Elliptic Partial Differential Equations

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Contents

1 Introduction and Acknowledgments

Second order elliptic partial differential equations are fundamentally modeled by Laplace's equation $\Delta u = 0$. This thesis begins with trying to prove existence of a solution u that solves $\Delta u = f$ using variational methods. In doing so, we introduce the theory of Sobolev spaces and their embeddings into L^p and $C^{k,\alpha}$. We then move on to applying our techniques to a non-linear elliptic equation on a compact Riemannian manifold. We introduce the method of continuity along the way to provide another way of solving the equation. We move onto proving Schauder estimates for general elliptic equations in divergence form: $\partial_i(a_{ij}\partial_j u) + c(x)u = f$ with various assumptions on a, c, and f. We conclude our study of equations in divergence form by proving the Harnack Inequality using Moser iteration. Personally I would have liked to have proved the Harnack inequality in my own flavor, but due to lack of time, I had to follow very closely the proof given in [1].

The second half of the thesis revolves around equations in non-divergence form: $a^{ij}u_{ij} = 0$. As a disclaimer, I wrote the second half separately from the first and so my notations change heavily. We first start with the proof of the ABP maximum principle which is used heavily in, not only the proof for the Harnack inequality for non-divergence equations, but for the section on curved $C^{1,\alpha}$ domains. In that section I go over a completely new way of getting regularity estimates: approximation by polynomials. We first show how this can be used for $\Delta u = f$ and then for general elliptic operators. We conclude the paper by introducing Krylov's regularity results for flat domains and generalize it to curved domains whose boundaries look locally like the graph of a $C^{1,\alpha}$ function.

As one more disclaimer, I was not able to prove every single detail in this book due to lack of time. I leave tiny bits and pieces as exercises for the reader, but the overwhelming majority is proved in rigorous detail.

The writing of this paper was a long and arduous process. It grew out of many discussions with Professor Daniela De Silva and Professor Ovidiu Savin. I am very grateful for their insights and *very very* patient guidance. I would not have been able to set my path as a mathematician without them. I'd like to give thanks to Professor Michael Weinstein for being an exceptional guidance counselor and instructor whom I've learned a lot from about singular integrals, methods of characteristics, and graduate schools. I'd like to give special thanks to Professor Duong Phong, whose Analysis II class helped refine the details of this thesis.

2 Variational Methods and Sobolev Embedding Theorems

2.1 Laplacian

We begin with the simplest problem. Let $\Omega \subset\subset \mathbb{R}^n$ be a bounded domain, and f a function on Ω. We wish to find a u such that

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

The way we will find the solution to this problem is by finding the minimum for a specific functional. This is the idea: let $g(x)$ be a function on $\Omega \subset \mathbb{R}$ such that $\exists G(x)$ on Ω with

$$G'(x) = g(x)$$

and we want to find an x_0 with $g(x_0) = 0$. Then one way to approach this problem is to find an $x_0 \in \Omega$ such that $G(x) \geq G(x_0)$ $\forall x \in \Omega \implies G'(x_0) = 0 \implies g(x_0) = 0$. By comparisons, $g(x_0) = 0$ would be the equivalent of $\Delta u = 0$. So now we try to find the equivalent of G. Define the functional

$$I(u) = \frac{1}{2}\int_\Omega |Du|^2 + \int_\Omega fu.$$

This functional is going to be defined for $u \in W^{1,2}(\Omega)$. Now lets go over some definitions.

Definition 2.1. Let B be a normed Banach space. Then its completion $\overline{B} := \big\{\{u_k\} \subset B,\ u_k \text{ is cauchy}\big\}$.

Example 2.2. $\overline{\mathbb{Q}} = \mathbb{R}$.

Example 2.3. $\overline{\{C_0^\infty(\Omega), \|u\|_p < \infty\}} = L^p(\Omega)$ (modulo the equivalence that $f \equiv g$ if $f \neq g$ on a set of measure zero).

Example 2.4. $W_0^{1,2}(\Omega) = \{u_k \in C_0^\infty, \|u_k - u_l\|_2 \to 0, \|Du_k - Du_l\|_2 \to 0\}$. Also, $W_0^{1,2}(\Omega) = \{u \in L^2(\Omega), L^2(\Omega) \ni u = \lim_{k\to\infty} u_k, \|Du_k - Du_l\|_2 \to 0, Du_k \to v \in L^2.\}$.

Now we go back to our question: does there exist a $u_0 \in W_0^{1,2}(\Omega)$ such that $I(u) \geq I(u_0)$ for any $u \in W_0^{1,2}(\Omega)$? We will begin to show this in two steps. Our first is to show that our functional is bounded below. We will show that $\exists C > 0$ such that $I(u) \geq -C$ for any $u \in W_0^{1,2}(\Omega)$. This will at least give us a starting point to find a minimum because we no longer have the ambiguity of $I(u)$ exploding to $-\infty$. Next, assuming this, if $I(u)$ did have a minimum, then $\min I(u) < \infty$. Now pick a minimizing sequence $\{u_j\}$ such that $I(u_j) \to \min I(u)$. The reason why this can be picked will be shown later. We wish to show that $u_j \to u_0$ and $I(u_0) = \min I(u)$.

Claim. We claim that $\exists C > 0$ such that $I(u) \geq -C$ $\forall u \in W_0^{1,2}(\Omega)$.

First recall the definition of I:

$$I(u) = \frac{1}{2}\int_\Omega |Du|^2 dx + \int_\Omega fu dx.$$

Then applying Hölder's inequality and Cauchy's inequality, we have

$$\begin{aligned} I(u) &\geq \frac{1}{2}\int_\Omega |Du|^2 dx - \|f\|_2\|u\|_2 \\ &\geq \frac{1}{2}\int_\Omega |Du|^2 dx - \left(\frac{\epsilon}{2}\|u\|_2^2 + \frac{1}{2\epsilon}\|f\|_2^2\right). \end{aligned}$$

Now note that we would be done with our claim if we show that $\frac{1}{2}\|Du\|_2^2 \geq \frac{\epsilon}{2}\|u\|_2^2$. This is saying that the gradient controls our function u. But we are in luck because we choose u to have compact support, so it is zero on the boundary. Then this implies that the gradient not only approximates values near u, but tells us what they are.

Lemma 2.5. *There exists* $\epsilon = \epsilon(\Omega)$ *such that* $\frac{\epsilon}{2}\|u\|_2^2 \leq \frac{1}{4}\|Du\|_2^2$.

Proof. First note that this will imply

$$I(u) \geq \int_\Omega \left(\frac{1}{4}|Du|^2 - \frac{1}{2\epsilon}f^2\right) dx.$$

Now we move onto a claim:

Claim. Let Ω be convex, and bounded in $\mathbb{R}^n$. Then I will show that $\forall u \in C^\infty(\Omega)$, $\forall x \in \Omega$, then

$$\int_\Omega |u(x) - u(y)|\, dy \leq \frac{(\operatorname{diam}\Omega)^n}{n}\int_\Omega \frac{|Du(y)|}{|x-y|^{n-1}}\, dy.$$

Proof of Claim. Let $x, y \in \Omega$ be arbitrary, and denote $r = |x - y|$. Let $\omega = (y-x)/|x-y|$ be the unit vector in this direction. Then we have that

$$\begin{aligned}
u(y) - u(x) &= \int_0^r \frac{d}{dt}u(x+t\omega)\, dt \\
&= \int_0^r \sum_{i=1}^n \frac{\partial u}{\partial x_i}(x+t\omega)\omega_i\, dt \\
|u(x) - u(y)| &\leq \int_0^r |Du(x+t\omega)||\omega|\, dt \\
\int_\Omega |u(x)-u(y)|\, dy &\leq \int_\Omega \int_0^r |Du(x+t\omega)|\, dt\, dy.
\end{aligned}$$

We transform the RHS of this integral inequality using the polar transformation centered at x: $y \mapsto (r, \omega)$. Let ℓ_ω denote the distance from x to $\partial\Omega$.

$$\begin{aligned}
\int_\Omega |u(x)-u(y)|\, dy &\leq \int_{S^{n-1}}\int_0^{\ell_\omega}\left[\int_0^r |Du(x+t\omega)|\, dt\right] r^{n-1} dr\, d\sigma(\omega) \\
&= \int_{S^{n-1}}\int_0^{\ell_\omega}\left[\int_t^r r^{n-1}\, dr\right]|Du(x+t\omega)|\, dt\, d\sigma(\omega) \\
&\leq \int_{S^{n-1}}\int_0^{\ell_\omega}\frac{\ell_\omega^n}{n}|Du(x+t\omega)| dt\, d\sigma(\omega) \\
&\leq \frac{(\operatorname{diam}\Omega)^n}{n}\int_{S^{n-1}}\int_0^{\ell_\omega}|Du(x+t\omega)|\, dt\, d\sigma(\omega) \\
&= \frac{(\operatorname{diam}\Omega)^n}{n}\int_{S^{n-1}}\int_0^{\ell_\omega}\frac{|Du(x+t\omega)|}{t^{n-1}}t^{n-1}\, dt\, d\sigma(\omega) \\
&= \frac{(\operatorname{diam}\Omega)^n}{n}\int_\Omega \frac{|Du(x+t\omega)|}{t^{n-1}}\, dy \\
&= \frac{(\operatorname{diam}\Omega)^n}{n}\int_\Omega \frac{|Du(y)|}{|x-y|^{n-1}}\, dy.
\end{aligned}$$

□

With the proof of the claim out of the way, we can prove the following corollary:

Corollary 2.6. *Let* $u \in C_0^\infty(\Omega)$. *Then for any* $x \in \mathbb{R}^n$, *we have*

$$|u(x)| \leq c_n \int_{\mathbb{R}^n} \frac{|Du|}{|x-y|^{n-1}}\, dy.$$

Proof of Corollary. We begin by introducing a little bit of notation:

$$\bar{u}_\Omega(y) := \fint u(y)\,dy.$$

Then we have for $\Omega = B_R(0)$

$$\begin{aligned}
|u(x) - \bar{u}_{B_R(0)}| &= \frac{1}{|B_R(0)|}\left|\int_{B_R(0)} (u(x) - u(y))\,dy\right| \\
&\leq \frac{1}{|B_R(0)|}\int_{B_R(0)} |u(x) - u(y)|\,dy \\
&\leq \frac{1}{|B_R(0)|}\frac{(2R)^n}{n}\int_{B_R(0)} \frac{|Du(y)|}{|x-y|^{n-1}}\,dy \\
&= c_n \int_{B_R(0)} \frac{|Du(y)|}{|x-y|^{n-1}}\,dy.
\end{aligned}$$

Notice that we will be done with our corollary if we show that $\bar{u}_{B_R(0)} \to 0$ as $R \to \infty$. But this is where we use the fact that u has compact support:

$$\bar{u}_{B_R(0)} = \fint_{B_R(0)} \frac{|Du(y)|}{|x-y|^{n-1}}\,dy = \fint_{\mathbb{R}^n} \frac{|Du(y)|}{|x-y|^{n-1}}\,dy \to 0 \text{ as } R \to \infty.$$

□

Now that we are done with this corollary, we will quote a lemma that will be proved later:

Lemma 2.7 (Estimates for Integral Operator)**.** *Assume*

$$|u(x)| \leq \int K(x,y)\cdot|v(y)|\,dy.$$

Then for any $1 \leq p \leq \infty$*, we have*

$$\int |u(x)|^p\,dx \leq A\int |v(y)|^p\,dy$$

where

$$A = \max\left\{\sup_x \int K(x,y)\,dy, \sup_y \int K(x,y)\,dx.\right\}.$$

Applying this lemma to Corollary 2.6 gives us $\|u\|_p \leq C\|Du\|_p$ for $u \in C_0^\infty(\Omega)$ (and in fact for $u \in W_0^{1,2}(\Omega)$ by approximations). We are finally done with our Lemma 2.5. □

Now that we are done with the first part of the problem, we go back to the infimum question. Assume we know that $\inf_{u\in W_0^{1,2}(\Omega)} I(u) > -\infty$ and pick a minimizing sequence, i.e. $u_k \in W_0^{1,2}(\Omega)$ such that $I(u_k) \to \inf I(u)$. By approximations, we may assume $u_k \in C_0^\infty(\Omega)$. The question now is do the u_k's converge? To do this we have to go through a few things:

Claim. I claim that $\{u_k\}$ is a bounded sequence, i.e. $\exists C > 0$ that is independent of k such that $\|u_k\|_2 \leq C, \|Du\|_2 \leq C$.

Proof of Claim. The fact that u_k is a minimizing sequence of $I(u)$ implies that $\forall k$,

$$C_1 \geq I(u_k) \geq \frac{1}{4|}\|Du\|_2^2 - \frac{1}{2\epsilon}\|f\|_2^2.$$

We can bring the f term to the other side and get $C_2 \geq \|Du\|_2$. Poincaré's inequality then implies that $C_3 \geq \|u\|_2^2$. □

Ok now we have that $\{u_k\}$ is a bounded sequence. Recall that in a finite dimensional vector space, boundedness implies pre-compactness. However, our functional space $W_0^{1,2}(\Omega)$ is infinite dimensional, so we need to find a weaker substitute called "weak compactness."

