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1 Notes on Brownian Motion

We present an introduction to *Brownian motion*, an important continuous-time stochastic process that serves as a continuous-time analog to the simple symmetric random walk on the one hand, and shares fundamental properties with the Poisson counting process on the other hand.

Throughout, we use the following notation for the real numbers, the non-negative real numbers, the integers, and the non-negative integers respectively:

$$\mathbb{R} \stackrel{\text{def}}{=} (-\infty, \infty) \tag{1}$$

$$\mathbb{R}_+ \stackrel{\text{def}}{=} [0, \infty) \tag{2}$$

$$\mathbb{Z} \stackrel{\text{def}}{=} \{\cdots, -2, -1, 0, 1, 2, \cdots\} \tag{3}$$

$$\mathbb{N} \stackrel{\text{def}}{=} \{0, 1, 2, \cdots\}. \tag{4}$$

1.1 Normal distribution

Of particular importance in our study is the normal distribution, $N(\mu, \sigma^2)$, with mean $-\infty < \mu < \infty$ and variance $0 < \sigma^2 < \infty$; the density and cdf are given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, \ x \in \mathbb{R}, \tag{5}$$

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-(y-\mu)^2}{2\sigma^2}} dy, \ x \in \mathbb{R}. \tag{6}$$

The normal distribution is also called the *Gaussian* distribution after the famous German mathematician and physicist Carl Friedrich Gauss (1777 - 1855).

When $\mu = 0$ and $\sigma^2 = 1$ we obtain the *standard* (or *unit*) normal distribution, $N(0, 1)$, and the density and cdf reduce to

$$\phi(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}, \tag{7}$$

$$\Phi(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-y^2}{2}} dy. \tag{8}$$

As we shall see over and over again in our study of Brownian motion, one of its nice features is that many computations involving it are based on evaluating $\Phi(x)$, and hence are computationally elementary.

If $Z \sim N(0, 1)$, then $X = \sigma Z + \mu$ has the $N(\mu, \sigma^2)$ distribution. Conversely, if $X \sim N(\mu, \sigma^2)$, then $Z = (X - \mu)/\sigma$ has the standard normal distribution.

It thus follows that that if $X \sim N(\mu, \sigma^2)$, then $F(x) = P(X \leq x) = \Phi((x - \mu)/\sigma)$.

Moment generating function of a normal distribution

Letting $X \sim N(\mu, \sigma^2)$, the moment generating function of the normal distribution is given by

$$\begin{aligned} M_X(s) &= E(e^{sX}) \\ &= \int_{-\infty}^{\infty} e^{sx} f(x) dx \\ &= e^{s\mu + s^2\sigma^2/2}, \ -\infty < s < \infty. \end{aligned} \tag{9}$$

Deriving (9): First we derive $M_Z(s) = e^{s^2/2}$, that is, the case when $X = Z$ is the unit normal.

$$\begin{aligned} M_Z(s) &= E(e^{sZ}) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{sx} e^{\frac{-x^2}{2}} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-(x^2 - 2sx)}{2}} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-(x-s)^2}{2}} + e^{s^2/2} dx \\ &= e^{s^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-(x-s)^2}{2}} dx \\ &= e^{s^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-u^2}{2}} du \\ &= e^{s^2/2} \times 1 \\ &= e^{s^2/2}. \end{aligned}$$

To obtain the general form in (9): If $X \sim N(\mu, \sigma^2)$, then it can be expressed as $X = \sigma Z + \mu$, and thus
$M_X(s) = E(e^{sX}) = e^{s\mu} E(e^{\sigma s Z})$
$= e^{s\mu} M_Z(\sigma s)$
$= e^{s\mu} e^{(\sigma s)^2/2}$
$= e^{s\mu + s^2\sigma^2/2}$; we have derived (9).

1.1.1 Central limit theorem (CLT)

Theorem 1.1 *If $\{X_i : i \geq 1\}$ are iid with finite mean $E(X) = \mu$ and finite non-zero variance $\sigma^2 = Var(X)$, then*

$$Z_n \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{n}} \Big(\sum_{i=1}^{n} X_i - n\mu \Big) \Longrightarrow N(0,1), \ n \to \infty, \ \textit{in distribution};$$

$\lim_{n\to\infty} P(Z_n \leq x) = \Phi(x), \ x \in \mathbb{R}$.

If $\mu = 0$ and $\sigma^2 = 1$, then the CLT becomes

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \Longrightarrow N(0,1).$$

Moreover, for any constant $c \neq 0$, since $cN(0,1) \sim N(0,c^2)$, we obtain:
If $\mu = 0$ and $\sigma^2 = 1$, then the CLT becomes for any constant $c \neq 0$

$$c\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i \Longrightarrow N(0,c^2).$$

1.2 Construction of Brownian motion from the simple symmetric random walk

Recall the simple *symmetric* random walk, $R_0 = 0$,

$$R_n = \Delta_1 + \cdots + \Delta_n = \sum_{i=1}^{n} \Delta_i,\ n \geq 1,$$

where the Δ_i are iid with $P(\Delta = -1) = P(\Delta = 1) = 0.5$. thus $E(\Delta) = 0$ and $Var(\Delta) = E(\Delta^2) = 1$.

