1 Fund theorems

In the Markowitz problem, we assumed that all n assets are risky; $\sigma_i^2 > 0,\ i \in \{1, 2, \dots, n\}$. This lead to the efficient frontier as a curve starting from the minimum variance point. We learned that in this case, this entire curve can be generated from just two distinct portfolios (*funds*) each with points on the curve. This is known as the *two-fund theorem* and will be reviewed again in the next section. Then, we will explore what happens when we allow one of the assets to be risk-free, and show that then the efficient frontier is simply a line connecting the risk-free asset to a particular fund of the risky assets. This is called the *one-fund theorem*, and it will be presented too.

1.1 Two-fund theorem

In Lecture Notes 3, Section 1.8 we learned that the entire efficient frontier can be generated from only two portfolios (funds). In other words if we let $\mathbf{w}^1 = (\alpha_1^1, \alpha_2^1, \dots, \alpha_n^1)$ be a solution to the Markowitz problem for a given expected rate of return $\overline{r}^1$, and $\mathbf{w}^2 = (\alpha_1^2, \alpha_2^2, \dots, \alpha_n^2)$ be a solution to the Markowitz problem for a different given expected rate of return $\overline{r}^2$, then for any number α, the new portfolio $\alpha\mathbf{w}^1 + (1-\alpha)\mathbf{w}^2$ is itself a solution to the Markowitz problem for expected rate of return $\alpha\overline{r}^1 + (1-\alpha)\overline{r}^2$. As α varies, $\overline{r} = \alpha\overline{r}^1 + (1-\alpha)\overline{r}^2$ takes on all feasible values for expected rate of return; thus all solutions to the Markowitz problem can be constructed. This is known as *the two-fund theorem.*

Treating each of the two fixed distinct solutions as portfolios and hence as "assets" in their own right, we conclude that we all can obtain any desired investment performance by investing in these two assets only. The idea is to think of each of these two assets as mutual funds, and create your investment by investing in these two funds only. In practice, one might first solve for the minimum variance portfolio as one such solution to be used as one of the two funds.

1.2 Allowing for a risk-free asset

In the previous analysis, we assumed all assets were risky. In reality there is always the opportunity to borrow/lend money at some fixed interest rate r. Lending refers to (say) the buying of bond or placing cash in a savings account, whereas borrowing refers to taking out a loan. For simplicity of analysis, we will imagine some fixed such rate r_f for both borrowing and lending. This type of risk-free investment can simply be treated as yet another kind of asset to be used in a portfolio. A positive weight refers to lending and a negative weight to borrowing: If you borrow x_0 at time $t = 0$, then you must return $x_1 = x_0(1+r_f)$ at time $t = 1$ (or $x_0e^{r_f}$ if compounding continuously). This risk-free asset has $\sigma^2 = 0$ and $\overline{r} = r_f$. Since $\sigma^2 = 0$, our previous analysis for finding the minimum variance portfolio makes no sense: The minimum variance is zero and can be obtained by investing all of your resources in the risk-free asset. But the risk-free asset has a lower rate of return than the risky assets which is why one would choose a portfolio combining both types of assets: the risk-free asset helps keep the risk down, whereas the risky assets help drive the expected rate of return up. Mixing the two still yields an efficient frontier in which a desired expected rate of return can be obtained with minimum variance, but, as we shall see, this new effficient frontier turns out to be a line; its analysis is thus quite easy, we discuss all this next.

1.3 New efficient frontier is a line

Consider n assets denoted by $A_1, \ldots, A_n$ with rates of return r_i having mean $\overline{r}_i$ and variance σ_i^2. Now suppose the risk-free asset with (deterministic) rate r_f, is joined in. We denote this asset by A_0.

