IEOR 4106 lec 6

Communication classes;

More on recurrence, transience

$$S = \bigcup_{k=1}^{\infty} C_k \quad \text{(finite or countably infinite)}$$

disjoint communication classes

positive / null recurrence

Recall: $i, j \in \mathcal{S}$;

i and j communicate

$\boxed{i \longleftrightarrow j}$ If $P_{ij}^{(n)} > 0$, $P_{ji}^{(m)} > 0$

for some $n \geq 0$, $m \geq 0$

1) all states communicate with themselves $i \longleftrightarrow i$ $\left(P_{ii}^{(0)} = 1 \right)$

2) symmetry; if $i \longleftrightarrow j$, then $j \longleftrightarrow i$

3) Transitivity: if $i \longleftrightarrow k$ <u>and</u> $k \longleftrightarrow j$

then $i \longleftrightarrow j$ $\left(P_{ij}^{(n+m)} \geq P_{ik}^{(n)} \cdot P_{kj}^{(m)} > 0 \right)$

Generalization of "=" for $\mathbb{R}$

1) $x = x$, $x \in \mathbb{R}$

2) if $x = y$, then $y = x$

3) if $x = y$ and $y = z$, then $x = z$

$\longleftrightarrow$ "equivalence relation"

Communication classes

Every state space $\mathcal{S}$ of a MC

can be uniquely written as

$$\mathcal{S} = \bigcup_{k=1}^{\infty} C_k \quad \left(\begin{array}{c}\text{finite or}\\ \text{countably}\\ \text{infinite union}\end{array}\right)$$

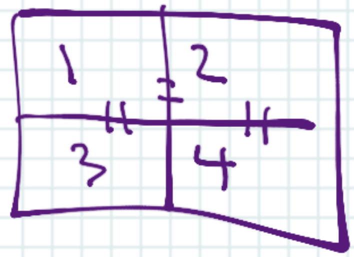
of sets C_k called communication classes

for which 1) they are disjoint subsets

2) all states within any class communicate, but don't with any states in another class

Examples

Rat in closed maze



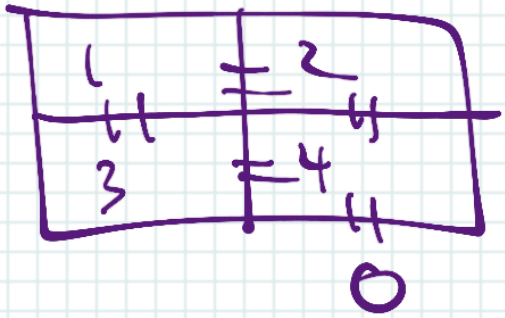
$\mathcal{S} = \{1, 2, 3, 4\}$

all states communicate

$$C = \mathcal{S}$$

an example of an "irreducible" MC

open maze



$\mathcal{S} = \{0, 1, 2, 3, 4\}$

NO state 1, 2, 3 or 4 is reachable from 0

So $C_1 = \{0\}$

$C_2 = \{1, 2, 3, 4\}$

$C_1 \cup C_2 = \mathcal{S}$

Gambler's Ruin MC

$$\mathcal{S} = \{0, 1, 2, \dots, N\}$$

$P_{00} = P_{NN} = 1$

$$C_1 = \{0\}$$

$$C_2 = \{N\}$$

$$C_3 = \{1, 2, \dots, N-1\}$$

$$C_1 \cup C_2 \cup C_3 = \mathcal{S}$$

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 2 & 1/3 & 1/6 & 1/6 & 1/3 \\ 3 & 0 & 0 & 0 & 1 \end{array}$$

$$\mathcal{S} = \{0, 1, 2, 3\}$$

$$C_1 = \{0, 1\}$$

$$C_2 = \{2\}$$

$$C_3 = \{3\}$$

$$C_1 \cup C_2 \cup C_3 = \mathcal{S}$$

Simple Random Walk

$0 < p < 1$

$$P_{i,i+1} = p$$

$$P_{i,i-1} = 1-p = q \quad , \quad i \in \mathbb{Z}$$

$$P^{(4)}_{2,6} = p^4 > 0$$

$$P^{(4)}_{6,2} = q^4 > 0 \; ; \quad P^{|i-j|}_{i,j} > 0 \quad n = |i-j|$$

$i \neq j$

irreducible MC
for <u>all</u> p.