We view time n in minutes, and R_n as the position at time n of a particle, moving on $\mathbb{R}$, which every minute takes a step, of size 1, equally likley to be forwards or backwards. Because $E(\Delta) = 0$ and $Var(\Delta) = 1$, it follows that $E(R_n) = 0$ and $Var(R_n) = n,\ n \geq 0$.

Choosing a large integer $k > 1$, if we instead make the particle take a step every $1/k$ minutes and make the step size $1/\sqrt{k}$, then by time t the particle will have taken a large number, $n = tk$, of steps and its position will be

$$B_k(t) = \frac{1}{\sqrt{k}}\sum_{i=1}^{tk} \Delta_i. \tag{10}$$

(By convention if tk is not an integer then we replace it by the largest integer less than or equal to it; denoted by $[tk]$.) This leads to the particle taking many many iid steps, but each of small magnitude, in any given interval of time. We expect that as $k \to \infty$, these small steps become a continuum and the process $\{B_k(t) : t \geq 0\}$ should converge to a process $\{B(t) : t \geq 0\}$ with continuous sample paths. We call this process Brownian motion (BM) after the Scottish botanist Robert Brown.[1] Its properties will be derived next.

Notice that for fixed k, any increment

$$B_k(t) - B_k(s) = \frac{1}{\sqrt{k}}\sum_{i=sk+1}^{tk} \Delta_i,\ 0 \leq s < t,$$

has a distribution that only depends on the length, $t - s$, of the time interval $(s,t]$ because it only depends on the number, $k(t-s)$, of iid Δ_i making up its construction. Thus we deduce that the limiting process (as $k \to \infty$) will possess *stationary increments*: *The distribution of any increment $B(t) - B(s)$ has a distribution that only depends on the length of the time interval $t - s$. In particular, $B(t) - B(s)$ has the same distribution as does $B(t-s)$.*

[1] Brown himself noticed in 1827, while carrying out some experiments, the unusual "motion" of particles within pollen grains suspended in water, under his microscope. The physical cause of such motion (bombardment of the particles by water molecules undergoing thermal motion) was not formalized via kinetic theory until Einstein in 1905. The rigorous mathematical construction of a stochastic process as a model for such motion is due to the mathematician Norbert Weiner; that is why it is sometimes called a Weiner process.

Notice further that given two non-overlapping time intervals, $(t_1, t_2]$ and $(t_3, t_4]$, $0 \le t_1 < t_2 < t_3 < t_4$, the corresponding increments

$$B_k(t_4) - B_k(t_3) = \frac{1}{\sqrt{k}} \sum_{i=t_3 k+1}^{t_4 k} \Delta_i, \tag{11}$$

$$B_k(t_2) - B_k(t_1) = \frac{1}{\sqrt{k}} \sum_{i=t_1 k+1}^{t_2 k} \Delta_i, \tag{12}$$

are independent because they are constructed from different Δ_i. Thus we deduce that the limiting process (as $k \to \infty$) will also possess *independent increments*: *For any non-overlapping time intervals, $(t_1, t_2]$ and $(t_3, t_4]$, the increment rvs $I_1 = B(t_2) - B(t_1)$ and $I_2 = B(t_4) - B(t_3)$ are independent.*

Observing that $E(B_k(t)) = 0$ and $Var(B_k(t)) = [tk]/k \to t,\ k \to \infty$, we infer that the limiting process will satisfy $E(B(t)) = 0,\ Var(B(t)) = t$ just like the random walk $\{R_n\}$ does in discrete-time n ($E(R_n) = 0,\ Var(R_n) = n$).

Finally, a direct application of the CLT yields (via setting $n = tk,\ \mu = 0,\ \sigma^2 = 1,\ c = \sqrt{t}$)

$$B_k(t) = \sqrt{t}\Big(\frac{1}{\sqrt{kt}} \sum_{i=1}^{tk} \Delta_i\Big) \Longrightarrow N(0, t),\ k \to \infty,\ \textit{in distribution},$$

and we conclude that for each fixed $t > 0$, $B(t)$ has a normal distribution with mean 0 and variance t. Similarly, using the stationary and independent increments property, we conclude that $B(t) - B(s)$ has a normal distribution with mean 0 and variance $t - s$, and more generally:

> *the limiting BM process is a process with continuous sample paths that has both stationary and independent normally distributed (Gaussian) increments: If $t_0 = 0 < t_1 < t_2 < \cdots < t_n$, then the rvs. $B(t_i) - B(t_{i-1}),\ i \in \{1, \ldots n\}$, are independent with $B(t_i) - B(t_{i-1}) \sim N(0, t_i - t_{i-1})$.*

If we define $X(t) = \sigma B(t) + \mu t$, then $X(t) \sim N(\mu t, \sigma^2 t)$, $\sigma \in \mathbb{R}_+,\ \mu \in \mathbb{R}$, and we obtain, by such scaling and translation, more generally, a process with stationary and independent increments in which $X(t) - X(s)$ has a normal distribution with mean $\mu(t - s)$ and variance $\sigma^2(t - s)$. We call this *Brownian motion with drift μ and variance term σ^2*, or just *Brownian motion with drift*, for short.

