4106 Lec 8

Stopping times

Strong Markov Property

Wald's Equation

Stopping times τ

For a stochastic Process $\{X_n : n \geq 0\}$

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a <u>random time</u> τ is a random variable on $n \in \{0, 1, 2, \ldots\}$

and we want to consider observing X_τ

if $\tau = n$, then

$$X_\tau = X_n$$

X_τ denotes the state of the process at time τ

Definition:

a Stopping time for $\{X_n\}$ is a random time such that for each time $n \geq 0$ the event $\{T = n\}$ is determined by (at most) only $X_0, \ldots X_n$

If a random time T is <u>independent</u> of the stochastic process $\{X_n : n \geq 0\}$

Then it is a special case of a stopping time:

for each n, the event $\{T = n\}$ is <u>independent</u> of $\{X_n\}$, hence depends <u>not at all</u> on it

Examples of stopping times: first passage/hitting times

for $i \in \mathcal{S}$ fixed

$$\tau = \min\{n \geq 0 : X_n = i\}$$

(first time the process visits state i)

$$\{\tau = 0\} = \{X_0 = i\} \checkmark$$

$$\{\tau = 1\} = \{X_0 \neq i, X_1 = i\} \checkmark$$

$$\{\tau = n\} = \{X_0 \neq i, \dots, X_{n-1} \neq i, X_n = i\} \checkmark$$

Let $A \subset \mathcal{S}$

$$T = \min\{n \geq 0 : X_n \in A\}$$

for example gamblers ruin MC $A = \{0, N\}$

T = time at which the gambler stopped gambling.

$$\{\tau = 0\} = \{X_0 \in A\} \quad \checkmark$$

$$\vdots$$

$$\{\tau = n\} = \{X_0 \notin A, \ldots, X_{n-1} \notin A, X_n \in A\} \quad \checkmark$$

Example of Not a Stopping time ;

"last exit time"

$$\tau = \max\{n \geq 0 : X_n = i\}$$

last time state i is visited

$$\{\tau = 0\} = \{X_0 = i, X_1 \neq i, X_2 \neq i, \ldots\}$$

We know for a MC $\{X_n\}$

$$\mathbb{P}\left(X_{n+1}=j \mid X_n=i, \{X_0 \cdots X_{n-1}\}\right)$$
$$= \mathbb{P}(X_{n+1}=j \mid X_n=i) = P_{ij}$$

If τ is a stopping time, then we also get the "Strong Markov Property"

$$\boxed{\mathbb{P}\left(X_{\tau+1}=j \mid X_\tau = i,\ \{X_n\}_{0\le n<\tau}\right)}$$

$$= p_{ij}$$

$\{X_{\tau+n} : n \geq 0\}$ is the

same MC as (X_n)

but started with $X_0 = X_\tau$

$$\mathbb{P}\big(X_{n+1} = j \;\big|\; \underbrace{T = n}_{\text{only can depend at most on}},\ X_n = i,\ \vdots\ X_0 = i_0\big)$$

$$= \mathbb{P}\big(X_{n+1} = j \mid X_n = i\big)$$

$$\{X_{n-1} = i_{n-1},\ \vdots\ X_0 = i_0\}$$

hence does <u>not</u> contain any new info about the MC

$$= \underline{p_{ij}}$$

Wald's Equation

Let $\{X_n : n \geq 1\}$ be any $\underline{\underline{iid}}$ Sequence with $E|X| < \infty$ and Suppose τ is Stopping time with $E(\tau) < \infty$.

then $$E\left[\sum_{n=1}^{\tau} X_n\right] = E(X)\,E(\tau)$$

in general that won't hold for a random time.

Also if τ is independent of $\{X_n\}$ (a special kind of stopping time)

then the proof is easy:

$$E\left[\sum_{n=1}^{\tau} X_n \,\middle|\, \tau = k\right] = E\left[\sum_{n=1}^{k} X_n\right] = k\,E(X)$$

$$E\left[\underbrace{\sum_{n=1}^{T} X_n}_{Y}\right] = \sum_{k=1}^{\infty} E[Y \mid T=k]\, \mathbb{P}(T=k)$$

$$= \sum k\, E(X)\, \mathbb{P}(T=k)$$

$$= E(X) \sum_{k=1}^{\infty} k\, \mathbb{P}(T=k)$$

$$= E(X)\, E(T)$$

General Proof:

$$E\left[\sum_{n=1}^{\tau} X_n\right] = E\left[\sum_{n=1}^{\infty} X_n I\{\tau \geq n\}\right]$$

independent for each $n \geq 1$

$$= E\left[\sum_{n=1}^{\infty} X_n \underbrace{I\{\tau > n-1\}}\right]$$

independent

depends on at most $X_1, \dots X_{n-1}$ via "stopping time"