Definition 2.8. Let B be a Banach space and B^* be its dual space (space of bounded linear functionals), i.e., $\ell \in B^*$ is linear and $|\langle l, u\rangle| \le C\|u\|$ for any $u \in B$. Let $\{u_k\} \subset B$. Then we say that $u_k \rightharpoonup u$ weakly if $\forall \ell \in B^*$,

$$\langle \ell, u_k\rangle \to \langle \ell, u\rangle.$$

It is easy to see that if $u_k \to u$ in the usual sense, then $u_k \rightharpoonup u$ weakly. Let $\ell \in B^*$ so we have $|\langle \ell, u_k - u\rangle| \le C\|u_k - u\| \to 0$. The converse is not true. For an easy example, let u_k be an orthonormal basis in an infinite dimensional Hilbert Space. Then $\|u_k - u_l\| = \sqrt{2}$, so we obviously do not have convergence. On the other hand, from Parseval's formula, we have

$$\sum_{u_k} |\langle \ell, u_k\rangle|^2 = \|\ell\|^2 \to 0.$$

Then this implies that $\langle l, u_k\rangle \to 0\ \forall \ell \Longrightarrow u_k \rightharpoonup 0$ but $u_k \not\to 0$. Now we go to a result from analysis:

Theorem 2.9 (Bamach-Alaoglu)**.** *Let B be a reflexive separable Banach space. Then for any bounded sequence $u_k \subset B$, there exists a subsequence $u_{k_l} \subset B$ such that $u_{k_l} \rightharpoonup u$ weakly.*

We will apply this to our problem. We have that $I(u_k) \to \inf I(u)$ and $\|u_k\|_2 + \|Du\|_2 \le C$. By passing through our subsequence, we have a $u_\infty \in W_0^{1,2}(\Omega)$ such that $u_k \rightharpoonup u_\infty$ and $Du_k \rightharpoonup Du_\infty$. Now the question that we have to answer is if u_∞ is the minimum that we seek after. However, we **can't** say that $u_k \rightharpoonup u_\infty, Du_k \rightharpoonup Du_\infty$ implies $I(u_k) \to I(u_\infty)$. The problem with this is that $I(u)$ is not continuous with respect to weak convergence. However, it is lower semi-continuous! i.e., $u_k \rightharpoonup u$ weakly in L^2 implies that $\|u\|_2 \le \liminf_{k\to\infty} \|u_k\|_2$. Here is the proof of this:

Proof.

$$\|u\|_2^2 = \int u\bar{u} = \lim_{k\to\infty} \int u\bar{u}_k \le \liminf_{k\to\infty} \|u\|_2 \|u_k\|_2.$$

In particular,

$$\begin{aligned} I(u_\infty) &= \frac{1}{2}\|Du_\infty\|_2^2 + \langle f, u_\infty\rangle \\ &\le \liminf_{k\to\infty} \left(\frac{1}{2}\|Du_k\|_2^2 + \langle f, u_k\rangle\right) \\ &= \liminf_{k\to\infty} I(u_k). \end{aligned}$$

However, since the following inequality is automatic,

$$\inf I(u) \le I(u_\infty) \le \liminf_{k\to\infty} I(u_k),$$

and we have that $I(u_\infty) = \inf I(u)$. □

We used the Banach-Alaoglu Theorem: if B is a reflective and separable Banach space and $\{u_j\}$ is a bounded sequence, then there exists a weakly precompact sequence. The missing steps we have are to show the proof of this, and completely show that $\|u\|_2 \le C\|Du\|_2$ for $u \in C_0^\infty(\Omega)$. The proof of Theorem 2.9 will be left as a black box. We move onto the proof of Lemma 2.7.

Proof of Lemma 2.7. Our main tool will be Hölder's inequality. Choose p and p^* such that $1/p + 1/p^* = 1$. Then we have

$$\begin{aligned}
|u(x)| &\leq \int |K(x,y)|^{\frac{1}{p^*}} |K(x,y)|^{\frac{1}{p}} |v(y)| dy \\
&\leq \left[\int |K(x,y)| dy\right]^{\frac{1}{p^*}} \left[\int |K(x,y)||v(y)|^p dy\right]^{\frac{1}{p}} \\
\int |u(x)|^p \, dx &\leq \int \left(\left[\int |K(x,y)| dy\right]^{\frac{p}{p^*}} \left[\int |K(x,y)||v(y)|^p dy\right] \right) dx \\
&\leq \sup_x \left[\int |K(x,y)| dy\right]^{\frac{p}{p^*}} \int \left(\int |K(x,y)||v(y)|^p \, dy \right) dx.
\end{aligned}$$

Since $\int |v(y)|^p dy$ is a constant in terms of x, we can take it out of the integral of the RHS. Then switching the order of integration and applying the same bounding trick we have

$$\begin{aligned}
\int |u(x)|^p \, dx &\leq \sup_x \left[\int |K(x,y)| dy\right]^{\frac{p}{p^*}} \sup_y \left[\int |K(x,y)| dx\right] \int |v(y)|^p dy \\
\|u\|_p^p &\leq A^{\frac{p}{p^*}+1} \|v\|_p^p \\
\|u\|_p &\leq A\|v\|_p.
\end{aligned}$$

□

Notice that this lemma would be pointless if $A = \infty$ because then we learn nothing know with $\|u\|_p \leq \infty$. We are in luck because we can actually deduce that A is finite in our case! The reason for this is because $|x-y|^{n-1}$ yields a singularity of dimension strictly less than n when taking the supremum over the y's, and so it integrable. Finally since x is over a set of compact support, it is bounded by a constant. Hence for $u \in C_0^\infty(\Omega) \implies \|u\|_p \leq C\|Du\|_p$. Now we are *finally* done with our result from earlier.

Now we propose a question: how do we sharpen our estimates? A better way to visualize this question is by noticing that the Kernel is integral for *any* power less than n. We actually used the worst power in our previous proof. The answer to our question comes from the following inequalities:

Theorem 2.10 (Sobolev Inequality). *Let $\Omega \subset\subset \mathbb{R}^n$ and $u \in C_0^\infty(\Omega)$. Then for any $p < n$ we have*

$$\|u\|_{\frac{np}{n-p}} \leq C_{n,p}\|Du\|_p.$$

Theorem 2.11 (Trudinger Ineqality). *For $p = n$, there exists constants $K, C > 0$ such that*

$$\int_\Omega \exp\left(\frac{|u(x)|}{K\|Du\|_n}\right)^{\frac{n}{n-1}} \leq C.$$

Theorem 2.12 (Morrey's Inequality). *For any $p > n$, we have*

$$\|u\|_{C^\alpha(\Omega)} \leq C\|Du\|_p$$

where $\alpha = 1 - n/p$ and

$$\|u\|_{C^\alpha(\Omega)} = \sup_\Omega |u| + \sup_{x \neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

Lets first think of why the first inequality is a better estimate than the Poincaré inequality that we have. It is better because $np/(n-p) > p$ and we know that the L^p norms grow bigger as p increases. This means that we have essentially closed the gap between u and Du that we got from Lemma 2.7. Ok now lets try to prove this. The proof is going to follow the same theme from Lemma 2.7's proof. From Hölder's inequality we have

$$\begin{aligned}|u(x)| &\leq \int |K(x,y)|^{\alpha}\left(|K(x,y)|^{1-\alpha}|v(y)|^{1-\beta}\right)|v(y)|^{\beta}dy \\ &\leq \left(\int |K(x,y)|^{\alpha a}dy\right)^{1/a}\left[\int |K(x,y)|^{(1-\alpha)c}|v(y)|^{(1-\beta)c}dy\right]^{1/c}\left(\int |v(y)|^{\beta b}dy\right)^{1/b}\end{aligned}$$

where $1/a+1/b+1/c=1$. We need to choose our parameters wisely so that we have our desired estimates. One obvious constraint to put is $\beta b=p$ and $(1-\beta)c=p$ because we want $|v(y)|$ on the RHS to have powers of p. Raising everything to the power q and integrating gives us

$$\begin{aligned}\int |u(x)|^q dx &\leq \int\left(\left(\int |K(x,y)|^{\alpha a}dy\right)^{q/a}\left[\int |K(x,y)|^{(1-\alpha)c}|v(y)|^p dy\right]^{q/c}\left(\int |v(y)|^p dy\right)^{\beta q}\right)dx \\ &= \int\left(\left(\int |K(x,y)|^{\alpha a}dy\right)^{q/a}\left[\int |K(x,y)|^{(1-\alpha)c}|v(y)|^p dy\right]^{q/c}\|v\|_p^{\beta q}\right)dx.\end{aligned}$$

In order to make our calculations a little bit easier take $q=c$. Then we can do the following:

$$\begin{aligned}\int |u(x)|^q dx &\leq \int\left(\left(\int |K(x,y)|^{\alpha a}dy\right)^{q/a}\left[\int |K(x,y)|^{(1-\alpha)c}|v(y)|^p dy\right]\|v\|_p^{\beta q}\right)dx \\ &\leq \sup_x\left(\int |K(x,y)|^{\alpha a}dy\right)^{q/a}\int\left[\int |K(x,y)|^{(1-\alpha)c}|v(y)|^p dy\right]\|v\|_p^{\beta q}dx \\ &\leq \sup_x\left(\int |K(x,y)|^{\alpha a}dy\right)^{q/a}\sup_y\left[\int |K(x,y)|^{(1-\alpha)c}dx\right]\|v\|_p^{p+\beta q}\end{aligned}$$

Recall that so far $0<\alpha<1$ is arbitrary. Choose it so that $\alpha a=(1-\alpha)c$. Now lets play around with these parameters. Recall that we chose $q=c$ and $(1-\beta)c=p \implies \beta=1-p/q$. We can then plug this into $\beta b=p \implies b=p/\beta$ to get

$$b=\frac{p}{\beta}=\frac{p}{1-\frac{p}{q}}=\frac{pq}{q-p}.$$

Now that we have parameters b and c, we can plug this into $1/a+1/b+1/c=1$ to get the parameter a. After skipping some steps we see that $1/a=1-1/p$. Finally recall that we have $\alpha a=(1-\alpha)c$. plugging in our value for a and c and solving for α gives us

$$\alpha=\frac{q(p-1)}{p+pq-q}, \quad \alpha a=\frac{pq}{p+pq-q}.$$

When applied to our gradient estimates, this means we need the integral

$$\int |K(x,y)|^{\alpha a}dy$$

to be finite. Following our explanation after our proof of Lemma 2.7 tells us that we need the integral

$$\int\left(\frac{1}{|x-y|^{n-1}}\right)^{\frac{pq}{p+pq-q}}dy$$

to be finite. Comparing the powers would require

$$(n-1)\frac{pq}{p+pq-q}<n.$$

After playing around with this inequality we get

$$q < \frac{pn}{n-p}.$$

What suffices to show the full proof of Theorem 2.10 is that our coefficient in front of $\|Du\|_p$ must depend only on n and p. Additionally, setting $K(x,y) = |x-y|^{-n+1}$ in our generalization of Lemma 2.7 would mean that A would only be finite when

$$\sup_x \int \left(\frac{1}{|x-y|^{n-1}}\right)^{\frac{pq}{pq+p-q}} dx < \infty.$$

One can see that our kernel will be integrable $\iff (n-1)\frac{pq}{pq+p-q} < n \iff \frac{1}{p} - \frac{1}{q} < \frac{1}{n}$. We now prove the following general lemma where u no longer has compact support:

Lemma 2.13. *Assume $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$. If u satisfies*

$$\int_\Omega |u(x) - u(y)| dy \le \frac{(\operatorname{diam}\Omega)^n}{n} \int_\Omega \frac{|Du(y)|}{|x-y|^{n-1}} dy$$

then

$$\|u - u_\Omega\|_q \le c_n \left(\frac{1 + \frac{1}{q} - \frac{1}{p}}{\frac{1}{n} + \frac{1}{q} - \frac{1}{p}}\right)^{1+\frac{1}{q}-\frac{1}{p}} \frac{(\operatorname{diam}\Omega)^n}{|\Omega|^{1-\frac{1}{n}+\frac{1}{p}}} \|Du\|_p.$$

Proof. It suffices to show, after dropping some constants,

$$\left\{\sup_x \int \left(\frac{1}{|x-y|^{n-1}}\right)^{\frac{pq}{pq+p-q}} dx\right\}^{1+\frac{1}{q}-\frac{1}{p}} \le \left(\frac{1 + \frac{1}{q} - \frac{1}{p}}{\frac{1}{n} + \frac{1}{q} - \frac{1}{p}}\right)^{1+\frac{1}{q}-\frac{1}{p}} |\Omega|^{\frac{1}{n}-\frac{1}{p}+\frac{1}{q}}. \tag{1}$$

The way to do this is from this simple fact: $\forall 0 \le \mu < 1$,

$$\int_\Omega \frac{dy}{|x-y|^{\mu n}} \le c_n |\Omega|^{1-\mu} \frac{1}{1-\mu}.$$

The way to prove this fact is to show that it is true for balls and then for general Ω through "rearrangement inequalities." Then (1) easily follows. □

We have finally finished the proof of the Sobolev inequality. Observe now that $\frac{1}{p} - \frac{1}{q} < \frac{1}{n} \iff q < \frac{np}{n-p}$. We can ask the question if this holds for $q \le \frac{np}{n-p}$? We'd think that the answer is no! Before we do that we give an alternative proof for the Sobolev Inequality:

Theorem 2.14. *Let $u \in C_0^\infty(\Omega)$ and $1 \le p < n$. Then*

$$\|u\|_{\frac{np}{n-p}} \le C_s \|Du\|_p.$$

Proof. Consider first $p = 1$ and $u \ge 0$. Notice that we can write $u(x)$ in the following way:

$$u(x) = \int_0^\infty \chi_{\{u>t\}} dt.$$

This makes sense because

$$\chi_{\{u>t\}} = \begin{cases} 1 & u > t \\ 0 & u \le t \end{cases}$$

and so the RHS becomes $\int_0^u dt$. Since $p = 1$ our goal is to show that $\|u\|_{\frac{n}{n-1}} \le C\|Du\|_1$. Then we see

$$\|u\|_{\frac{n}{n-1}} \le \int_0^\infty \|\chi_{\{u>t\}}(\cdot)\|_{\frac{n}{n-1}} dt.$$

But the inside of this inequality is

$$\|\chi_{\{u>t\}}(\cdot)\|_{\frac{n}{n-1}} = \left(\int_\Omega \left(\chi_{\{u>t\}}(\cdot)\|_{\frac{n}{n-1}}\right)^{\frac{n}{n-1}} dx\right)^{\frac{n-1}{n}} = \left(\int_{\{u>t\}} dx\right)^{\frac{n}{n-1}} = (\mathrm{Vol}\{u>t\})^{\frac{n-1}{n}}$$