When $\sigma^2 = 1$ and $\mu = 0$, the process is called *standard Brownian motion*, and denoted by $\{B(t) : t \ge 0\}$.

Definition 1.1 *A stochastic process $\mathbf{B} = \{B(t) : t \ge 0\}$ possessing (wp1) continuous sample paths is called standard Brownian motion if*

1. $B(0) = 0$.
2. $\mathbf{B}$ *has both stationary and independent increments.*
3. $B(t) - B(s)$ *has a normal distribution with mean* 0 *and variance* $t - s,\ 0 \le s < t$.

For Brownian motion with drift μ and variance term σ^2, $X(t) = \sigma B(t) + \mu t$, the definition is the same except that *3* must be modified to

> $3'$. $X(t) - X(s)$ *has a normal distribution with mean* $\mu(t - s)$ *and variance* $\sigma^2(t - s)$.

Remark 1.1 It can in fact be proved that Condition *3* above is redundant: a stochastic process with stationary and independent increments that possesses (wp1) continuous sample paths must be Brownian motion, that is, the increments must be normally distributed. This is analogous to the Poisson counting process $\{N(t) : t \geq 0\}$ which is the unique *simple* counting process that has both stationary and independent increments: the stationary and independent increments property forces the increments to be Poisson distributed. (*Simple* means that the arrival times of the underlying point process are strictly increasing, only one arrival at a time; no batches.)

Donsker's theorem

Our construction of Brownian motion as a limit is in fact a rigorous one, but requires more advanced mathematical tools (beyond the scope of these lecture notes) in order to state it precisely and to prove it. We have (due to the CLT) proved that fixed increments of $B_k(t)$ converge to a normal rv, but more generally it can be proved that the stochastic process $\{B_k(t) : t \geq 0\}$ as defined by (10) converges in distribution (weak convergence in path (function) space), as $k \to \infty$, to Brownian motion $\{B(t) : t \geq 0\}$. This is known as *Donsker's theorem* or the *functional central limit theorem*. The point is that it is a generalization of the central limit theorem because it involves an entire stochastic process (with all its multi-dimensional joint distributions, for example) as opposed to just a one-dimensional limit (such as for fixed $t > 0$, $B_k(t) \to N(0, t)$ in distribution). Donsker's theorem implies, for example, that the vector $(B_k(t_1), \ldots, B_k(t_n))$ converges (jointly) in distribution to the vector $(B(t_1), \ldots, B(t_n))$: for any time points $0 \leq t_1 < t_2 < \cdots < t_n$ and $x_i \in \mathbb{R},\ 1 \leq i \leq n$, it holds that (as $k \to \infty$)

$$P(B_k(t_1) \leq x_1, \ldots, B_k(t_n) \leq x_n) \to P(B(t_1) \leq x_1, \ldots, B(t_n) \leq x_n).$$

1.3 Hitting times for standard BM

Consider two values $a > 0$, $b > 0$ and the two-point set $\{a, -b\}$. Let

$$\tau = \min\{t \geq 0 : B(t) \in \{a, -b)\}\},$$

the first time that BM hits either a or $-b$. (The contiuity of sample paths implies that $B(t)$ either hits a or hits $-b$ (e.g, there is no overshoot/jumps).)

Recall from the gambler's ruin problem that for the simple *symmetric* random walk $\{R_n\}$, $p_a = \frac{b}{a+b}$, where $a > 0$, $b > 0$ (integers), and p_a denotes the probability that the random walk starting at $R_0 = 0$ first hits a before hitting $-b$;

$$\tau = \min\{n \geq 0 : R_n \in \{a, -b\} | R_0 = 0\},$$

and $p_a = P(R_\tau = a)$. Thus p_a denotes the probability that R_n goes up a steps before going down b steps.

For the process $\{B_k(t) : t \geq 0\}$ to hit a before $-b$ would require the random walk

$$R_n^{(k)} = \frac{1}{\sqrt{k}} \sum_{i=1}^{n} \Delta_i,\ n \geq 0$$

to hit a before $-b$. This would require (approximately) that the original random walk,

$$R_n = \sum_{i=1}^{n} \Delta_i,\ n \geq 0,$$

hits $a\sqrt{k}$ before $-b\sqrt{k}$; yielding the same answer

$$p_a = \frac{b\sqrt{k}}{a\sqrt{k}+b\sqrt{k}} = \frac{b}{a+b}.$$

We thus deduce (via letting $k \to \infty$) that the same holds for standard BM (where a and b need not be integers now). Letting $p_a = P(B(\tau) = a)$, where
$\tau = \min\{t \geq 0 : B(t) \in \{a, -b\} \mid B(0) = 0\}$, we state this as

Proposition 1.1 *For standard BM, the probability that a is first hit before $-b$ is given by*

$$p_a = \frac{b}{a+b},\ a > 0, b > 0.$$

Proof : A rigorous proof is provided by the optional stopping theorem for martingales (MG) (see Section 1.6):

$E(B(\tau)) = E(B(0)) = 0$ implies that $aP_a - b(1-p_a) = 0$; solving yields $p_a = \frac{b}{a+b}$.