How does this added risk-free asset effect the feasible region and efficient frontier? We will refer to the feasible region and efficient frontier of only the n risky assets as the *old* feasible region and the *old* efficient frontier, and the feasible region and frontier including A_0 as the *new*. Recall that the feasible region is by definition all achievable points $(\sigma, \overline{r})$ in the two-dimensional $\sigma - \overline{r}$ plane. By achievable we mean obtained by some portfolio. We will use $(\beta_1, \ldots, \beta_n)$ to denote a portfolio of the n risky assets; $\beta_1 + \cdots + \beta_n = 1$, and we will refer to any such portfolio as a *fund*. We will use $(\alpha_0, \alpha_1, \ldots, \alpha_n)$ to denote a portfolio of all $n+1$ assets; $\alpha_0 + \alpha_1 + \cdots + \alpha_n = 1$.

Clearly the new feasible region contains the old since by choosing $\alpha_0 = 0$ we obtain all points in the old.

Since $\alpha_0 + \alpha_1 + \ldots + \alpha_n = 1$, we can, noting that $1 - \alpha_0 = \alpha_1 + \ldots + \alpha_n$, re-write the portfolio as

$$(\alpha_0, (1-\alpha_0)(\beta_1, \ldots, \beta_n)),$$

where

$$\beta_i = \frac{\alpha_i}{1-\alpha_0}.$$

Since $\beta_1 + \cdots + \beta_n = 1$ we conclude that the portfolio has been re-written as a portfolio of only two "assets": A_0 and the given fund $(\beta_1, \ldots, \beta_n)$. The weights are α_0 and $1-\alpha_0$. On the other hand, every fund $(\beta_1, \ldots, \beta_n)$ can be viewed as an asset and then used to construct a portfolio of itself with A_0. We conclude that the collection of all "two-asset" portfolios made up of A_0 and a fund is the same as all portfolios of $A_0, \ldots, A_n$. So we can determine the new feasible set and efficient frontier by looking at the $(\sigma, \overline{r})$ points of all the two-asset (A_0, fund) portfolios.

Each fund has its own rate of return $\beta_1 r_1 + \cdots + \beta_n r_n$ and hence its own mean

$$m = \sum_{i=1}^{n} \beta_i \overline{r}_i$$

and variance

$$\gamma^2 = Var(\beta_1 r_1 + \cdots + \beta_n r_n) = \sum_{i,j=1}^{n} \beta_i \beta_j \sigma_{ij}.$$

Thus a portfolio $(\alpha_0, \alpha_1, \ldots, \alpha_n)$ has mean and variance of the form

$$\begin{aligned} \overline{r} &= \alpha_0 r_f + (1-\alpha_0)m \\ \sigma^2 &= (1-\alpha_0)^2 \gamma^2. \end{aligned}$$

Risk-free A_0 did not contribute to the variance since r_f is deterministic (a constant); $\rho = 0$ between r_f and any r_i. The portfolio thus yields point

$$(\sigma, \overline{r}) = (|1-\alpha_0|\gamma, \alpha_0 r_f + (1-\alpha_0)m).$$

For $\alpha_0 \leq 1$, $|1-\alpha_0| = 1-\alpha_0$ so the point can be re-written as

$$(1-\alpha_0)(\gamma, m) + \alpha_0(0, r_f).$$

As α_0 varies from 1 down to 0, the point spans out the line connecting point $(0, r_f)$ to point (γ, m), corresponding to the two extremes of investing all in A_0 or all in the fund. The line is given by the equation

$$\overline{r} = \frac{(m - r_f)}{\gamma}\sigma + r_f,$$

and has slope

$$\frac{(m - r_f)}{\gamma}. \tag{1}$$

Assuming that $m > r_f$ (as it must be for a rational investor; why invest in the fund otherwise?) the line has a positive slope and continues off to $+\infty$ as $\alpha \to -\infty$ (this corresponds to borrowing more and more of the risk-free asset so as to invest it in the fund). By choosing funds yielding a higher slope, we get a more efficient line (higher rate of return for the same given variance). From (1) we see that the slope can be made steepest by choosing funds with points lying on the old efficient frontier (smallest variance γ for a given mean m). Thus the line from $(0, r_f)$ to that point $F = (\gamma, m)$ which yields a line tangent to the old efficient frontier is the most efficient (see Figure 6.14 on Page 167 in the Text). We thus conclude that:

> *The new efficient frontier is the line connecting the point $(0, r_f)$ to the unique*[1] *point F (on the old efficient frontier) yielding a line tangent to the old efficient frontier.*