$$C = \mathcal{S} = \mathbb{Z}$$

State $i \in \mathcal{S}$ is either $\boxed{\text{recurrent } (f_i = 1)}$ or $\boxed{\text{transient } (f_i < 1)}$

$$T_{i,i} = \begin{cases} \min\{n \geq 1 : X_n = i \mid X_0 = i\} \\ \infty \quad \text{if no return to } i \end{cases}$$

$$f_i = \mathbb{P}(T_{ii} < \infty)$$

N_i = Total number of visits to i $| X_0 = i$

$$\mathbb{P}(N_i = k) = f_i^{k-1}(1 - f_i), \quad k \geq 1$$

geometric distr.

$$N_i = \sum_{n=0}^{\infty} I\{X_n = i \mid X_0 = i\}$$

$$\mathbb{P}(N_i < \infty) = 1 \quad \text{if} \quad f_i < 1$$

$$\mathbb{P}(N_i = \infty) = 1 \quad \text{if} \quad f_i = 1$$

$$E(N_i) = \frac{1}{1-f_i} = \begin{cases} \infty & \text{if } f_i = 1 \\ < \infty & \text{if } f_i < 1 \end{cases}$$

If $i \longleftrightarrow j$ and if i is recurrent, then j is recurrent

$\left(P^{(n)}_{ij} > 0\right)$

If $i \longleftrightarrow j$ and if i is transient, then j is transient

$\Longrightarrow$ all states in a communication class either all are recurrent or all transient

Simple Random Walk $\quad 0 < p < 1$

irreducible

$$C = S$$

$\Rightarrow$ all states together are recurrent or all are transient

$p = \frac{1}{2}$ recurrent

$p \neq \frac{1}{2}$ transient

Rat in closed maze
(irreducible)

$C = \{1, 2, 3, 4\}$

Since $|\mathcal{S}| < \infty$,

must be recurrent

$$\boxed{\text{all irreducible finite state MCs are recurrent.}}$$

If j is recurrent

then $\left(\underline{\mathbb{P}(T_{jj} < \infty) = 1} \right)$

either

$\boxed{a) \ E(T_{jj}) < \infty \quad \text{positive recurrent}}$

b) $E(T_{jj}) = \infty$ Null recurrent

Consider a rv

$$\mathbb{P}(X = k) = \frac{c}{k^2}, \quad k \geq 1,$$

$\boxed{\mathbb{P}(X < \infty) = 1}$

$$c^{-1} = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty, \quad E(X) = \sum_{k=1}^{\infty} \frac{c}{k} = \infty$$

If $i \longleftrightarrow j$ then

if i is pos. rec, then

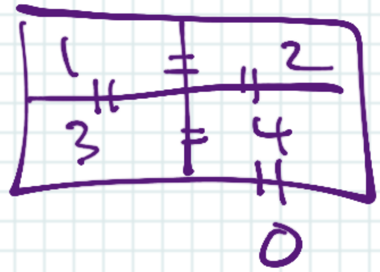
so is j

$\rightarrow$ all states in a communication class are either

1) pos. rec.
2) Null. rec. } both recurrent
3) transient

Examples

open

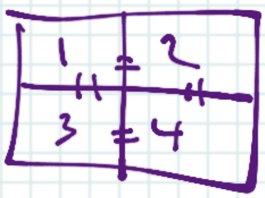


$C_1 = \{1, 2, 3, 4\}$ transient

$C_2 = \{0\}$ pos recurrent

$\tau_{0,0} \equiv 1$

closed



$C = \{1, 2, 3, 4\}$ recurrent (Positive)

Simple Symmetric RW $p = \frac{1}{2}$

recurrent (Null recurrent $E(T_{0,0}) = \infty$)

($T_{0,0}$ is an example of a rv X such that $\mathbb{P}(X < \infty) = 1$ but $E(X) = \infty$.)