Uniform integrability (UI) of the stopped MG $\{\overline{B}(t)\} = \{B(t \wedge \tau)\}$ follows since it is bounded between $-b$ and a, thus justifying the use of the optional stopping theorem. (Proving that τ is proper, $P(\tau < \infty) = 1$, can be deduced in a variety of ways; one such way is given in Proposition 1.3 later on in these notes.) ■

We also can derive:

Proposition 1.2 *For standard BM, if $\tau = \min\{t \geq 0 : B(t) \in \{a, -b\} | B(0) = 0\}$, then*

$$E(\tau) = ab.$$

Note that if a variance term is introduced, $\sigma B(t)$, $\sigma > 0$, then $\sigma B(t) \in \{a, -b\}$ if and only if $B(t) \in \{a/\sigma, -b/\sigma\}$ yielding $E(\tau) = \frac{ab}{\sigma^2}$.

Proof : Here we use optional stopping on the MG $B^2(t) - t$ yielding $E(B^2(\tau)) = E(\tau)$, or $E(\tau) = a^2 p_a + b^2(1-p_a) = ab$, where we use the fact that $p_a = \frac{b}{a+b}$. UI of the stopped martingale $\overline{X}(t) = B^2(\tau \wedge t) - \tau \wedge t$ holds: The proof is identical to the discrete-time version of the simple symmetric random walk, where we used the martinagle $R_n^2 - n$: The stopped process $\overline{X}(t) = B^2(\tau \wedge t) - \tau \wedge t$ is a MG, so $E(\overline{X}(t)) = 0$ for all t or equivalently $E(\tau \wedge t) = E(B^2(\tau \wedge t)) \leq (a+b)^2 < \infty$. But $\tau \wedge t$ is monotone increasing to τ and non-negative, so by the monotone convergence theorem we conclude that $E(\tau) \leq (a+b)^2 < \infty$; $\tau \in L^1$. Thus $|\overline{X}(t)| \leq B^2(\tau \wedge t) + \tau \wedge t \leq (a+b)^2 + \tau = Y \in L^1$; we conclude that $\{\overline{X}(t)\}$ is UI. ■

(In a later section, we will learn the corresponding (more complicated) formulas for BM with drift μ and variance term σ^2.)

Examples

1. A particle moves on a line according to a standard BM, $B(t)$. What is its expected position at time $t = 6$? What is the variance of its position at time $t = 6$?

   SOLUTION: $B(t)$ has a normal distribution with mean $E(B(t)) = 0$ and variance $Var(B(t)) = t$, hence the answers are 0 and 6.

2. *Continuation:*

   If the particle is at position 1.7 at time $t = 2$, what is its expected position at time $t = 4$?

   SOLUTION: $B(4) = B(2) + B(4) - B(2) = (B(2) - B(0)) + (B(4) - B(2))$, where the two increments are independent; $B(2)$ is independent of $(B(4) - B(2))$.

   $E(B(4)|B(2) = 1.7) = 1.7 + E(B(4) - B(2)|B(2) = 1.7)$
   $= 1.7 + E(B(4) - B(2))$ (independent increments)
   $= 1.7 + 0 = 1.7$, since all increments have mean 0, $E(B(t) - B(s)) = 0$.

3. *Continuation:*

   What is the probability that the particle hits level 10 before level -2? What is the expected length of time until either 10 or -2 are hit?

   SOLUTION: $a = 10$, and $b = 2$ in the formula for $p_a = b/(a + b) = 1/6$, and $E(\tau) = ab = 20$.

4. The price of a commodity moves according to a BM, $X(t) = \sigma B(t) + \mu t$, with variance term $\sigma^2 = 4$ and drift $\mu = -5$. Given that the price is 4 at time $t = 8$, what is the probability that the price is below 1 at time $t = 9$?

   SOLUTION:

   $P(X(9) < 1|X(8) = 4) = P(X(9) - X(8) < -3|X(8) = 4) = P(X(9) - X(8) < -3)$ (independent increments, $X(9) - X(8)$ is independent of $X(8) - X(0) = X(8)$)
   $= P(X(1) < -3)$ (stationary increments)
   $= P(2Z - 5 < -3)$ (since $X(1) \sim N(-5, 4)$ can be represented in terms of a unit normal, Z, as $2Z - 5$)
   $= P(Z < 1) = \Theta(1) = 0.8413$ (via a Table for the standard normal distribution, as found in any statistics textbook for example).

5. A stock price per share moves according to *geometric BM*,

$$S(t) = S_0 e^{B(t)},\ t \geq 0.$$

   Suppose that $S_0 = 4$, $S(t) = 4e^{B(t)}$. What is the probability that the stock price will reach a high of 7 before a low of 2?