The beauty of this is that in an open market (everyone has the same assets to choose from and the same r_f) we conclude that every individual's portfolio can be obtained as a mixture of the same unique fund F and the risk-free asset. (All that differs are the weights for the mixture; different people have a different risk tolerance.) This is called the *one-fund theorem*

Note that we need not worry about the $\alpha_0 > 1$ case since this yields a line (starting at $(0, r_f)$) with negative slope:

$$\overline{r} = \frac{(r_f - m)}{\gamma}\sigma + r_f.$$

This is the lower boundary of the new feasible region and involves shorting of the fund so as to invest in the risk free asset; an inefficient (and irrational) thing to do since it raises the variance while lowering the expected rate of return.

It is apparent that we must figure out how to compute this special fund F, we do so next.

[1] Uniqueness is ensured if no two of the n risky assets are perfectly correlated; e.g. as long as $|\rho_{ij}| < 1$ for all $i \neq j$.

1.4 Determining F

To find F we simply need to find the fund $(\beta_1, \ldots, \beta_n)$ that corresponds to the pair (γ, m) that maximizes the slope (1). In other words we must maximize the function

$$f(\beta_1, \ldots, \beta_n) = \frac{(m - r_f)}{\gamma},$$

where

$$m = \sum_{i=1}^{n} \beta_i \overline{r}_i, \tag{2}$$

$$\gamma = \sqrt{Var(\beta_1 r_1 + \cdots + \beta_n r_n)}$$

$$= \Big(\sum_{i,j=1}^{n} \beta_i \beta_j \sigma_{ij} \Big)^{1/2}. \tag{3}$$

Since the β_j sum to 1, $r_f = \beta_1 r_f + \cdots + \beta_n r_f$ and we can re-write f as

$$f(\beta_1, \ldots, \beta_n) = \frac{\sum_{i=1}^{n} \beta_i (\overline{r}_i - r_f)}{\left(\sum_{i,j=1}^{n} \beta_i \beta_j \sigma_{ij} \right)^{1/2}}. \tag{4}$$

Noting that for any $c > 0$, replacing β_i by $c\beta_i$ would not change the value of f (c would cancel from numerator and denominator), we conclude that the constraint $\beta_1 + \cdots + \beta_n = 1$ can be dealt with later by normalizing; we need not use a Lagrange multiplier to accommodate this constraint. The differentiation, $\frac{\partial f}{\partial \beta_i} = 0, \ i \in \{1, \ldots n\}$, yields n linear equations

$$\sum_{j=1}^{n} v_j \sigma_{ji} = \overline{r}_i - r_f, \ i \in \{1, \ldots n\},$$

where $v_i = c\beta_i$ with (unknown) constant c given by

$$c = \frac{\sum_{i=1}^{n} \beta_i (\overline{r}_i - r_f)}{\left(\sum_{i,j=1}^{n} \beta_i \beta_j \sigma_{ij} \right)}, \tag{5}$$

where the β_i are from the optimal solution. Summarizing:

Theorem 1.1 *The fund $F = (\beta_1, \ldots, \beta_n)$ in the one-fund theorem is given by*

$$\beta_i = \frac{v_i}{\sum_{j=1}^{n} v_j}, \ i \in \{1, 2 \ldots n\},$$

where $(v_1, \ldots, v_n)$ is the solution to the set of n linear equations

$$\sum_{j=1}^{n} v_j \sigma_{ji} = \overline{r}_i - r_f, \ i \in \{1, \ldots n\}.$$

The equations are particularly simple to solve when all assets are uncorrelated, for then they reduce to

$$v_i \sigma_i^2 = \overline{r}_i - r_f,\ i \in \{1, \dots n\},$$

with solution

$$v_i = \frac{\overline{r}_i - r_f}{\sigma_i^2},$$

yielding weights

$$\beta_i = \frac{\frac{\overline{r}_i - r_f}{\sigma_i^2}}{\sum_{j=1}^n \frac{\overline{r}_j - r_f}{\sigma_j^2}}.$$