Notice that this has dimension of surface area because we can interpret Vol as having n dimensions, then we take the n^{th} root of it leaving one spacial dimension, and then we raise it to the $n+1^{\text{th}}$ dimension again. Then we can use the following isoperimetric inequality:

$$(\mathrm{Vol}\{u>t\})^{\frac{n-1}{n}} \leq C_S \,\mathrm{Area}(\partial\{u>t\}).$$

This gives us now the inequality

$$\|u\|_{\frac{n-1}{n}} \leq C_s \int_0^\infty \mathrm{Area}(\partial\{u>t\})dt.$$

Now we will apply iterated integrals using the coarea formula: let u be a real valued function that isn't constant. Then

$$dx = \frac{1}{|Du|} d\sigma_t dt \tag{2}$$

is the coarea formula. What we are doing is integrating the $\{u=t\}$ level set with the $d\sigma_t$ measure and then integrating with respect to dt. Applying this gives us

$$\begin{aligned}\|u\|_{\frac{n-1}{n}} &\leq C_s \int_0^\infty \mathrm{Area}(\partial\{u>t\})dt \\ &= C_s \int_0^\infty \left(\int_{u=t} d\sigma_t\right) dt \\ &= C_s \int_\Omega |Du| dx.\end{aligned}$$

We are done with the proof because for $p \neq 1$ we notice that $u \in L^p \iff u^p \in L^1$. □

With this being said we turn back to the Trudinger inequality: for $p=n$, there exists constants $K_n, C_n > 0$ such that

$$\int_\Omega \exp\left(\frac{|\Omega|}{K_n(\mathrm{diam}\,\Omega)^n}\frac{|u(x)-u_\Omega|}{\|Du\|_n}\right)^{\frac{n}{n-1}} dx \leq C_n|\Omega|.$$

Notice that this is stronger than what the Sobolev inequality says for $p<n$ because exponentials are always greater than polynomials, and so this is stronger than $\|u-u_\Omega\|_p < \infty$. However it is weaker than $\|u-u_\Omega\|_\infty \leq C$ because this deals with the supremum, which is the best bound we can have.

Proof of the Trudinger Inequality. We will recall the integral inequality

$$\|u-u_\Omega\|_q \leq c_n \left(\frac{1+\frac{1}{q}-\frac{1}{p}}{\frac{1}{n}+\frac{1}{q}-\frac{1}{p}}\right)^{1+\frac{1}{q}-\frac{1}{p}} \|Du\|_p \tag{3}$$

and drop terms involving Ω because in the end they're just constants. We expand the exponential as a power series to get

$$\begin{aligned}\int_\Omega \exp\left(\frac{1}{K_n}\frac{|u(x)-u_\Omega|}{\|Du\|_n}\right)^{\frac{n}{n-1}} dx &= \sum_{k=0}^\infty \frac{1}{k!}\int_\Omega \left(\frac{|u(x)-u_\Omega|}{K_n\|Du\|_n}\right)^{\frac{kn}{n-1}} dx \\ &= \sum_{k=0}^\infty \frac{1}{k!}\left(\frac{\|u-u_\Omega\|_{\frac{kn}{n-1}}}{K_n\|Du\|_n}\right)^{\frac{kn}{n-1}} \end{aligned} \tag{4}$$

Now we go back to (3) and plug in $p = n$ to get the following inequality:

$$\frac{\|u - u_\Omega\|_q}{\|Du\|_n} \leq c_n \left(q \left(1 + \frac{1}{q} - \frac{1}{n} \right) \right)^{1 + \frac{1}{q} - \frac{1}{n}}$$
$$\leq c_n q^{1 - \frac{1}{n}}.$$

This was verified on paper by playing little tricks. Anyway, we plug this into (4) with $q = \frac{kn}{n-1}$:

$$\begin{aligned}\int_\Omega \exp\left(\frac{1}{K_n} \frac{|u(x) - u_\Omega|}{\|Du\|_n} \right)^{\frac{n}{n-1}} dx &\leq \sum_{k=0}^\infty \frac{1}{k!} \left(\frac{1}{K_n} \left(c_n \frac{kn}{n-1} \right)^{1-\frac{1}{n}} \right)^{\frac{kn}{n-1}} \\ &= \sum_{k=0}^\infty \frac{1}{k!} \frac{1}{K_n}^{\frac{kn}{n-1}} \left(c_n \frac{kn}{n-1} \right)^k \\ &= \sum_{k=0}^\infty \frac{k^k}{k!} \left(\frac{n}{K_n^{\frac{n}{n-1}} (n-1)} \right)^k .\end{aligned}$$

Then something called Stirling's formula makes this converge. ☐

Recall that we have shown this basic inequality before: Consider p, q and assume $1 \leq p \leq q \leq \infty$ and $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$. Then

$$\|u - u_\Omega\|_q \leq c_n \left(\frac{1 + \frac{1}{q} - \frac{1}{p}}{\frac{1}{n} + \frac{1}{q} - \frac{1}{p}} \right)^{1 + \frac{1}{q} - \frac{1}{p}} \frac{(\operatorname{diam} \Omega)^n}{|\Omega|^{1 - \frac{1}{n} + \frac{1}{p}}} \|Du\|_p$$

for Ω convex and u satisfying one of our integral inequalities. Then certainly we have

$$|u - u_\Omega| \leq c_n \left(\frac{1 + \frac{1}{q} - \frac{1}{p}}{\frac{1}{n} + \frac{1}{q} - \frac{1}{p}} \right)^{1 + \frac{1}{q} - \frac{1}{p}} \frac{(\operatorname{diam} \Omega)^n}{|\Omega|^{1 - \frac{1}{n} + \frac{1}{p}}} \|Du\|_p \tag{5}$$

We have used this to show the Sobolev Inequality and Trudinger inequality. Now we prove Morrey's inequality.

Theorem 2.15 (Morrey's Inequality)**.** *Let $p > n, \alpha = 1 - \frac{n}{p}, \Omega = S^{n-1}$. Then*

$$\|u\|_{C^\alpha} \leq C \|Du\|_p.$$

Proof. Recall that

$$\|u\|_{C^\alpha} = \|u\|_\infty + \sup_{x \neq y} \frac{u(x) - u(y)}{|x - y|^\alpha}$$

where $0 < \alpha < 1$. It the suffices to show that each term is bounded by $\|Du\|_p$ i.e. $\|u\|_\infty \leq C\|Du\|_p$ and $[u]_{C^\alpha} \leq C\|Du\|_p$. Clearly the first part follows from (5) with $q = \infty$. Now we prove the second part. Fix $x, y, x \neq y$ and let $\delta = |x - y|$. Now define $\tilde{\Omega} := B_\delta(x) \cap B_\delta(y)$. Clearly this is convex. Then we have the following inequalities

$$\begin{aligned}|u(x) - u(y)| &\leq |u(x) - u_{\tilde{\Omega}}| + |u_{\tilde{\Omega}} - u(y)| \\ &\leq \frac{(\operatorname{diam} \tilde{\Omega})^n}{|\tilde{\Omega}|} \left(\int_{\tilde{\Omega}} \frac{|Du(z)|}{|x - z|^{n-1}} dz + \int_{\tilde{\Omega}} \frac{|Du(z)|}{|y - z|^{n-1}} dz \right)\end{aligned}$$

Now notice that $\operatorname{diam}(\tilde{\Omega}) \leq 2\delta$ and $|\tilde{\Omega}| \leq |B_\delta(x)|$. Finally since all the terms inside the integral are positive then we can write

$$|u(x) - u(y)| \leq C \left(\int_{B_{\delta(x)}} \frac{Du(z)}{|x - z|^{n-1}} dz + \int_{B_\delta(y)} \frac{|Du(z)|}{|y - z|^{n-1}} dz \right).$$

Now at this point we don't have to reinvent the wheel so we see now that our function satisfies the correct requirements of (5) so we can write

$$\begin{aligned}|u(x)-u(y)| &\le C\|Du\|\frac{\operatorname{diam}(B_\delta(x))^n}{|B_\delta(x)|^{1-\frac{1}{n}+\frac{1}{p}}} \\ &\le C\|Du\|_p(2\delta)^n\cdot(\omega_n\delta^n)^{\frac{1}{n}-\frac{1}{p}-1} \\ &\le C\|Du\|_p\delta^{1-\frac{n}{p}-n}\delta^n \\ &\le C\|Du\|_p\delta^\alpha\end{aligned}$$

And we are done with the proof. □

Theorem 2.16 (Sobolev Embedding Theorem)**.** *Let* $u \in W^{k,p}(\Omega)$. *Then* $\|u\|_{L^\infty(\Omega)} \le \|u\|_{W^{k,p}(\Omega)}$ *for* $1/p < k/n$.

Proof. Lets recall Morrey's inequality. This says that if $1/p > 1/n$, then

$$\|u\|_\infty + \sup_{x\neq y}\frac{|u(x)-u(y)|}{|x-y|^\alpha} = \|u\|_{C^\alpha} \le C\|Du\|_p$$

for $u \in C_0^\infty$ and $\alpha = 1 - n/p$.

Let $k = 1$. Then the theorem follows from Morrey's inequality because $1/p < 1/n$ and $\|u\|_\infty$ is obviously bounded by $\|u\|_{C^\alpha}$. Now assume the theorem holds for k and $\forall B$, consider $D^\beta u$ where $|\beta| \le k$. Set $v = D^\beta u$. Then by definition we have $\|Dv\|_p \le \|u\|_{W^{k+1,p}}$. Since by we are assuming $\frac{1}{p} < \frac{k+1}{n}$ pick some $\epsilon > 0$ such that $\frac{1}{p} = \frac{k+1}{n} - \epsilon$. Now recall the Sobolev inequality: $\|v\|_q \le \|Dv\|_p$ for $\frac{1}{p} - \frac{1}{q} < \frac{1}{n} \iff \frac{1}{p} - \frac{1}{n} \le \frac{1}{q}$. We will choose a q satisfying (for some $\epsilon' < \epsilon$)

$$\begin{aligned}\frac{1}{q} &= \frac{1}{p} - \frac{1}{n} + \epsilon' \\ &= \frac{k+1}{n} - \frac{1}{n} + \epsilon' - \epsilon \\ &= \frac{k}{n} + \epsilon' - \epsilon < \frac{k}{n}.\end{aligned}$$

Now we have $\|v\|_q \le \|u\|_{W^{k+1,p}}$ for $\frac{1}{q} < \frac{k}{n}$ and by definition $\|u\|_{W^{k,q}} \le \|u\|_{W^{k+1,p}}$ for $\frac{1}{q} < \frac{k}{n}$. And so our induction hypothesis tells us

$$\|u\|_\infty \le \|u\|_{W^{k,q}} \le \|u\|_{W^{k+1,p}}.$$

□

As a consequence we have then that if $u \in W_0^{k,p}(\Omega)$ then $u \in C^0(\Omega)$ if $1/p < k/n$. The way to see this is the following: Let $\{u_j\} \subset C_0^\infty(\Omega)$ and $u_j \to u$ with respect to $\|\cdot\|_{W^{k,p}(\Omega)}$. This of course exists because $W_0^{k,p}(\Omega) = \overline{\{C_0^\infty(\Omega)\|\|\cdot\|_{W^{k,p}(\Omega)} < \infty\}}$. Then we apply the Sobolev embedding theorem to $u_j - u_m$. Then we have that $\|u_j - u_m\|_{C^0} \le C\|u_j - u_m\|_{W^{k,p}(\Omega)} \to 0$. This implies that u_j converges uniformly and so $\lim u_j$ is continuous. In fact, $u_j \to u$ uniformly in the usual sense because uniform convergence implies that $u_j \to u$ in L^p. However, since $W^{k,p}(\Omega)$ convergence is stronger, we have $u_j \to u$ in the usual sense.

Lets summarize what we've done. Let $\Omega \subset\subset \mathbb{R}^n, f \in L^2(\Omega)$. Define

$$I(u) = \int_\Omega \left(\frac{1}{2}|Du|^2 + fu\right)dx$$

for $u \in W_0^{1,2}(\Omega)$. Then we showed that $\exists u_\infty \in W_0^{1,2}(\Omega)$ with $I(u_\infty) = \inf I(u)$. We will observe that the minimum of a functional $I(u)$ is going to be a generalized solution of the Euler-Lagrange equation for $I(u)$. The basis on which we will set our ground on is that if $x_\infty = \min_{x\in\Omega} f(x) \Longrightarrow f'(x_\infty) = 0$.

Let $\varphi \in C_0^\infty(\Omega)$ and consider for $t << 1$ the function $\mathbb{R} \ni t \mapsto A(t) = I(u_\infty + t\varphi)$. Then we see that $t = 0$ is going to be a minimum for $A(t) \Longrightarrow$

$$\begin{aligned}
0 = \frac{dA}{dt}\bigg|_{t=0} &= \frac{d}{dt}\bigg|_{t=0} \int \left(\frac{1}{2}|D(u_\infty + t\varphi)|^2 + f \cdot (u_\infty + t\varphi)\right) dx \\
&= \frac{d}{dt}\bigg|_{t=0} \int \left(\frac{1}{2}(|Du_\infty|^2 + 2tDu_\infty D\varphi + t^2|D\varphi|^2) + f \cdot (u_\infty + t\varphi)\right) dx \\
&= \int (Du_\infty D_\varphi + f\varphi)\, dx \\
&= \int \left(\sum_{j=1}^n \frac{\partial u_\infty}{\partial x_j}\frac{\partial \varphi}{\partial x_j} + f\varphi\right) dx.
\end{aligned}$$

Now if we assume temporarily that $u_\infty \in C^2(\Omega)$ then we are allowed to integrate by parts and get

$$0 = \int \left(-\sum \frac{\partial^2 u}{\partial x_j^2} + f\right) \varphi dx.$$

Since this is true for any φ then we have $-\Delta u_\infty + f = 0$, which is the Laplace equation that we wanted to solve from the first page.