   SOLUTION:

   Taking natural logarithms, we can convert the problem into *What is the probability that* $\ln(4) + B(t)$ *hits a high of* $\ln(7)$ *before a low of* $\ln(2)$*?*

   This is equivalent to

   What is the probability that $B(t)$ *hits a high of* $\ln(7/4)$ *before a low of* $\ln(1/2)$*?*

   Noting that $\ln(1/2) = -\ln(2)$ we can set $a = \ln(7/4)$ and $b = \ln(2)$ in the formula $p_a = b/(a + b) = \ln(2)/(\ln(7/2))$.

1.4 BM as a Gaussian process

We observe that the vector $(B(t_1), \ldots, B(t_n))$ has a multivariate normal distribution because the event

$$\{B(t_1) = x_1, \ldots, B(t_n) = x_n\}$$

can be re-written in terms of independent increment events

$$\{B(t_1) = x_1, B(t_2) - B(t_1) = x_2 - x_1, \ldots, B(t_n) - B(t_{n-1}) = x_n - x_n\},$$

yielding the joint density of $(B(t_1), \ldots, B(t_n))$ as

$$f(x_1, \ldots, x_n) = f_{t_1}(x_1) f_{t_2 - t_1}(x_2 - x_1) \cdots f_{t_n - t_{n-1}}(x_n),$$

where

$$f_t(x) = \frac{1}{\sqrt{2\pi t}} e^{\frac{-x^2}{2t}}$$

is the density for the $N(0, t)$ distribution.

The finite dimensional distributions of BM are thus multivariate normal, or Gaussian, and BM is an example of a *Gaussian process*, that is, a process with continuous sample paths in which the finite dimensional distributions are multivariate normal, that is, for any fixed choice of n time points $0 \le t_1 < t_2 < \cdots < t_n,\ n \ge 1$, the joint distribution of the vector $(X(t_1), \ldots, X(t_n))$ is multivariate normal.

Since a multivariate normal distribution is completely determined by its mean and covariance parameters, we conclude that a Gaussian process is completely determined by its mean and covariance function $m(t) \stackrel{\text{def}}{=} E(X(t)),\ a(s,t) \stackrel{\text{def}}{=} cov(X(s), X(t)),\ 0 \le s \le t$.

For standard BM, $m(t) = 0$ and $a(s,t) = s$:

$$\begin{aligned} cov(B(s), B(t)) &= cov(B(s), B(s) + B(t) - B(s)) \\ &= cov(B(s), B(s)) + cov(B(s), B(t) - B(s)) \\ &= var(B(s)) + 0 \ \textit{(via independent increments)} \\ &= s. \end{aligned}$$

Thus standard BM is the unique Gaussian process with $m(t) = 0$ and $a(s,t) = \min\{s,t\}$. Similarly, BM with variance σ^2 and drift μ, $X(t) = \sigma B(t) + \mu t$, is the unique Gaussian process with $m(t) = \mu t$ and $a(s,t) = \sigma^2 \min\{s,t\}$.

1.5 BM as a Markov Processes

If B is standard BM, then the independent increments property implies that $B(s+t) = B(s) + (B(s+t) - B(s))$, in which $B(s)$ and $(B(s+t) - B(s))$ are independent. The independent increments property implies further that $(B(s+t) - B(s))$ is also independent of the past before time s, $\{B(u) : 0 \le u < s\}$.

Thus the future, $B(s+t)$, given the present state, $B(s)$, only depends on a rv, $B(s+t) - B(s)$, that is independent of the past. Thus we conclude that BM satisfies the Markov property. Since the increments are also stationary, we conclude that BM is a time-homogenous Markov process.

Letting $p(x,t,y)$ denote the probability density function for $B(s+t) = y$ given $B(s) = x$, we see, from $B(s+t) = x + (B(s+t) - B(s))$, that $p(x,t,y)$ is the density for $x + B(s+t) - B(s)$. But $x + B(s+t) - B(s) = y$ if and only if $(B(s+t) - B(s)) = y - x$, yielding

$$p(x,t,y) = f_t(y-x) = \frac{1}{\sqrt{2\pi t}} e^{\frac{-(y-x)^2}{2t}}. \tag{13}$$

More generally, $X(t) = \sigma B(t) + \mu t$ is a Markov process with

$$p(x,t,y) = \frac{1}{\sigma\sqrt{2\pi t}} e^{\frac{-((y-x-\mu t)^2}{2\sigma^2 t}}. \tag{14}$$

Brownian motion also possesses the *strong Markov property:* If T is a (proper) stopping time with respect to a Brownian motion $X(t) = \sigma B(t) + \mu t$, then

$\{X(T+t) - X(T) : t \ge 0\}$ is yet again a Brownian motion (started at 0) with the same σ and μ, and is independent of $\{X(s) : 0 \le s \le T\}$, the past up to time T.

