \mathbb{P}(X > x)\,dx$

$= \int_0^\infty e^{-\lambda x}\,dx$

$= \frac{1}{\lambda}$

when X is a discrete rv

pmf

$$\left(\begin{array}{l} \mathbb{P}(X=k) = p(k) \\ k \geq 0 \\ \sum_{k=0}^{\infty} p(k) = 1 \end{array} \right)$$

then $\int_0^{\infty} \mathbb{P}(X>x)\,dx$

collapses into $\sum_{k=0}^{\infty} \mathbb{P}(X>k)$

Example: X is geometric with success prob. p

$$\mathbb{P}(X=k) = (1-p)^{k-1}p, \quad k \geq 1$$

$$E(X) = \sum k\,p(k) = \frac{1}{p}$$

$$\mathbb{P}(X>k) = (1-p)^k, \quad k \geq 0$$

$$E(X) = \sum_{k=0}^{\infty} (1-p)^k = \frac{1}{1-(1-p)} = \frac{1}{p} \checkmark$$

$$E(X^2) = \int_0^\infty 2x \, \mathbb{P}(X > x) \, dx$$

etc.

can be derived too

Markov Chains

a stochastic process $\{X_n : n \geq 0\}$ with discrete state space

$$\mathcal{S} \subseteq \mathbb{Z}$$

is called a Markov chain if

$$\mathbb{P}(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0)$$

$i, j \in \mathcal{S}$ $= \mathbb{P}(X_{n+1} = j \mid X_n = i) = P_{ij}$

The future $\{X_{n+1}, X_{n+2}, \cdots\}$
is independent of the
past $\{X_0, \cdots, X_{n-1}\}$
given the present X_n