2.2 A non-linear PDE and Method of Continuity

2.2.1 Non-linear variations on Manifolds

Before we can extend our current groundwork to general manifolds, we need to expand our theory to more general boundary values. Let $\mathbb{R}_+^{n+1} = \{x \in \mathbb{R}^{n+1} | x_{n+1} \geq 0\}$ and let $\Omega \subset\subset \mathbb{R}_+^{n+1}$. Assume $u \in C_0^\infty(\Omega)$. Then we will attempt to show

$$\|u(\cdot, 0)\|_{L^p(\mathbb{R}^n)} \leq C_\Omega \|\partial_{n+1} u\|_{L^p(\mathbb{R}_+^{n+1})}. \tag{6}$$

Pick $d > \operatorname{diam}\Omega$. Since u has compact support we have

$$\begin{aligned}
|u(x,0)| = |u(x,0) - u(x,d)| &= \left| -\int_0^d \frac{\partial u}{\partial x_{n+1}}(x, x_{n+1}) dx_{n+1} \right| \\
&\leq \left(\int_0^d \left|\frac{\partial u}{\partial x_{n+1}}(x, x_{n+1})\right|^p dx_{n+1}\right)^{1/p} \left(\int_0^d 1^q dx_{n+1}\right)^{1/q} \\
|u(x,0)|^p &\leq C \int_0^d \left|\frac{\partial u}{\partial x_{n+1}}(x, x_{n+1})\right|^p dx_{n+1} \\
\int |u(x,0)|^p dx &\leq C_\Omega \int \int_0^d \left|\frac{\partial u}{\partial x_{n+1}}(x, x_{n+1})\right|^p dx_{n+1} dx.
\end{aligned}$$

Taking the pth root gives us the desired claim.

We will now attempt to show that this inequality will allow us to define $u|_{\partial\Omega}$ for an arbitrary $u \in W^{1,p}(\Omega)$. Take $u_j \in C^\infty(\Omega)$ where $u_j \to u$ in $W^{1,p}$. We know this sequence converges because $W^{1,p}$ is the completion of C^∞ with respect to the p norm. Then by our inequality (6), we can see

$$\|(u_j - u_k)|_{\partial\Omega}\|_p \leq C\|u_j - u_k\|_{W^{1,p}} \to 0$$

by definition and so u_j is cauchy. But since L^p is complete we can then formally define

$$u|_{\partial\Omega} := \lim u_j|_{\partial\Omega}$$

where the limit is taken over the L^p norm. Recall that we still had Ω being some some simple semi-circle in $\mathbb{R}^{n+1}_+$.

Lets extend the notion of boundary values in more general Ω with $\Omega \in C^\infty$. Let $\tilde{\Omega}$ be a small subset of Ω such that $\tilde{\Omega} \cap \partial\Omega \neq \emptyset$. Let $v \in C_0^\infty(\tilde{\Omega})$. We will define (using norms) $v|_{\partial\tilde{\Omega}}$. Let $y \in \tilde{\Omega}$. Since the boundary is smooth we can map $\tilde{\Omega}$ into an upper half sphere as before with $y \mapsto x$. Of course, we can go backwards. So then we can say $v(y) = v(y(x)) =: u(x)$ and note that via our definitions, $v|_{\partial\tilde{\Omega}} = u(x,0)$ and from our previous observations we have

$$\begin{aligned} \|u(\cdot,0)\|_{L^p(\mathbb{R}^n)} &\leq C \left\| \frac{\partial u}{\partial x_{n+1}} \right\|_{L^p(\mathbb{R}^{n+1})} \\ &= C \left\| \frac{\partial}{\partial x_{n+1}} v(y(x)) \right\|_{L^p(\mathbb{R}^{n+1})} \\ &\leq C \sum_{l=1}^{n+1} \left\| \frac{\partial v}{\partial y^l} \right\|_{L^p(\mathbb{R}^{n+1})} \\ &\leq C \|v\|_{W^{1,p}(\mathbb{R}^{n+1})}. \end{aligned} \tag{7}$$

Noticed that (7) we did something very fishy that I will now justify. We did the change of variables from integrating with respect to the x coordinates to integrating with respect to the y coordinates. The problem with this is that the integrals might not be bounded in the correct way. Recall

$$dy = \det \left| \frac{\partial y^l}{\partial x^j} \right| dx$$

and $c \leq \det \left| \frac{\partial y^l}{\partial x^j} \right| \leq C$. And so we have for a general function f

$$\begin{aligned} \|f(\cdot)\|_{L^p_y}^p &= \int |f(y)|^p dy \\ &= \int |(f \circ y)(x)|^P \det \left| \frac{\partial y^l}{\partial x^j} \right| dx \end{aligned}$$

and so we have

$$c\|f \circ y\|_{L^p_x}^p \leq \|f\|_{L^p_y}^p \leq C\|f \circ y\|_{L^p_x}^p$$

and (7) is valid.

Claim. The following inequality holds for any v (not just those supported in a boundary neighborhood):

$$\left\|v\big|_{\partial\Omega}\right\|_{L^p(\partial\Omega)} \leq C\|v\|_{W^{1,p}(\Omega)}.$$

Proof. Note that $cdx \leq d\sigma \leq Cdx$ and we are able to apply the same argument as above (i.e. norms are equivalent under change of variable). So now the problem is to deal with the full Ω. Since Ω is compact we cover it $\Omega = \cup_{\alpha=1}^N \Omega_\alpha$ and pick a partition of unity $\chi_\alpha \in C_0^\infty(\Omega_\alpha)$ and $\sum \chi = 1$. Then

$$\begin{aligned} \left\|v\big|_{\partial\Omega}\right\|_{L^p(\partial\Omega)} &\leq \sum_{\alpha=1}^N \|\chi_\alpha v\|_{L^P(\partial\Omega)} \leq \sum_{\alpha=1}^N \|\chi_\alpha v\|_{W^{1,p}(\Omega)} \\ &\leq \sum_{\alpha=1}^N \big(\|\chi_\alpha v\|_{L^p(\Omega)} + \|D(\chi_\alpha v)\|_{L^p(\Omega)}\big) \\ &\leq C\|v\|_{W^{1,p}(\Omega)} \end{aligned}$$

Where we have bounded $\|\chi_\alpha v\|_p \leq \|v\|$ and expanded the second term using the Leibniz rule. □

With this done we can now extend our previous work to more general boundary conditions. Say you wanted to solve $\Delta u = f$ with $u|_{\partial\Omega} = g$. Then if $g \in L^p(\partial\Omega)$ then choose a $G \in W^{1,p}(\Omega)$ whose restriction is equal to g and then consider $v = u - G$ and the problem $\Delta v = \Delta u - \Delta G$ and $v|_{\partial\Omega}$.

We will begin by analyzing a Non-Linear PDE. Let M be a compact Riemannian manifold of dimension 2. Let

$$ds^2 = \sum_{i,j=1}^{2} g_{ij} dx^i dx^j$$

be the Riemannian metric. Let $v \in C^\infty(M)$ and let R be a given negative constant. We want to solve

$$\Delta u + \lambda e^{u+v} - R = 0. \tag{8}$$

Notice that the exponential of u makes this a very non-linear equation. Let $\sqrt{g} := \det g_{ij}$. Then we will define the laplacian as follows

$$\begin{aligned}
\Delta u &= \frac{1}{\sqrt{g}} \sum_{j,k=1}^{2} \partial_k \left(\sqrt{g} g^{ij} \partial_j u\right) \\
&= \sum_{j,k=1}^{2} \frac{1}{\sqrt{g}} \left(\sqrt{g} g^{jk} \partial_k \partial_j u\right) + \text{ first order terms} \\
&= \sum_{j,k=1}^{2} g^{jk} \partial_k \partial_j u + \text{ first order terms.}
\end{aligned}$$

Example 2.17. In Euclidean space, we have $ds^2 = \sum (dx^i)^2 \implies g_{ij} = g^{ij} = \delta_{ij}$ and so

$$\Delta u = \sum \frac{\partial^2 u}{\partial x_j^2} + \text{ first order terms.}$$

In order to solve this, we would have to consider the following functional:

$$I(u) = \int_M |Du|^2 \sqrt{g} dx + R \int_M u \sqrt{g} dx$$

subject to the constraint

$$\frac{1}{V} \int_M e^{u+v} \sqrt{g} dx = 1, \quad V = \int_M \sqrt{g} dx$$

and attempt to mimic our work for $\Delta u = f$ (show that the function is bounded from below and find a minimum).

Note that $Du = \partial_j u$ and $|Du|^2 = \partial_j u \partial_k u$ then we need the metric to contract the indices so we introduce g^{jk}. Note that we can assume

$$\frac{1}{V} \int v \sqrt{g} dx = 0 \tag{9}$$

because we can shift everything about constants. We will prove boundedness from below for our functional, but first lets compare with $\Omega \subset\subset \mathbb{R}^n$ and

$$I(u) = \frac{1}{2} \int_\Omega |Du|^2 dx + \int_\Omega f u dx$$

for any $u \in W_0^{1,2}(\Omega)$ and $f \in L^2(\Omega)$. The way we proved this was by showing that

$$\int_\Omega |u|^2 dx \leq C \int_\Omega |Du|^2 dx,$$

but can we do this on a general compact manifold? No! In general we have the following Poincaré inequality

$$\|u-\bar{u}\|^2_{L^2(M)} \leq C\|Du\|^2_{L^2(M)}, \quad \bar{u} = \frac{1}{V}\int_M u\sqrt{g}dx.$$

Now we can write our functional in the following way

$$I(u) = \underbrace{\frac{1}{2}\int_M |Du|^2\sqrt{g}dx + R\int_M (u-\bar{u})\sqrt{g}dx}_{\text{bounded as before}} + \underbrace{R\bar{u}\int_M \sqrt{g}dx}_{\text{can blow up}}.$$

In order to deal with this difficulty we exploit our constraint

$$1 = \frac{1}{V}\int_M e^{u+v}\sqrt{g}dx$$
$$0 = \log\left(\frac{1}{V}\int_M e^{u+v}\sqrt{g}dx\right)$$

and note that $\frac{1}{V}\sqrt{g}dx$ has total measure 1. Now recall consider Jensen's Inequality: If $\int d\mu = 1$ then

$$\int \log(f)d\mu \leq \log\left(\int f d\mu\right).$$

We will also use the fact that

$$\log\left(\frac{a+b}{2}\right) \geq \frac{1}{2}\log a + \frac{1}{2}\log b$$

in the following way

$$0 = \log\left(\frac{1}{V}\int_M e^{u+v}\sqrt{g}dx\right) \geq \frac{1}{V}\int_M \log(e^{u+v})\sqrt{g}dx$$
$$= \frac{1}{V}\int_M (u+v)\sqrt{g}dx = \frac{1}{V}\int_M u\sqrt{g}dx$$

where we have used (9) in the last step. So now we have that our constraint implies $\bar{u} \leq 0$ and so we do indeed have our bound

$$I(u) = \frac{1}{2}\int |Du|^2\sqrt{g}dx + R\int u\sqrt{g}dx \geq 0.$$

Now as before lets pick a minimizing sequence $u_j \in C^\infty(M)$ such that $I(u_j) \to \min I(u)$ and each u_j satisfies the constraint. Clearly $\|Du\|^2_{L^2(M)} \leq C$ and so

$$\|u_j\|_2 \leq \|u_j - \bar{u}_j\|_2 + \|\bar{u}_j\| \leq C\|Du_j\|_2 + \frac{1}{R}I(u_j) \leq C.$$

This means that our minimizing sequence is bounded and so we can apply the Banach-Alaoglu Theorem to show that $\exists u_\infty \in W^{1,2}(M)$ such that $Du_j \rightharpoonup Du$ weakly and $u_j \rightharpoonup u$ weakly. Now the question is if u_∞ is the minimum or not? We do have

$$I(u_\infty) \leq \liminf \frac{1}{2}\int |Du_j|^2\sqrt{g}dx + \int Ru_j\sqrt{g}dx$$
$$= \liminf I(u_j) = \lim_{j\to\infty} I(u_j).$$

The real question is if u_∞ satisfies the constraint i.e. we wish to show

$$\frac{1}{V}\int_M e^{u_\infty + v}\sqrt{g}dx = 1 \tag{10}$$

The bottom-line question here is if we can pass through the limit. We need to strengthen our converging hypothesis as much as possible (i.e. from weak convergence to point-wise convergence). In fact, we claim that for the minimizing sequence we have, $Du_j \rightharpoonup Du_\infty$ weakly but $u_j \to u$ in L^2. For this we will need

Lemma 2.18 (Rellich's Lemma)**.** *Let M be a compact manifold and $\{u_j\} \subset W^{1,p}(M)$ with $\|u_j\|_{W^{1p}(M)} \leq C$ with C independent of j, then there exists $u_\infty \in W^{1,p}(M)$ and a subsequent $\{u_{j_k}\}$ such that $u_{j_k} \to u_\infty$ in $L^p(M)$.*

Note. Note that this also holds for $W^{1,p}(\mathbb{R}^n)$ if $\operatorname{supp} u_j \subset k \subset\subset \mathbb{R}^n \; \forall j$.