1.6 BM as a martingale

Standard BM is a *martingale*:

$$E(B(s+t)|B(u):0\le u\le s)=B(s),\ t\ge 0,\ s\ge 0,$$

which asserts that the conditional expectation of BM at any time in the future after time s equals the original value at time s. This of course follows from the independent increments property and using $B(s+t)=B(s)+(B(s+t)-B(s))$:

$$\begin{aligned}
E(B(s+t)|B(u):0\le u\le s) &= E(B(s+t)|B(s)),\ \textit{via the Markov property}\\
&= E(B(s)+(B(s+t)-B(s))|B(s))\\
&= B(s)+E(B(s+t)-B(s)|B(s))\\
&= B(s)+E(B(s+t)-B(s)),\ \textit{via independent increments}\\
&= B(s)+0\\
&= B(s).
\end{aligned}$$

A martingale captures the notion of a fair game, in that regardless of your current and past fortunes, your expected fortune at any time in the future is the same as your current fortune: on average, you neither win nor lose any money. This property of BM is the continuous-time analog of the simple symmetric RW in discrete-time: The simple symmetric random walk is a martingale (and a Markov chain) in discrete time;

$$E(R_{n+k}|R_n,\ldots,R_0)=E(R_{n+k}|R_n)=R_n,\ k\ge 0,\ n\ge 0,$$

because

$$R_{n+k}=R_n+\sum_{i=1}^{k}\Delta_{n+i},$$

and $\sum_{i=1}^{k}\Delta_{n+i}$ is independent of R_n (and the past before time n) and has mean 0.

The optional stopping theorem and the martingale convergence theorem hold for continuous-time martingales too; we will not re-prove such things here. In general, a stochastic process $\{X(t)\}$ is called a martingale if $E|X(t)|<\infty,\ t\ge 0$ and

$$E(X(s+t)|X(u):0\le u\le s)=X(s),\ t\ge 0,\ s\ge 0.$$

As in discrete time we denote

$$\mathcal{F}_t=\sigma\{X(s):0\le s\le t\},$$

and we can re-write the martingale property as

$$E(X(s+t)|\mathcal{F}_s)=X(s),\ t\ge 0,\ s\ge 0.$$

The standard framework used is to assume that the sample-paths of the stochastic processes are *cadlag* (e.g. continuous from the right with left-hand limits[2]). This allows for some jumps in the process, as is the case in many applications (such as in queueing theory).

[2] A function $x(t)$, $t\in[0,\infty)$ is cadlag if $x(t)=x(t+)\stackrel{\text{def}}{=}\lim_{h\downarrow 0}x(t+h)$ for all $t\ge 0$ (right continuity at each point t), and the limit $x(t-)\stackrel{\text{def}}{=}\lim_{h\downarrow 0}x(t-h)$ exists for all $t>0$.

1.7 The distribution of $T_x = \min\{t > 0 : B(t) = x\}$ and $M = \max_{t \geq 0} B(t)$

Now let $T_x = \min\{t \geq 0 : B(t) = x \mid B(0) = 0\}$, the hitting time to $x \neq 0$. From our study of the simple symmetric random walk, we expect $P(T_x < \infty) = 1$, but $E(T_x) = \infty$: although any level $x \neq 0$ will be hit with certainty, the mean length of time required is infinite. We will prove this directly and derive the cdf $P(T_x \leq t),\ t \geq 0$ along the way.

Without loss of generality, we can assume that $x > 0$, since by symmetry of standard BM about 0, T_x and T_{-x} have the same distribution for any $x \neq 0$: $\{-B(t) : t \geq 0\}$ is still a standard BM, so $\min\{t > 0 : B(t) = x\}$ must have the same distribution as $\min\{t > 0 : -B(t) = x\} = \min\{t > 0 : B(t) = -x\}$.

The key to our analysis is based on a simple observation involving the symmetry of standard BM, known as *the reflection principle*: If $T_x < t$, then $B(s) = x$ for some $s < t$. Thus the value of $B(t)$ is determined by where the BM went in the remaining $t - s$ units of time after hitting x. But BM, having stationary and independent Gaussian increments, will continue having them after hitting x (strong Markov property): $\{B(T_x + t) - x : t \geq 0\}$ is yet again a standard Brownian motion. So by symmetry (about x), the path of BM during the time interval $(s, t]$ with $B(s) = x$ is just as likely to lead to $B(t) > x$ as to $B(t) < x$. So the events $\{B(t) > x |\ T_x \leq t\}$ and $\{B(t) < x |\ T_x \leq t\}$ are equally likely; both have probability 1/2. ($P(B(t) = x) = 0$ since $B(t)$ has a continuous distribution.) To be precise, if $T_x = s < t$, then $B(t) = x + B(t) - B(s)$ which has the $N(x, t - s)$ distribution (which is symmetric about x). Thus $P(B(t) > x \mid T_x \leq t) = 1/2$. On the other hand $P(B(t) > x \mid T_x > t) = 0$ (if $x > 0$ as we assume) because BM (having continuous sample paths) can't be above x at time t if it never hit x prior to t. Summarizing yields