① $\pi_j \stackrel{\text{def}}{=} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \boxed{I\{X_n = j \mid X_0 = i\}}$

wp1

long-run proportion of time the chain spends in state j

we want π_j to <u>not</u> depend on $X_0 = i \in \mathcal{S}$

constant

also want $\pi_j > 0$, $j \in \mathcal{S}$

and $\sum_{j\in\mathcal{S}} \pi_j = 1$ $\quad \pi = (\pi_j)_{j\in\mathcal{S}}$ is a Prob. dist.

if ① exists, we can take expected values of both sides to set $\left(E\left(I\{X_n = j \mid X_0 = i\}\right) = p_{ij}^{(n)} \right)$

For all $j \in \mathcal{S}$

$$\boxed{\pi_j = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} p_{ij}^{(n)}}$$

for all $i \in \mathcal{S}$

$\pi = (\pi_j)_{j\in\mathcal{S}}$ row vector

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} P^n = \begin{pmatrix} \pi \\ \pi \\ \vdots \\ \vdots \end{pmatrix}$$

If a Markov chain is irreducible and all states are pos. rec.

then ① exists for all j

and $\boxed{\pi_j = \dfrac{1}{\boxed{E(T_{jj})}}}$

$$\left(\begin{array}{l} \pi_j > 0 \\ \sum \pi_j = 1 \end{array}\right)$$

(π is called the limiting or stationary dist. of the MC)

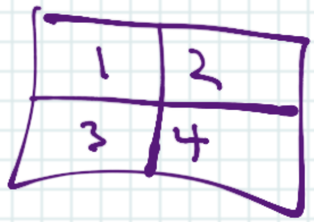
(pos. rec. : $1 \le E(T_{jj}) < \infty$)

$\pi = (\pi_j)_{j \in \mathcal{S}}$

if the chain (irreducible) and
is <u>Null rec</u>, or
<u>transient</u>, then

limits in (1) $= 0$;

<u>No</u> limiting prob. dist.
exists



Pos. rec.

$$\pi \stackrel{?}{=} \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$$

$$\left(\left(E(T_{jj}) = 4\right) \quad 1 \le j \le 4 \right)$$

for an irreducible MC

it is posi rec. iff

there exists a prob. solution $\left(\begin{array}{l}\pi_j > 0 \quad j \in \mathcal{S} \\ \sum \pi_j = 1\end{array}\right)$

to the set

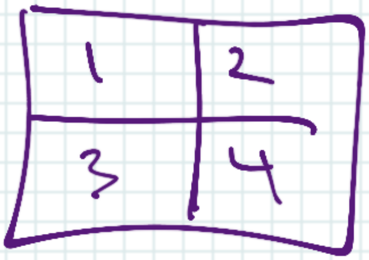
of linear equations $\Pi = \pi P$

Row vector

in which case

π <u>is</u> the limiting dist

$\left(\pi_j = \frac{1}{E(T_{ji})}\right)$ (unique)



$$P = \begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \end{array}$$

$$\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$$

$$\pi = \pi P \qquad (\pi_1 + \cdots + \pi_4 = 1)$$

$$\pi_1 = \frac{1}{2}\pi_2 + \frac{1}{2}\pi_3, \quad \pi_2 = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_4$$

$$\pi_3 = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_4, \quad \pi_4 = \frac{1}{2}\pi_2 + \frac{1}{2}\pi_3$$

$\boxed{\pi = (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4})}$ is the Solution