Now recall a theorem from measure theory:

Theorem 2.19. *If $u_j \to u_\infty$ in L^p for $1 \leq p < \infty$, then there exists $u_{j_k} \to u_\infty$ point wise a.e.*

Note that all together, we can take a subsequence of our subsequence to find a sequence $\|Du_j\| \leq C, u_j \to u_\infty$ almost everywhere. We will see that in two dimensions these properties imply that $e^{u_j+v} \to e^{u_\infty+v}$ in L^1. Since these values are always positive, the L^1 norm is just convergence of the integral, which is what we needed. To see this,

$$\begin{aligned} e^{u_j+v} - e^{u_\infty+v} &= -\int_0^1 \frac{d}{dt}\left[e^{tu_\infty+(1-t)u_j+v}\right] dt \\ &= -\int_0^1 (u_\infty - u_j) e^{tu_\infty+(1-t)u_j+v} dt. \end{aligned}$$

Taking the L^1 norm will give us

$$\begin{aligned} \int_M |e^{u_j+v} - e^{u_\infty+v}|\sqrt{g}dx &\leq \int_0^1 \left[\int_M |u_\infty - u_j| e^{tu_\infty+(1-t)u_j+v}\sqrt{g}dx\right] dt \\ &\leq \int_0^1 \left[\int_M |u_\infty - u_j|^2\sqrt{g}dx\right]^{1/2} \left[\int_M e^{2(tu_\infty+(1-t)u_j+v)}\sqrt{g}dx\right]^{1/2} dt \\ &= \|u_\infty - u_j\|_{L^2} \int_0^1 \left[\int_M e^{2(tu_\infty+(1-t)u_j+v)}\sqrt{g}dx\right]^{1/2} dt \end{aligned}$$

and now note that we need the second integral to be uniformly bounded for the RHS of the inequality to go to zero.

We claim that $\int e^{w_j}\sqrt{g}dx \leq C$ (independent of j) if $\|w_j\|_{L^2} \leq C$ and $\|Du_j\|_{L^2} \leq C$. In this case we have $w_j = 2(tu_\infty + (1-t)u_j + v)$ satisfying the condition because $\|Du_j\|_{L^2} \leq C$ and $\|Du_\infty\|_{L^2} \leq \liminf \|Du_j\|_{L^2} \leq C$ and similarly the L^2 norm of u_j and u_∞ is bounded. Now lets see why this claim is true.

Recall the Trudinger inequality: $u \in C_0^\infty(B)$ implies that

$$\int \exp\left(\frac{|u(x)|}{K\|Du\|_{L^n}}\right)^{\frac{n}{n-1}} \leq C$$

and so for $n = 2$ we have

$$\int \exp\left(\frac{|u(x)|}{K\|Du\|_{L^2}}\right)^2 \leq C.$$

Now note that we can write

$$|u(x)| = \frac{|u(x)|}{K\|Du\|_{L^2}} K\|Du\|_{L^2} \leq \frac{1}{2}\left(\frac{|u(x)|}{K\|Du\|_{L^2}}\right)^2 + \frac{1}{2}(K\|Du\|_{L^2})^2.$$

Since the exponential function is increasing we can write

$$\begin{aligned} \exp|u(x)| &\leq \exp\left[\frac{1}{2}\left(\frac{|u(x)|}{K\|Du\|_{L^2}}\right)^2 + \frac{1}{2}(K\|Du\|_{L^2})^2\right] \\ \int \exp|u(x)| &\leq e^{\frac{1}{2}(K\|Du\|_{L^2})^2} \int \exp\frac{1}{2}\left(\frac{|u(x)|}{K\|Du\|_{L^2}}\right)^2 \\ &\leq e^{\frac{1}{2}(K\|Du\|_{L^2})^2} C \end{aligned}$$

And this concludes the proof. A great exercise is as follows. Let M be a compact Riemannian n-manifold and show that

$$\int e^w \sqrt{g} dx \leq C \exp\left(C\|Du\|_{L^n}^n + \|w\|_{L^n}^n\right).$$

As a summary, we have shown that for a sequence satisfying $I(u_j) \to \min I(u)$ and $\frac{1}{V}\int e^{u_j+v}\sqrt{g}dx = 1$ then $u_j \to u_\infty$ (as explained above) and $\frac{1}{V}\int e^{u_\infty+v}\sqrt{g}dx = 1$ so we have

$$\inf I(u) \leq I(u_\infty) \leq \inf I(u).$$

Now we claim that u_∞ satisfies our partial differential equation in the Euler-Langrange sense. Fix $\varphi \in C^\infty(M)$ and consider $u_\infty + t\varphi + c_t$. We add the constant c_t so that this function still satisfies the constraint. To see what c_t has to be note

$$\begin{aligned} 1 &= \frac{1}{V}\int_M e^{u_\infty+t\varphi+c_t+v}\sqrt{g}dx \\ e^{c_t} &= \left(\frac{1}{V}\int_M e^{u_\infty+t\varphi+v}\sqrt{g}dx\right)^{-1} \\ c_t &= -\log\left(\frac{1}{V}\int_M e^{u_\infty+t\varphi+v}\sqrt{g}dx\right) \end{aligned}$$

and thus $I(u_\infty + t\varphi + c_t) \geq I(u_\infty)$ for any t and so we leave it as an exercise to show in detail

$$\left.\frac{d}{dt}\right|_{t=0} I(u_\infty + t\varphi + c_t) = 0.$$

Lets recap our problem. Our functional was

$$I(u) = \frac{1}{2}\int_M |Du|^2\sqrt{g}dx + R\int_M u\sqrt{g}dx$$

subject to the constraint

$$\frac{1}{V}\int_M e^{u+v}\sqrt{g}dx = 1 \tag{11}$$

where v is some smooth function and V is the volume over this compact manifold. We obtained a u_∞ subject to the same constraint that satisfies $\Delta u_\infty + \lambda e^{u_\infty+v} - R = 0$. Recall that this general laplacian has the form $\Delta u = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}\partial_j u\right)$. Where does this come from? If we assume u_t is smooth then we can check

$$\begin{aligned} \frac{d}{dt}\int_M |Du_t|^2\sqrt{g}dx &= \frac{d}{dt}\left(\int g^{ij}\partial_i u_t \partial_j u_t \sqrt{g}dx\right) \\ &= -\frac{d}{dt}\left(\int u_t \partial_i\left(g^{ij}\partial_j u_t\right)\sqrt{g}dx\right) \\ &= -\frac{d}{dt}\left(\int u_t \Delta u_t \sqrt{g}dx\right) \end{aligned}$$

So we see that this laplacian is the perfect analogue of the one taken on $\mathbb{R}^n$.

We want to derive the generalized Euler-Lagrange equation for our PDE, so let $\varphi \in C_0^\infty(M)$ and $u_t = u_\infty + t\varphi + c_t$, where c_t is picked so that u_t also satisfies the constraint (11). Lets figure out what λ has to be in order to for our equation to be solved correctly. Integrate on both sides of our PDE to get

$$\begin{aligned} 0 &= \int_M \Delta u_\infty\sqrt{g} + \lambda\int_M e^{u_\infty+v}\sqrt{g} - R\int_M \sqrt{g}dx \\ &= \int_M \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\partial_j u_\infty\right) + \lambda\int_M e^{u_\infty+v}\sqrt{g} - R\int_M \sqrt{g}dx \\ &= \lambda V - RV \end{aligned}$$

so we see that λ has to equal R. The reason the first term vanishes is because we are integrating an exact form over a compact manifold. So then we have that our PDE is

$$\Delta u + Re^{u+v} - R = 0 \tag{12}$$

and u satisfies (12) in the general sense if $\forall \varphi \in c_0^\infty$ we have

$$\begin{aligned} 0 &= \int \left(\Delta u + Re^{u+v} - R\right) \varphi \sqrt{g} \\ &= \int \left(-g^{ij}\partial_i \partial_j \varphi + Re^{u+v} - R\varphi\right) \sqrt{g} dx \end{aligned} \tag{13}$$

and (13) is the generalized Euler-Lagrange equation.

2.2.2 Regularity Theory

Lets finally start some regularity theory. Let g_{ij} be a smooth Riemannian metric and assume $\Lambda \geq g_{ij} \geq \lambda > 0$. If $u \in W^{1,2}_{\text{loc}}$ satisfies $\Delta u = f$ in the weak sense then

1. If $f \in W^{k,p}$ and $1 < p < \infty$ the $u \in W^{k+2,p}$ and for any $\Omega \subset\subset \Omega'$

$$\|u\|_{W^{k+2,p}(\Omega)} \leq C_{\Omega,\Omega'} \left(\|f\|_{W^{k,p}(\Omega')} + \|u\|_{W^{k,p}(\Omega')}\right)$$

2. If $f \in C^{k,\alpha}$ and $1 < p < \infty$ the $u \in C^{k+2,\alpha}$ and for any $\Omega \subset\subset \Omega'$

$$\|u\|_{C^{k+2,\alpha}(\Omega)} \leq C_{\Omega,\Omega'} \left(\|f\|_{C^{k,p}(\Omega')} + \|u\|_{W^{k,p}(\Omega')}\right)$$

Claim. It follows easily that if u satisfies (13) in the generalized sense, then $u \in C^\infty$ and actually satisfies (12) in the standard sense.

Proof. Indeed we can let $-Re^{u+v} + R = f$ and $f \in L^2 = W^{0,2}$ by the Trudinger's inequality (it tells us that this exponential is bounded by the L^2 norm of the weak derivative) then by regularity we will get $u \in W^{2,2}$. Now recall that Morrey's inequality tells us that if $\frac{1}{p} < \frac{k}{n}$ then $W^{k,p}$ is embedded in a Hölder space, but since $p = n = 2$ we have $u \in C^\alpha$. But then since $\Delta u = f$ this implies $f \in C^\alpha$ and we can apply regularity to get $u \in C^{2,\alpha}$. We can iterate this and find of course that $u \in C^\infty$. □

We will prove these regularity statements by the method of continuity and a priori estimates. Consider

$$\Delta u + Re^{u+v} - R = 0$$

on $(M, g_{ij}(x))$ where v is a smooth function. Note that $\Delta u + Re^u - R = 0$ admits a solution $u \equiv 0$ so we see that the difficulty comes from the v term. Let $t \in [0,1]$ and introduce the family of equations

$$\Delta u + Re^{u+tv} - R = 0 \tag{14}$$

And consider the set $I = \{t \in [0,1] | (14) \text{ admits a solution } u_t\}$. Note that $I \neq \emptyset$ because $0 \in I$ is a solution. Then we obviously want to show $I = [0,1]$, so we need to show that I is open and closed. Lets discuss this very briefly (to be made concise later).

Say we want to show I is open. We will do this by an analogue of the implicit function theorem. Recall that it says given $f(x_0, y_0) = 0$ and $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$ then $\exists \epsilon > 0$ such that for $|x - x_0| < \epsilon$, then $\exists! y$ such that $f(x, y) = 0$. Now let $f(t, u) = \Delta u + Re^{u+tv} - R$. Then we want to solve $f(x, u) = 0$ where we know $f(t, u_0) = 0$. So our goal is going to be an implicit function theorem for Banach spaces and we need to check that $\frac{\partial f}{\partial u}(t_0, u_0) \neq 0$ in a way we will define later.

Now lets briefly discuss how we will show that I is closed. Take $t_j \in I$ such that (14) will admit a solution called u_j, and assume $t_j \to T$. Then closeness of I is equivalent to showing that (14) will admit a solution for T. It will suffice to have a subsequence converge in C^2.

Proof. Lets start by showing that I is closed. Let $t_j \to T$ with $t_j \in I$ and let u_j be the corresponding solution of (15). Observe that if suffices to show that $\exists C$ independent of j so that $\|u_j\|_{C^3} \leq C$ where in general

$$\|u\|_{C^3(\Omega)} = \sum_{|\alpha| \leq 3} \|\partial^\alpha u\|_{C^0(\Omega)}.$$

The reason why this will help is that this would imply that $\forall \beta \leq 2$, then $\partial^\beta u_j$ is an equicontinuous family. Then the Arzela-Ascolati Theorem tells us that if we have an equicontinuous family, then by going through a subsequence $\partial^\beta u_j$ we have uniform convergence to $D^\alpha u_T$ where $u_T \in C^2$. However we don't necessarily have that $T \in I$ because we need u_T to be smooth and so far it is only C^2. This is fixed however by our regularity observations.

Before we can apply our regularity conditions we must show that our equation is uniformly elliptic. Recall that a second order PDE is said to be uniformly elliptic if the leading coefficient satisfies

$$\lambda |\xi|^2 \leq \left| \sum_{|\alpha|=2} a_\alpha \xi^\alpha \right| \leq \Lambda |\xi|^2.$$

This we need to show that our Laplacian is elliptic.

$$\begin{aligned} \Delta u &= \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g} \partial_j u \right) \\ &= \frac{1}{\sqrt{g}} \sqrt{g} g^{ij} \partial_i \partial_j u + \text{first order terms.} \end{aligned}$$

So the symbol (the middle term of the ellipticity inequality requirement) of our Laplacian is going to be

$$\sigma_\Delta(x, \xi) = g^{ij} \xi_i \xi_j$$

and since g is positive definite we definitely have the ellipticity requirement. So then we can apply our regularity theorems by viewing $\Delta u = f \in C^2 \subset C^{1,\alpha}$.

So now we have to prove the a priori estimate $\|u_j\|_{C^3} \leq C$. We will use the maximum principle. Since there are so many different formulations of maximum principles, it is a good idea to simply examine what happens near a maximum. Let $u \in C^\infty$ satisfy $\Delta u + R^{tw+u} - R = 0$ (by denoting $u_t = u$) and then I claim that $\|u\|_{C^0} \leq C$ where C is independent of t. Let $x_0 \in \Omega$ such that $u(x_0)$ is a maximum. Then $\Delta u(x_0) \leq 0$ and since $R < 0$ we have

$$\begin{aligned} 0 &\leq -\Delta u = Re^{tw+u} - R \\ R &\leq Re^{tw+u} \\ -|R| &\leq -|R| e^{tw+u} \\ 1 &\geq e^{tw+u} \\ 0 &\geq tw + u \\ u &\leq -tw \\ &\leq \|w\|_{C^0}. \end{aligned}$$

But since x_0 is a maximum we have that $\forall x, u(x) \leq u(x_0) \leq \|w\|_{C_0}$ and applying the same argument for the minimum we have $\|u\|_{C_0} \leq \|w\|_{C_0}$. In order to get higher derivatives we write $\Delta u = f \in C^0$ and so for $f \in L^p$ $\forall 1 \leq p < \infty$ we have $u \in W^{2,p} \subset C^\alpha$ for some α when $n < kp$.