$$\begin{aligned} P(B(t) > x) &= P(B(t) > x \mid T_x \leq t)P(T_x \leq t) + P(B(t) > x \mid T_x > t)P(T_x > t) \\ &= P(B(t) > x \mid T_x \leq t)P(T_x \leq t) + 0 \\ &= \frac{1}{2}P(T_x \leq t), \end{aligned}$$

or

$$P(T_x \leq t) = 2P(B(t) > x) = \frac{2}{\sqrt{2\pi t}} \int_x^\infty e^{\frac{-y^2}{2t}} dy,$$

because $B(t) \sim N(0, t)$. Changing variables $u = y/\sqrt{t}$ then yields

Proposition 1.3 *For standard BM, for any fixed $x > 0$*

$$P(T_x \leq t) = 2P(B(t) > x) = \frac{2}{\sqrt{2\pi}} \int_{\frac{x}{\sqrt{t}}}^\infty e^{\frac{-y^2}{2}} dy = 2(1 - \Phi(x/\sqrt{t})),\ t \geq 0.$$

In particular T_x is a proper random variable; $P(T_x < \infty) = 1$ because $P(T_x < \infty) = \lim_{t \to \infty} P(T_x < \infty) = 2(1 - \Phi(0)) = 2(1 - 0.5) = 1$.

Corollary 1.1 *For standard BM, for any fixed $x \neq 0$, $E(T_x) = \infty$.*

Proof : We shall proceed by computing $E(T_x) = \infty$ by integrating the tail $P(T_x > t)$;

$$E(T_x) = \int_0^\infty P(T_x > t) dt.$$

To this end, $P(T_x > t) = 1 - P(T_x \leq t) = \frac{2}{\sqrt{2\pi}} \int_0^{\frac{x}{\sqrt{t}}} e^{\frac{-y^2}{2}} dy$. Since the constant factor $\frac{2}{\sqrt{2\pi}}$ plays no role in wether the integrated tail is infinite or finite, we leave it out for simplicity. It thus suffices to show that

$$\int_0^\infty \int_0^{\frac{x}{\sqrt{t}}} e^{\frac{-y^2}{2}} dydt = \infty.$$

Changing the order of integration, we re-write as

$$\begin{aligned}
\int_0^\infty \int_0^{\frac{x^2}{y^2}} e^{\frac{-y^2}{2}} dtdy &= x^2 \int_0^\infty \frac{1}{y^2} e^{\frac{-y^2}{2}} dy \\
&\geq x^2 \int_0^1 \frac{1}{y^2} e^{\frac{-y^2}{2}} dy \\
&\geq x^2 e^{-1/2} \int_0^1 \frac{1}{y^2} dy \\
&= \infty.
\end{aligned}$$

The second inequality is due to the fact that the decreasing function $e^{\frac{-y^2}{2}}$ is minimized over the interval $(0, 1]$ at the end point $y = 1$.

■

Let $M_t \stackrel{\text{def}}{=} \max_{0 \leq s \leq t} B(t)$ denote the maximum value of BM up to time t. Noting that $M_t \geq x$ if and only if $T_x \leq t$, we conclude that $P(M_t \geq x) = P(T_x \leq t)$ yielding (from Proposition 1.3) a formula for the distribution of M_t:

Corollary 1.2 *For standard BM, for any fixed $t \geq 0$,*

$$P(M_t > x) = \frac{2}{\sqrt{2\pi}} \int_{\frac{x}{\sqrt{t}}}^\infty e^{\frac{-y^2}{2}} dy = 2(1 - \Phi(x/\sqrt{t})), \ x \geq 0.$$

Remark 1.2 Note that the above results hold for a BM with variance term, $\{\sigma B(t)\}$, $\sigma > 0$, because $\sigma B(t) = x$ if and only if $B(t) = x/\sigma$; just replace x by x/σ.

1.8 Hitting times for BM with drift

For $X(t) = \sigma B(t) + \mu t$, let's assume that $\mu < 0$ so that the BM has *negative drift*. This is analogous to the simple random walk with negative drift, that is, $\{R_n\}$ when the increments have distribution $P(\Delta = 1) = p, \ P(\Delta = -1) = q = 1 - p$ and $q > p$. Recall from the gambler's ruin problem that in this case

$$p_a = \frac{1 - (p/q)^b}{(p/q)^{-a} - (p/q)^b},$$

and thus by letting $b \to \infty$ we obtain the probability that the random walk will ever exceed level a;

$$P(\max_{n \geq 0} R_n \geq a) = \lim_{b \to \infty} p_b = (p/q)^a.$$

We conclude that the maximum of the random walk has a geometric distribution with "success" probability $1 - p/q$. The point is that the negative drift random walk will eventually drift off

to $-\infty$, but before it does there is a positive probability, $(p/q)^a$, that it will first reach level $a > 0$.