$$\Rightarrow E\left(X_n I\{T > n-1\}\right)$$

$$= E(X)\,\mathbb{P}(T > n-1)$$

if we are allowed to interchange $E, \sum$ then

we set $\displaystyle\sum_{n=1}^{\infty} E(X)\,\mathbb{P}(T > n-1)$

$$= E(X) \sum_{n=0}^{\infty} \mathbb{P}(T > n) = E(X)\, E(T)$$

interchange is allowed by Fubini's Thm via using the conditions

$$\boxed{\begin{array}{l} E(T) < \infty \\ E|X| < \infty \end{array}}$$

Example: Suppose

X_n = outcome of rolling 1 dice nth time

$$E(X) = \frac{6+1}{2} = \frac{7}{2} = \boxed{3.5}$$

$$\boxed{\mathbb{P}(X = i) = \frac{1}{6} \quad 1 \le i \le 6}$$

discrete uniform dist. over $\{1, 2, \cdots, 6\}$

$$T = \min\{n \geq 1 : X_n = 6\}$$

a stopping time

$$\mathbb{P}(T = k) = \left(\frac{5}{6}\right)^{k-1} \frac{1}{6}, \quad k \geq 1$$

$$\boxed{E(T) = 6} < \infty$$

$$E\left(\sum_{n=1}^{T} X_n\right) = E(T)\,E(X) \quad \text{Wald's}$$

$$= 6\,(3.5) = \boxed{21}$$

If $T = 3$, then we know that $X_1 \neq 6$, $X_2 \neq 6$, $X_3 = 6$

the values of X_k $\quad k < n$ are biased

Brute Force calculation Not using Wald's

$E(T-1)$
$= E(T) - 1 = 6 - 1$
$= 5$

$$E\left(\sum_{n=1}^{T} X_n\right) = E(T-1)\,E(X \mid X \neq 6) + 6$$

$(X_T = 6)$

i.i.d uniform over $\{1, 2, 3, 4, 5\}$

$E(X \mid X \neq 6)$
$= 3 = \frac{5+1}{2}$

$$6 + 5(3) = 21$$

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Application to Proving Null recurrence of the Simple symmetric ($p=\frac{1}{2}$) random Walk

We know all states are recurrent ($\mathbb{P}(T_{jj}<\infty)=1$, $j\in\mathbb{Z}$)

We also proved $\boxed{E(T_{jj})=\infty}$ Null rec. using other method.

Now we will re-prove.

Sufficient to

Prove $E(T_{j,1}) = \infty$

$$E\{T_{j,1} \mid \Delta_1 = 1\}\frac{1}{2} + E\{T_{j,1} \mid \Delta_1 = -1\}\frac{1}{2}$$

$$\left(1 + E(T_{2,j,1})\right)\frac{1}{2} + \left(1 + E(T_{0,j,1})\right)\frac{1}{2}$$

$$= 1 + \frac{1}{2}E(T_{2,j,1}) + \frac{1}{2}\boxed{E(T_{0,j,1})}$$

it thus suffices to show that

$$E(\tau) = \infty$$

where $\tau = \min\{n \geq 1 : R_n = 1 \mid R_0 = 0\}$

$= \tau_{0,1}$ a stopping time for $\Delta_1, \Delta_2, \ldots$ (Δ_n)

$$1 = R_\tau = \sum_{n=1}^{\tau} \Delta_n$$

$$\left(\begin{array}{c} E|\Delta| = 1 < \infty \\ |\Delta| = 1 \end{array} \right)$$

If $E(T) < \infty$,

then we can

use Wald's Equation

$$1 = E\left(\sum_{n=1}^{T} \Delta_n\right) = \underline{E(T)\,\underbrace{E(\Delta)}_{\substack{p=\frac{1}{2} \\ =0}}}$$

$$= 0$$

Contradiction; hence

$E(T) = \infty$

Expected Total number of visits of a finite state MC to a transient state i

for example the $E\left(\begin{array}{l}\text{Total \# vistits} \\ \text{to Room 2} \\ \text{by the rat} \\ \mid X_0 = 1\end{array}\right)$

(open maze)

Suppose a MC (X_n) has finite state space with N states

$$\mathcal{S} = \{1, 2, \dots, N\}$$

and there are $1 \le b < N$ <u>transient</u> states

Set of transient states $= T = \{1, 2, \dots, b\}$ (hence $N-b$ recurrent states)

$$S_{ij} = E\left\{ \begin{array}{l} \text{Total \# visits of} \\ (X_n) \text{ to state } j \end{array} \middle| X_0 = i \right\}$$

$i, j \in T$

$$S_{ij} = E\left[\sum_{n=0}^{\infty} I\{X_n = j \mid X_0 = i\}\right) = \sum_{n=0}^{\infty} p_{ij}^{(n)}$$

$$P_T \stackrel{def}{=} (P_{ij})_{i,j \in T}$$

$b \times b$ matrix (but not stochastic)

$$S = (S_{ij})_{i,j \in T}$$

is also a $b \times b$ matrix

Prop. Let I denote the $k \times k$ identity matrix $\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

then $S = I + P_T S$

$$\Rightarrow \boxed{S = (I - P_T)^{-1}}$$

We will prove at next lecture