Now lets show that I is open. Let $t_0 \in I$ i.e. there exists u_0 which is a smooth solution of (14). Let $(u, t) \mapsto F(u, t) = \Delta u + Re^{tw+u} - R$. Then we want to show that $\exists \delta > 0$ so that $|t - t_0| < \delta$ implies $\exists u_t \in C^\infty | F(u_t, t) = 0$. The main tool will be the Implicit Function Theorem for Banach Spaces, which goes as follows.

Let B_1 and B_2 be Banach spaces. Let $F \in C^1$ and consider $B_1 \times \mathbb{R} \supset \Omega \ni (u,t) \mapsto F(u,t) \in B_2$. Assume that $F(u_0,t_0) = 0$. Let $\frac{\partial F}{\partial u}(u_0,t_0)$ be the derivative of F at (t_0,u_0) viewed as a linear operator $B_1 \to B_2$. Note that if

$$\|h\|_{B_1} \leq C \left\| \frac{\partial F}{\partial u}(u_0,t_0)h \right\|_{B_2} \qquad \forall h \in B_1 \tag{15}$$

then $\frac{\partial F}{\partial u}(u_0,t_0)$ is injective and surjective. So assume that (15) holds. Then the Implicit Function Theorem for Banach Spaces says that $\exists \delta > 0, \exists V$ that is a neighborhood of u_0 such that $\exists! u_t \in V$ with $F(u_t,t) = 0$.

Before we can even apply this theorem, we need to mae sense of derivatives in terms of Banach spaces. Let $B_1 \ni u \mapsto F(u) \in B_2$. Then F is differentiable at u_0 if $\exists L : B_1 \to B_2$ that is a bounded linear operator satisfying

$$F(t,u+h) = F(t,u) + Lh + E(t,u,h)$$

with

$$\lim_{h \to 0} \frac{\|E(u,h)\|_{B_2}}{\|h\|_{B_1}} = 0.$$

In order to apply our theorem we need so specify our Banach spaces. We will want to have $(t,u) \in \mathbb{R} \times C^{2,\alpha} \to F(u,t) \in C^{0,\alpha}$. What we will then need to check are the assumptions of the implicit function theorem. Let $t_0, u_0 \in \mathbb{R} \times C^{2,\alpha}$ satisfy the conditions of the IFT. Lets determine L. Consider the expression $F(t_0, u_0 + h)$:

$$\begin{aligned} F(t_0, u_0 + h) &= \Delta(u_0 + h) + Re^{t_0 w + u_0 + h} - R \\ &= F(t_0, u_0) + \Delta h + Re^{t_0 w + u_0 + h} - Re^{t_0 w + u} \\ &= F(t_0, u_0) + \Delta h + Re^{t_0 w + u}\left(e^h - 1\right) \\ &= F(t_0, u_0) + \left(\Delta + Re^{t_0 + u}\right)h + Re^{t_0 w + u}(e^h - 1 - h). \end{aligned}$$

So we have figured out our L. Now we leave it as an exercise to show that

$$\|Re^{t_0 w + u_0}(e^h - 1 - h)\|_{C^{0,\alpha}} \leq C\|h\|^2_{C^{2,\alpha}}.$$

The main tool for this is the integral form of Taylor's Remainder Theorem, which starts as follows

$$\begin{aligned} h\int_0^1 (1-t)\frac{d}{dt}\left[f'(u+th)\right]dt &= \int_0^1 f'(u+th)dt + h(1-t)f'(u+th)\Big|_0^1 \\ &= \int_0^1 \frac{d}{dt}f(u+th)dt - hf'(u) \\ &= f(u+h) - f(u) - hf'(u). \end{aligned}$$

However we can write the LHS as

$$h\int_0^1 (1-t)\frac{d}{dt}\left[f'(u+th)\right]dt = \int_0^1 (1-t)f''(u+th)dt$$

and so we have the final part as

$$f(u+h) = f(u) + f'(u)h + h^2\int_0^t (1-t)f''(u+th)dt.$$

Thus we have that $L = \frac{\partial F}{\partial u}(t_0,u_0) : C^{2,\alpha} \ni h \mapsto \Delta h + Re^{t_0 w + u}h \in C^{0,\alpha}$, which is clearly a bounded operator. It is trickier to show that it is injective and bijective.

Assume $0 = Lh = \Delta h + Re^{t_0 w + u}h$. Let x_0 me a maximum (again this means $\Delta h \leq 0$). Then we have

$$-Re^{t_0 w + u}h = \Delta h(x_0) \leq 0$$

and since the coefficients are both strictly positive, it means that $h(x_0) \leq 0$. Since this is a maximum it means that $\forall x, h(x) \leq h(x_0) \leq 0$. Applying the same process to the minimum we get that $h(x) \geq 0$ and so $h \equiv 0$. This implies that the kernel of L is zero, and so it is injective.

In order to finally complete the proof we will show that L is onto i.e. $\forall f \in C^{0,\alpha}$, we want to show $\exists h \in C^{2,\alpha}$ so that $Lh = f$. We will show this by variational methods again. Set

$$I(h) = \int \left[\sqrt{g} g^{ij} \partial_i h \partial_j h - R e^{t_0 w + u_0} h^2 + fh\right] \sqrt{g} dx$$

for some $h \in W^{1,2}(M)$. We leave it as an exercise to show that $I(h)$ attains its minimum for some h_∞. One way to do this is by showing that

$$I(h) \geq \|Dh\|_2^2 + \epsilon \|h\|_2^2 - \frac{1}{\epsilon}\|f\|_2^2$$

and using our tricks. Assuming the exercise, we invoke the black box to make h smooth. Recall that it says if $f \in C^{k,\alpha}$ then

$$\|h\|_{C^{k+2,\alpha}} \leq C\left(\|Lh\|_{C^{k,\alpha}} + \|h\|_{C^{k,\alpha}}\right).$$

We now improve the black box by saying that if $\ker L = 0$, then

$$\|h\|_{C^{k+2},\alpha} \leq C\|Lh\|_{C^{k,\alpha}}. \tag{16}$$

Now we will prove this little lemma. We will use weak compactness. If $\{u_l\} \subset C^{k,\alpha}$ with $\|u_l\|_{C^{k,\alpha}} \leq C$ with C independent of l, then *either* $\forall k < k, \forall \beta$ *or* $k' = k$ with $0 < \beta < \alpha$ then there exists a convergent subsequence in $C^{k',\beta}$ by the routine application of the Arzela-Ascolti Theorem. Now assume (17) does not hold. Then for any $N, \exists h_N \in C^{k+2,\alpha}$ such that $\|h_N\|_{C^{k+2,\alpha}} > N\|Lh_n\|_{C^{k,\alpha}}$. Set

$$\tilde{h}_N = \frac{h_N}{\|h_N\|_{C^{k+2,\alpha}}}$$

and notice that $\|\tilde{h}_N\|_{C^{k+2,\alpha}} = 1$ and so this implies that $\|L\tilde{h}\|_{C^{k,\alpha}} < \frac{1}{N} \to 0$. By the weak compactness, going through a subsequence, we can assume that $\tilde{h}_N \to h_\infty$ in $C^{k,\alpha}$. Applying the black box gives

$$\|\tilde{h}_N - \tilde{h}_M\|_{C^{k+2,\alpha}} \leq C\left(\|L\tilde{h}_N - \tilde{h}_M\|_{C^{k,\alpha}} + \|\tilde{h}_N - \tilde{h}_M\|_{C^{k,\alpha}}\right)$$

which implies that $\tilde{h}_N \to h_\infty$ in $C^{k+2,\alpha}$. Thus $L\tilde{h}_N \to Lh_\infty$ in $C^{k,\alpha}$ and so $h_\infty \in \ker L$. Since

$$\|h_\infty\|_{C^{k+2,\alpha}} = \lim \|\tilde{h}_N\|_{C^{k+2,\alpha}} = 1$$

this contradicts the fact that $\ker L = 0$. $\square$

3 Harnack Inequality for Divergence Equations

3.1 Regularity Estimates

Suppose that $u \in W^{1,2}(\Omega)$ solves $\partial_i(a_{ij}\partial_j u) + cu = f$ in the generalized sense, where a_{ij} is uniformly elliptic i.e. $0 < \lambda \leq a_{ij} \leq \Lambda$. The question we want to answer is: when is u "regular" i.e. $u \in C^\alpha, W^{k,2}, C^\infty$, etc?

Lets look at the simplest case where a_{ij} is constant and $c = f = 0$. In this case u solving $a_{ij}\partial_i\partial_j u = 0$ in the generalized sense. This means that $\forall v \in W_0^{1,2}(\Omega)$,

$$\int a_{ij}\partial_i u \partial_j v = 0. \tag{17}$$

Then we will show that $u \in C^\infty(\Omega)$ and for $|\alpha| = k, 0 < r < R$, we have

$$\int_{B_r(x_0)} |D^\alpha u|^2 \leq \frac{C_{\lambda,\Lambda,k}}{(R-r)^{2k}} \int_{B_R(x_0)} |u|^2. \tag{18}$$

Note that this inequality is very powerful because we have that the derivative is being bounded by the function, where w usually have it the other way around.

Proof. We apply (17) with $v = \chi^2 u$ with $0 \leq \chi \leq 1, \chi \equiv 1$ on $B_r(x_0)$ and $\chi \in C_0^1(B_R(x_0))$. Additionally assume

$$D\chi(x_0) \leq \frac{2}{R-r}.$$

Applying (17) gives us

$$\int \chi^2 a_{ij}\partial_i u \partial_j u = -2\int a_{ij}(\chi\partial_j\chi)u\partial_i u$$

and note that ellipticity gives us $|a_{ij}u_i v_j| \leq (a_{ij}u_i u_j)^{\frac{1}{2}}(a_{ij}v_i v_j)^{\frac{1}{2}}$. Putting absolute values gives us

$$\begin{aligned}
\left|\int \chi^2 a_{ij}\partial_i u \partial_j u\right| &\leq 2\left(\int a_{ij}\partial_i u\partial_j u \chi^2\right)^{\frac{1}{2}}\left(\int a_{ij}\partial_i\chi\partial_j\chi|u|^2\right)^{\frac{1}{2}} \\
&\leq \frac{1}{2}\int a_{ij}\partial_i u \partial_j u\chi^2 + 4\int a_{ij}\partial_i\chi\partial_j\chi|u|^2 \\
\frac{1}{2}\int a_{ij}\partial_i u\partial_j u\chi^2 \leq 4\int a_{ij}\partial_i\chi\partial_j\chi|u|^2
\end{aligned}$$

where we have used the standard fact that $ab \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2$ where we chose ϵ to give us the right coefficients. Then from the ellipticity condition we have

$$\frac{\lambda}{2}\int_{B_r(x_0)} |Du|^2 \leq \frac{1}{2}\int a_{ij}\partial_i u\partial_j u\chi^2 \leq 4\int a_{ij}\partial_i\chi\partial_j\chi|u|^2 \leq 16\int \Lambda|D\chi|^2|u|^2 \leq \frac{8\Lambda}{(R-r)^2}\int_{B_R(x_0)}|u|^2$$

which proves the inequality for $k = 1$. To prove $|\alpha| = k \in \mathbb{Z}$, we proceed by induction. Assume $u \in C^\infty$. Then $D^\alpha u \in C^\infty$ and satisfies the same equation $a_{ij}\partial_i\partial_j(D^\alpha) = 0$. Applying the previous case with $k = 1$ gives (by induction)

$$\begin{aligned}
\int_{B_r(x_0)} |D(D^\alpha u)|^2 &\leq \frac{C}{\left(\frac{r+R}{2} - r\right)^2}\int_{B_{R-r}(x_0)} |D^\alpha u|^2 \\
&\leq \frac{C}{\left(\frac{R+r}{2} - r\right)\left(R - \frac{R+r}{2}\right)^{2k}}\int_{B_R(x_0)} |u|^2 \\
&= \frac{C}{(R-r)^{2(k+1)}}\int_{B_R(x_0)} |u|^2.
\end{aligned}$$

As one can expect, for the non-smooth case, we use mollifiers. Take $\eta \in C_0^\infty(|x|<1)$ with

$$\int_{\mathbb{R}^n} \eta = 1$$

and define $\eta_\epsilon = \frac{1}{\epsilon^n}\eta(\frac{x}{\epsilon})$. Then define

$$u_\epsilon(x) = \int u(x-y)\eta_\epsilon(y)dy = \int u(y)\eta_\epsilon(x-y)dy$$

which is well defined for $\text{dist}(x, \partial\Omega) > \epsilon$. By an exercise we leave to the reader, note that if $u \in L^p(\Omega)$ for $1 < p < \infty$, then for any $K \subset\subset \Omega$ and $\epsilon < \epsilon_K$, then $u_\epsilon \to u$ in $L^p(K)$. Using this, and by a single exercise that shows

$$\int a_{ij}\partial_i u_\epsilon \partial_j v = \int a_{ij}\partial_i u \partial_j v_\epsilon$$

we can conclude that if u satisfies $a_{ij}\partial_i\partial_j u = 0$ in the generalized sense then so does u_ϵ. Thus we have that $u_\epsilon \in C^\infty$ and $a_{ij}\partial_i\partial_j u_\epsilon = 0$ and thus we can apply our estimate (18) to find

$$\int_{B_r(x_0)} |D^\alpha u_\epsilon| \leq \frac{1}{(R-r)^{2k}} \int_{B_R(x_0)} |u_\epsilon|^2.$$

□

Lets try to generalize this. We will do this with the following theorem.

Theorem 3.1. *Let $u \in W^{1,2}(\Omega)$ be a weak solution to*

$$\partial_i(a_{ij}\partial_j u) + c(x)u = f \tag{19}$$

in $\Omega \subset \mathbb{R}^n$. Assume that

i) $0 < \lambda \leq a_{ij} \leq \Lambda$

ii) a_{ij} are continuous with modulus of continuity τ i.e. $|a_{ij}(x) - a_{ij}(y)| \leq \tau(|x-y|)$

iii) $c \in L^n, f \in L^q$ where $\frac{n}{2} < q < n$.