$X(t)$ is similar. We let $M = \max_{t\geq 0} X(t)$ denote the maximum of the BM:

Proposition 1.4 *For BM with negative drift, $X(t) = \sigma B(t) + \mu t$, $\mu < 0$,*

$$p_a = \frac{1 - e^{-\alpha b}}{e^{\alpha a} - e^{-\alpha b}},$$

where $\alpha = 2|\mu|/\sigma^2$; thus (letting $b \to \infty$)

$$P(M > a) = e^{-\alpha a},\ a \geq 0,$$

and we conclude that M has an exponential distribution with mean $\alpha^{-1} = \sigma^2/2|\mu|$.

In general, $\mu > 0$ or $\mu < 0$, the formula for p_a is

$$p_a = \frac{1 - e^{(2\mu/\sigma^2)b}}{e^{(-2\mu/\sigma^2)a} - e^{(2\mu/\sigma^2)b}}.$$

Proof :

Here we use an exponential martingale of the form

$$Z(t) = e^{\lambda X(t) - (\lambda\mu + \frac{1}{2}\lambda^2\sigma^2)t}.$$

This is a mean 1 MG for any value of λ: First note that from the results from Page 2 on the moment generating function of the normal distribution, we conclude that in fact $E(Z(t)) = e^0 = 1,\ t \geq 0$. Then note that $Z(t)$ is a Markov process, since it is an invertible function $f(x) = e^x$, of a Markov process, the Brownian motion $Y(t) = \lambda X(t) - (\lambda\mu + \frac{1}{2}\lambda^2\sigma^2)t$; $Z(t) = e^{Y(t)}$. Thus it suffices to show that $E(Z(s+t) \mid Z(s)) = Z(s))$, for all $s, t \geq 0$. But $Z(s+t) = Z(s)[\frac{Z(s+t)}{Z(s)}]$, and by independent and stationary increments of the BM $Y(t)$, $[\frac{Z(s+t)}{Z(s)}]$ is independent of $Z(s)$ and has the same distribution as $Z(t)$. Thus $E(Z(s+t) \mid Z(s)) = Z(s)E(Z(t)) = Z(s) \times 1 = Z(s)$. We have thus proved that $\{Z(t)\}$ is a MG for any λ.

Choosing $\lambda = \alpha = -2\mu/\sigma^2$, so that the second term in the exponent vanishes, that is, $\lambda\mu + \frac{1}{2}\lambda^2\sigma^2 = 0$, we obtain the MG

$$U(t) = e^{\alpha X(t)}.$$

This is an example of what is called a *Wald's martingale*. Then for $\tau = \min\{t \geq 0 : X(t) \in \{a, -b\} | X(0) = 0\}$, we use optional sampling to obtain $E(U(\tau)) = 1$, or $e^{\alpha a}p_a + e^{-\alpha b}(1-p_a) = 1$; solving for p_a yields the result. ($U(t\wedge\tau)$ is bounded hence UI.) That $P(\tau < \infty) = 1$: We already know from Prop 1.3 that for any $x \neq 0$, $P(T_x < \infty) = 1$ for standard BM $B(t)$, and hence for $\sigma B(t)$ ($\sigma B(t) = x$ if and only if $B(t) = x/\sigma$). If $\mu < 0$ then $X(t) = \sigma B(t) + \mu t$ will hit $-b$ sooner than would $\sigma B(t)$; $\tau \leq T_{-b} < \infty$. Similarly, if $\mu > 0$ then $X(t) = \sigma B(t) + \mu t$ will hit a sooner than would $\sigma B(t)$; $\tau \leq T_a < \infty$. Thus in general, $\tau \leq \min\{T_{-b}, T_a\} < \infty$.

■

We also have (proofs omitted):

Proposition 1.5 *For BM with drift, $X(t) = \sigma B(t) + \mu t$, $\mu \neq 0$, if $\tau = \min\{t \geq 0 : X(t) \in \{a, -b\} | X(0) = 0\}$, then*

$$E(\tau) = \frac{a(1 - e^{\frac{2\mu b}{\sigma^2}}) + b(1 - e^{\frac{-2\mu a}{\sigma^2}})}{\mu(e^{\frac{-2\mu a}{\sigma^2}} - e^{\frac{2\mu b}{\sigma^2}})}$$

What about $T_x = \min\{t > 0 : X(t) = x\}$? If the drift is negative, then we already know that for $x > 0$, the BM might not ever hit x; $P(T_x = \infty) = P(M < x) = 1 - e^{-\alpha x} > 0$. But if the drift is positive, x will be hit with certainty (because this is so even when $\mu = 0$; Proposition 1.3). In this case the mean is finite:

Proposition 1.6 *For BM with positive drift,*$X(t) = \sigma B(t) + \mu t$, $\mu > 0$, *if* $T_x = \min\{t \geq 0 : X(t) = x | X(0) = 0\}$, *then*

$$E(T_x) = \frac{x}{\mu}, \ x > 0.$$

Note how, as $\mu \to 0$, $E(T_x) \to \infty$, and this agrees with our previous calculation (Corollary 1.1) that $E(T_x) = \infty$ when $\mu = 0$ (even though $P(T_x < \infty) = 1$).