Then for any $B_R(x) \subset\subset \Omega$, $u \in C^\alpha(B_R(x))$ with $\alpha = 2 - \frac{n}{q}$ and $0 < \alpha < 1$ with the estimate

$$\|u\|_{C^\alpha(B)} \leq C_{n,\lambda,\Lambda,\tau,\|c\|_p}\left(\|f\|_{L^q(\Omega)} + \|u\|_{W^{1,2}(\Omega)}\right).$$

In order to prove this we will need a few lemmas. They are as follows.

Lemma 3.2. *Assume $u \in W^{1,2}(B)$ and*

$$\int_{B_r(x_0)} |u - \bar{u}_{x_0}|^2 dx \leq M^2 r^{n+2\alpha}.$$

Then $u \in C^\alpha$ and $\|u\|_{C^\alpha(B)} \leq C(M + \|u\|_{L^1})$.

Proof. First note that what we are doing makes some sort of sense. In the assumption, we have the difference between u and its average at x_0 being squared, so that explains the 2α. Then since we are integrating over the ball, we have the factor of r^n.

Let $0 < r < R$ and look at the difference between averages

$$|\bar{u}_r(x_0) - \bar{u}_R(x_0)| \leq |\bar{u}_r(x_0) - u(x)| + |u(x) - \bar{u}_R|$$
$$|\bar{u}_r(x_0) - \bar{u}_R(x_0)|^2 \leq 2\left(|\bar{u}_r - u(x)|^2 + |u(x) - \bar{u}_R|^2\right).$$

We now integrate over the ball of radius r and using the fact that the LHS is a constant and the assumption of the lemma we have

$$
\begin{aligned}
r^n|\bar{u}_r(x_0)-\bar{u}_R(x_0)|^2 &\le 2\left(\int_{B_r(x_0)}|\bar{u}_r-u(x)|^2+\int_{B_r(x_0)}|u(x)-\bar{u}_R|^2\right)\\
&\le 2\left(M^2r^{n+2\alpha}+\int_{B_R(x_0)}|u(x)-\bar{u}_R|^2\right)\\
&\le CM^2\left(r^{n+2\alpha}+R^{n+2\alpha}\right)\\
|\bar{u}_r(x_0)-\bar{u}_R(x_0)|^2 &\le CM^2\left(r^{2\alpha}+\left(\frac{R}{r}\right)^n R^{2\alpha}\right)\\
&\le CM^2\left(1+\left(\frac{R}{r}\right)^n\right)R^{2\alpha}\\
|\bar{u}_r(x_0)-\bar{u}_R(x_0)| &\le CM\left(1+\left(\frac{R}{r}\right)^{\frac{n}{2}}\right)R^{\alpha}. \qquad (20)
\end{aligned}
$$

Let $r=2^{-l-1}L, R=2^{-l}L$ and plug this into (20) to see that

$$
\begin{aligned}
|\bar{u}_r-\bar{u}_R| &\le CM\left(1+\left(\frac{2^{-l}L}{2^{-l-1}L}\right)^{\frac{n}{2}}\right)(2^{-l}L)^{\alpha}\\
&= CM(1+2^{\frac{n}{2}})(2^{-l}L)^{\alpha}\\
&= CM(2^{-l}L)^{\alpha}. \qquad (21)
\end{aligned}
$$

Now use (21) to see that

$$
\begin{aligned}
|\bar{u}_{2^{-m-1}L}-\bar{u}_{2^{-l}L}| &\le \sum_{i=l}^{m}|\bar{u}_{2^{-i-1}L}-\bar{u}_{-i}|\\
&\le CM\sum_{i=l}^{m}(2^{-i}L)^{\alpha}\\
&\le CM(2^{-l}L)^{\alpha} \qquad (22)
\end{aligned}
$$

where we have used the fact that we had a telescoping sequence and that we have a geometric sequence that we have bounded by the highest term. This implies that $\{\bar{u}_{2^{-l}L}\}$ is a cauchy sequence and so we can define

$$u^* := \lim_{l\to\infty}\bar{u}_{2^{-l}L}(x_0).$$

We will now show that this is independent of L. Take $L<L'$. Then (22) implies that

$$|\bar{u}_{2^{-l}L}(x_0)-\bar{u}_{2^{-l}L'}| \le CM\left(1+\left(\frac{L'}{L}\right)^n\right)(2^{-l}L')^{\alpha}$$

and taking $l\to\infty$ shows that u^* is independent of L.

From all of our hard work, we can say that $u^*=u$ almost everywhere by the Lebesgue Differentiation Theorem which says that if $u\in L^1$, then $\lim_{r\to 0}\bar{u}_r(x_0)=u(x_0)$ for almost every x_0. Now taking $m\to\infty$ in (22) we are able to get

$$
\begin{aligned}
|\bar{u}_{2^{-m-1}L}(x_0)-\bar{u}_{2^{-l}L}(x_0)| &\le CM(2^{-l}L)^{\alpha}\\
|u^*(x_0)-\bar{u}_{2^{-l}L}(x_0)| &\le CM(2^{-l}L)^{\alpha}\\
|u^*(x_0)-\bar{u}_r(x_0)| &\le CMr^{\alpha}. \qquad (23)
\end{aligned}
$$

Letting $r = 1$ implies that $\|u^*(x_0)\|_{L^\infty(B_1)} \le C(M + \|u\|_{L^1(B_1)})$

Now in order to complete this theorem we need to estimate $[u^*]_{C^\alpha}$. Let $x, y \in \Omega$ such that $B_r(x), B_r(y) \subset\subset \Omega$ and $B_r(x) \cap B_r(y) \neq \emptyset$. Denote $\delta = |x - y|$ and let z be the point midway between x and y. By convexity we can see that $B_\delta(z) \subset\subset B_r(x) \cap B_r(y)$. Then we write

$$\begin{aligned}|u^*(x) - u^*(y)| &\le |u^*(x) - \bar{u}_\delta(x)| + |u^*(y) - \bar{u}_\delta(y)| + |\bar{u}_\delta(x) - u(z)| + |\bar{u}_\delta(y) - u(z)| \\ |u^*(x) - u^*(y)|^2 &\le C\left(|u^*(x) - \bar{u}_\delta(x)|^2 + |u^*(y) - \bar{u}_\delta(y)|^2 + |\bar{u}_\delta(x) - u(z)|^2 + |\bar{u}_\delta(y) - u(z)|^2\right) \\ &\le CM^2\delta^{2\alpha} + C|\bar{u}_\delta(x) - u(z)|^2 + C|\bar{u}_\delta(y) - u(z)|^2\end{aligned}$$

where we have used (23) on the first two terms. Integrating the inequality with respect to z and using the assumptions of the lemma yields the required results. ☐

Lemma 3.3. *Assume that*

$$\int_{B_r(x_0)} |Du|^2 \le M^2 r^{n-2+2\alpha}.$$

Then $u \in C^\alpha$.

Proof. The details of the proof will be left as an exercise. Here is a sketch of it. Recall the Poincaré inequality:

$$\int_S |u - \bar{u}_S|^2 \le \lambda_S \int_S |Du|^2.$$

Then you need to show that

$$\int_{B_r(x_0)} |u - \bar{u}_r(x_0)|^2 \le Cr^2 \int_{B_r(x_0)} |Du|^2$$

where c is independent of r. The way to do this is to apply the Poincaré Inequality with $r = 1$. Then consider the rescaling $\tilde{u} = u(rx)$. After doing this, apply the assumption of the lemma and you'll find yourself in the position of Lemma 3.2. ☐

Lets recall our goals. We were proving Schauder estimates to get regularity. Assume $u \in W^{1,2}$ is a weak solution of $\partial_i(a_{ij}\partial_j u) = 0$ i.e.

$$\int a_{ij}\partial_j u \partial_i v = 0 \quad \forall v \in W_0^{1,2}.$$

Then we will show that $u \in C^\alpha$ where $0 < \lambda \le a_{ij} \le \Lambda$ and $|a_{ij}(x) - a_{ij}(y)| \le \tau(|x - y|)$ with $\tau \downarrow 0$ as $R \downarrow 0$. We will be using the following key estimate for $0 < r < R$

$$\int_{B_r(x_0)} |Du|^2 \le C\left[\left(\frac{r}{R}\right)^n + \tau(R)\right] \int_{B_R(x_0)} |Du|^2. \tag{24}$$

Proof. We first prove the case where we have a constant coefficient i.e. $\tau \equiv 0$. Then we will show that

$$\int_{B_r(x_0)} |Du|^2 \le C\left(\frac{r}{R}\right)^n \int_{B_R(x_0)} |Du|^2 \tag{25}$$

and use a the Lemma of De Giorgi (seen later). By rescaling $v(x) = u(Rx)$ what we need to show becomes

$$R^{-2}\int_{B_{\frac{r}{R}}(x_0)} |Dv|^2 \le C\left(\frac{r}{R}\right)^n \int_{B_1(x)} |Dv|^2 \cdot R^{-2}$$

$$\Updownarrow$$

$$\int_{B_s(x_0)} |Dv|^2 \le Cs^n \int_{B_1(x_0)} |Dv|^2.$$

This will follow from previous work. Take s small ($s < \frac{1}{2}$). Then By the Sobolev Embedding Theorem and by the first theorem proved in this section we are able to say

$$\begin{aligned}
\int_{B_s(x_0)} |Dv|^2 &\leq \sup_{B_s} |Dv|^2 \cdot s^n \\
&\leq \left(\sum_{|\alpha| \leq k} \int_{B_{\frac{3}{4}}(x_0)} D^\alpha (Dv)^2 \right) s^n \\
&\leq C s^n \int_{B_1(x_0)} |Dv|^2
\end{aligned}$$

and we are done showing (25), which corresponds to the case where a_{ij} are constants. Since this case will follow when we prove the Lemma of De Giorgi, we move on to the non-constant case.

We will prove the non-constant case as a perturbation of the equation $a_{ij}(x_0)\partial_i\partial_j w = 0$, which we have already finished. To carry this out, consider the following Dirichlet problem

$$\begin{cases} a_{ij}(x_0)\partial_i\partial_j w = 0 & \text{weak sense} \\ w - u \in W_0^{1,2}(B_R(x_0)). \end{cases}$$

Set $v = u - w$ and $u = v + w$. Then

$$\begin{aligned}
\int_{B_r(x_0)} |Du|^2 &\leq 2\left[\int_{B_r(x_0)} |Dv|^2 + \int_{B_r(x_0)} |Dw|^2 \right] \\
&\leq 2\left[\int_{B_r(x_0)} |Dv|^2 + c\left(\frac{r}{R}\right)^n \int_{B_R(x_0)} |Dw|^2 \right] \\
&\leq 2\left[\int_{B_r(x_0)} |Dv|^2 + c\left(\frac{r}{R}\right)^n \int_{B_R(x_0)} |Du|^2 + |Dv|^2 \right] \\
&\leq C\left[\left(\frac{r}{R}\right)^n \int_{B_r(x_0)} |Du|^2 + \int_{B_R(x_0)} |Dv|^2 \right] \qquad (26)
\end{aligned}$$

where we have used the fact that w solves our PDE with constant coefficients, saw that the integral over r is $\leq$ than the integral over R, and saw $\max\{1, \frac{r^n}{R^n}\} = 1$. Now we need to control the integral of $|Dv|^2$, so we use the fact that $v = u - w$, where u solves $\partial_i(a_{ij}\partial_j u) = 0$ and w solves $a_{ij}(x_0)\partial_i\partial_j w = 0$ in the weak sense. Now since $v \in W_0^{1,2}$, we are able to use it as a test function in the definition of weak derivatives to get

$$\begin{aligned}
\int a_{ij}(x_0) D_j v D_i v &= \int a_{ij}(x_0)(D_j u + D_j w) D_i v \\
&= \int a_{ij}(x_0) D_j u D_i v \\
&= \int (a_{ij}(x_0) - a_{ij}(x)) D_j u D_i v + \int a_{ij}(x) D_j u D_i v \\
&= \int (a_{ij}(x_0) - a_{ij}(x)) D_j u D_i v.
\end{aligned}$$

We now use the fact that a_{ij} is elliptic and use its modulus of continuity to see

$$\begin{aligned}
\lambda \int_{B_R(x_0)} |Dv|^2 &\leq \tau(R) \int_{B_R(x_0)} |Du||Dv| \\
&\leq \tau(R) \int_{B_R(x_0)} \frac{1}{2}|Du|^2 + \frac{1}{2}|Dv|^2 \\
\int_{B_R(x_0)} |Dv|^2 &\leq C\tau(R) \int_{B_R(x_0)} |Du|^2
\end{aligned}$$

which we can plug into (26) to see that we have finally proved (24) for the case of non-constant a_{ij}.

We still need to relate everything to get that $u \in C^{\alpha}$. Now we finally state the Lemma of De Girogi:

Lemma 3.4 (Lemma of De Giorgi)**.** *Let $\varphi(r) \geq 0$ with $\varphi(r) \downarrow$ as r $\downarrow$, and $A, B \geq 0$. Assume that $0 < r < R$ and $\exists \alpha > \beta > 0$ such that*

$$\varphi(r) \leq A\left[\left(\frac{r}{R}\right)^{\alpha} + \epsilon\right]\varphi(R) + BR^{\beta}. \tag{27}$$

Then $\forall 0 < \beta < \gamma < \alpha, \exists \epsilon_0$ such that $\epsilon < \epsilon_0$ implies

$$\varphi(r) \leq C\left[\left(\frac{r}{R}\right)^{\gamma}\varphi(R) + Br^{\beta}\right]. \tag{28}$$

We will apply the Lemma with

$$\varphi(r) = \int_{B_r(x_0)} |Du|^2$$

because (24) implies that $\varphi(r)$ satisfies (27) with $B = 0$ and $\alpha = n$. Then by the Lemma of De Giorgi we have

$$\varphi(r) \leq C\left(\frac{r}{R}\right)^{\gamma}\varphi(R)$$

and by taking $R = 1$ we imply that $\varphi(r) \leq Cr^{\gamma}$ after absorbing $\varphi(1)$ into the constant. We are now in the case of Lemma 3.3 because we are free to choose any $\gamma < \alpha = n$ and

$$\int |u - \bar{u}_r(x_0)|^2 dx \leq Cr^{\gamma+2}.$$

Hence $u \in C^{\alpha}$. ☐

Recall that we consider weak solutions of

$$\partial_i(a_{ij}\partial_j u) + c(x)u = f$$

where $0 < \lambda \leq a_{ij} \leq \Lambda$ and $|a_{ij}(x) - a_{ij}(y)| \leq \tau(|x-y|)$ where $\tau \downarrow 0$ as $R \downarrow 0$, $c \in L^n$ and $f \in L^q$ for $\frac{n}{2} < q < n$. We want to show that $u \in C^{\alpha}$ for $\alpha = 2 - \frac{n}{q}$.

Proof. We already got the case for $c = f = 0$, where for $0 < r < R$ the main tool was showing

$$\int_{B_r(x_0)} |Du|^2 \leq C\left\{\left(\frac{r}{R}\right)^n \int_{B_R(x_0)} |Du|^2 + \tau(R)\int_{B_R(x_0)} |Dv|^2\right\}. \tag{29}$$

Recall that we defined $v = u - w$ where w solves

$$\begin{cases} a_{ij}(x_0)\partial_i\partial_j w = 0 & \text{on } B_R(x_0) \\ u - w \in W_0^{1,2}(B_R(x_0)) \end{cases}$$

in the generalized sense. By the same argument as before, (29) can also be established for non-trivial lower ordered terms. Lets now estimate $|Dv|$ in (29):

$$\begin{aligned} \int a_{ij}(x_0)D_ivD_jv &= \int a_{ij}(x_0)(D_iu - D_iw)D_jv \\ &= \int a_{ij}(x_0)D_iuD_jv \\ &= \int a_{ij}(x)D_iuD_jv + \int (a_{ij}(x_0) - a_{ij}(x))D_iuD_jv \\ &= -\int c(x)uv + \int fv + \int (a_{ij}(x_0) - a_{ij}(x))D_iuD_jv. \end{aligned} \tag{30}$$

These calculations show two new terms that were not there before in the proof of the previous theorem. We handle these terms by bounding their derivatives. Recall the Sobolev inequality:

$$\|v\|_{L^{\frac{2n}{n-2}}} \leq C\|Dv\|_{L^2}$$

and use the Holder's and the Sobolev inequality

$$\begin{aligned}\left|\int_{B_R(x_0)} c(x)uv\right| &\leq \|u\|_{L^2}\left(\int_{B_R(x_0)} |cv|^2\right)^{1/2}\\ &\leq \|u\|_{L^2}\left[\left(\int |v|^{\frac{2n}{n-2}}\right)^{?}\left(\int |c^2|\right)^{?}\right]^{1/2}\end{aligned}$$

where we have to figure out which powers to use. We need Lesbegue conjugates $\frac{1}{p}+\frac{1}{q}=1$ and so we let $2p=\frac{2n}{n-2} \implies p=\frac{n}{n-2}, q=\frac{n}{2}$. This implies

$$\int |cv|^2 \leq \left(\int |v|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\left(\int (|c|^2)^{\frac{n}{2}}\right)^{2/n} = \|v\|^2_{L^{\frac{2n}{n-2}}}\|c\|^2_{L^n}$$

Putting this together yields

$$\begin{aligned}\left|\int c(x)uv\right| &\leq \|u\|_{L^2}\|v\|_{L^{\frac{2n}{n-2}}}\|c\|_{L^n}\\ &\leq \|u\|_{L^2}\|Dv\|_{L^2}\|c\|_{L^n}\\ &\leq \frac{\epsilon}{2}\|Dv\|^2_{L^2}+\frac{1}{2\epsilon}\|u\|^2_{L^2}\|c\|^2_{L^n}\end{aligned}$$

and now the first term gets absorbed in (29). The second term in (30) can be bounded as follows

$$\begin{aligned}\left|\int_{B_R} fv\right| &\leq \|f\|_{L^{\frac{2n}{n+2}}}\|v\|_{L^{\frac{2n}{n-2}}}\\ &\leq \|f\|_{L^{\frac{2n}{n+2}}}\|Dv\|_{L^2}\\ &\leq \frac{\epsilon}{2}\|Dv\|^2_{L^2}+\frac{1}{2\epsilon}\|f\|^2_{L^{\frac{2n}{n+2}}}.\end{aligned}$$

Once again the Dv term gets absorbed in (29). Putting everything together gives us the following estimate

$$\int_{B_r(x_0)} |Du|^2 \leq C\left\{\left(\left(\frac{r}{R}\right)^n+\tau(R)\right)\int_{B_R(x_0)}|Du|^2\right\}+C\|u\|^2_{L^2(B_R)}\|c\|^2_{L^n(B_R)}+C\|f\|^2_{L^{\frac{2n}{n+2}}}. \tag{31}$$

Since we want to show $u \in C^\alpha$, we look for things of the form

$$\int_{B_r(x_0)} |Du|^2 \leq Mr^{n+2-2\alpha}.$$

Now recall the Lemma of De Giorgi: $\varphi(r) \downarrow 0$ as $r \downarrow 0, \varphi(r) \geq 0$. Assume that $\forall 0<r<R, \beta<\alpha$ and

$$\varphi(r) \leq A\left(\left(\frac{r}{R}\right)^\alpha+\epsilon\right)\varphi(R)+BR^\beta.$$

Then $\forall 0<\beta<\gamma<\alpha, \exists \epsilon_0>0$ such that $\epsilon<\epsilon_0$ implies

$$\varphi(r) \leq C\left(\frac{r}{R}\right)^\gamma \varphi(R)+Br^\beta.$$

In particular we can let $R = 1$ above and have

$$\varphi(r) \le \tilde{C} r^{\beta}.$$

The analysis tells us that in order to apply the lemma, we need to estimate the additional terms in (31) by R^γ for $\gamma = n - 2 + 2\alpha$. Lets begin with the f term

$$\|f\|^2_{L^{\frac{2n}{n+2}}(B_R(x_0))} = \left(\int_{B_R(x_0)} |f|^{\frac{2n}{n+2}} \right)^{\frac{n+2}{n}} \le \left(\left(\int |f|^q \right)^? \left(\int 1 \right)^? \right)^{\frac{n+2}{n}}.$$

In order to figure out the exponents, we need $\frac{2n}{n+2} p = q \implies p = \frac{q(n+2)}{2n} \implies \frac{1}{m} = 1 - \frac{1}{p} = 1 - \frac{2n}{(n+2)q}$. Thus

$$\begin{aligned} \|f\|^2_{L^{\frac{2n}{n+2}}(B_R(x_0))} &\le \left(\left(\int |f|^q \right)^{1/p} \left(\int 1 \right)^{1/m} \right)^{\frac{n+2}{n}} \\ &= \left(\|f\|^{q/p}_{L^q} R^{n/m} \right)^{\frac{n+2}{n}} \\ &= \|f\|^2_{L^q} R^{\frac{n+2}{m}} = \|f\|^2_{L^q} R^{n-2+2\alpha} \end{aligned}$$

where $\alpha = 2 - \frac{n}{q}$. Now consider the other term in (31):

$$\begin{aligned} \|u\|^2_{L^2(B_R(x_0))} &= \int_{B_R(x_0)} |u|^2 \\ &\le \left(\int_{B_R} |u|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \left(\int_{B_R} 1 \right)^{2/n} \\ &\le \|u\|^2_{L^{\frac{2n}{n-2}}} (R^n)^{2/n} \\ &\le \|u\|^2_{W^{1,2}(B_R)} R^2. \end{aligned}$$

In the end the lower ordered terms are all bounded by

$$R^{\min\{n-2+2\alpha,2\}} \underbrace{(\|u\|^2_{W^{1,2}(B_R)} + \|f\|^2_{L^q})}_{B}$$

Then (31) and the De Giorgi Lemma tell us

$$\begin{aligned} \int_{B_r(x_0)} |Du|^2 &\le C \left(\frac{r}{R} \right)^\gamma \int_{B_R(x_0)} |Du|^2 + B r^{\min\{n-2+2\alpha,2\}} \\ &\le C r^{\min\{n-2+2\alpha,2\}} \end{aligned}$$

where again we have used the trick of letting $R = 1$.

Now there are two cases. If $\min\{n-2+2\alpha, 2\} = n - 2 + 2\alpha$, we are done by Lemma 3.3. If it is not, consider the integral of $|u|^2$ over $B_r(x_0)$. I claim that

$$\int_{B_r(x_0)} |u|^2 \le \left(\frac{r}{R} \right)^n \int_{B_r(x_0)} |u|^2 + C R^{2+\min\{n-2+2\alpha,2\}}.$$

If I manage to show this, observe that we are done by the De Giorgi Lemma. Consider the triangle type inequality $|u|^2 \le |u - \bar{u}_R(x_0)|^2 + \bar{u}_R^2$ and integrate over the ball of radius r to see

$$\begin{aligned}
\int_{B_r(x_0)} |u|^2 &\le \int_{B_r(x_0)} |u - \bar{u}_R(x_0)|^2 + \int_{B_r(x_0)} \bar{u}_R^2 \\
&\le \int_{B_R(x_0)} |u - \bar{u}_R(x_0)|^2 + \int_{B_r(x_0)} \bar{u}_R^2 \quad (r \le R) \\
&\le R^2 \int_{B_R(x_0)} |Du|^2 + \int_{B_r(x_0)} \bar{u}_R^2 \quad \text{(Poincaré)} \\
&\le R^{2+\min\{n-2+2\alpha,2\}} + \int_{B_r(x_0)} \bar{u}_R^2 \\
&= R^{2+\min\{n-2+2\alpha,2\}} + \int_{B_r(x_0)} \left(\frac{1}{R^n} \int_{B_R(x_0)} u \right)^2 \\
&\le R^{2+\min\{n-2+2\alpha,2\}} + \int_{B_r(x_0)} \left[\frac{1}{R^n} \left(\int_{B_R(x_0)} u^2 \right)^{1/2} \left(\int_{B_R(x_0)} 1 \right)^{1/2} \right]^2 \\
&= R^{2+\min\{n-2+2\alpha,2\}} + \int_{B_r(x_0)} \left(\frac{1}{R^{n/2}} \left(\int_{B_R(x_0)} u^2 \right)^{1/2} \right)^2 \\
&= R^{2+\min\{n-2+2\alpha,2\}} + \frac{r^n}{R^n} \int_{B_R(x_0)} u^2.
\end{aligned}$$

□

3.2 Harnack Inequality

The Harnack Inequality is essential in proving Regularity estimates in a different way as the previous section. We follow the proof of presented in Gilbarg and Trudinger very closely. We begin by defining our differential operator

$$Lu = D_i(a^{ij} D_j u + b^i u) + c^i D_i u + du$$

and force boundedness of the coefficients on L:

$$\sum |a^{ij}(x)|^2 \le \Lambda^2, \qquad \lambda^{-2} \left(\sum |b^i(x)|^2 + |c^i(x)|^2 \right) + \lambda^{-1} |d(x)| \le \nu^2.$$

We also make this an elliptic equation by enforcing

$$a^{ij}(x)\xi_i \xi_j \ge \lambda |\xi|^2, \quad \forall x \in \Omega, \xi \in \mathbb{R}^n.$$

Then u is a weak solution to $Lu = 0 (\ge 0, \le 0)$ in the domain Ω if

$$\int_\Omega \left\{ \left(a^{ij} D_j u + b^i u \right) D_i v - (c^i D_i u + du) v \right\} \, dx = 0 (\le 0, \ge 0)$$

for all non-negative functions $v \in C_0^1(\Omega)$. Let f^i, g be locally integrable functions in Ω. Then u is a *weak* solution of the inhomogeneous equation

$$Lu = g + D_i f^i$$

in Ω if it satisfies

$$\int_\Omega \left\{ \left(a^{ij} D_j u + b^i u \right) D_i v - (c^i D_i u + du) v \right\} \, dx = \int_\Omega \left(f^i D_i v - gv \right) \, dx$$

for all $v \in C_0^1(\Omega)$.

3.2.1 Structural Inequalities

We rewrite $Lu = g + D_i f^i$ as

$$D_i A^i(x,u,Du) + B(x,u,Du) = 0, \tag{32}$$

where

$$A^i(x,z,p) = a^{ij}p_j + b^i z - f^i$$
$$B(x,z,p) = c^i p_i + dz - g$$

for $(x,z,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$. Then we say that u is a weak subsolution (supersolution, solution) of (32) in Ω if $A^i(x,u,Du)$ and $B(x,u,Du)$ are locally integrable and

$$\int_\Omega \left(D_i v A^i(x,u,Du) - vB(x,u,Du)\right)\, dx \le (\ge,=)0 \tag{33}$$

for all non-negative $v \in C_0^1(\Omega)$. Writing $b = (b^1,\dots,b^n), c = (c^1,\dots,c^n), f = (f^1,\dots,f^n)$ and using the Schwarz inequality, we have the inequalities

$$p_i A^i(x,z,p) \ge \frac{\lambda}{2}|p|^2 - \frac{1}{\lambda}(|bz|^2 + |f|^2)$$
$$|B(x,z,p)| \le |c||p| + |dz| + |g|.$$

We can consolidate this further by writing

$$\bar{z} = |z| + k, \qquad \bar{b} = \lambda^{-2}(|b|^2 + |c|^2 + k^{-2}|f|^2) + \lambda^{-1}(|d| + k^{-1}|g|)$$

for some $k > 0$. Then for an $0 < \epsilon < 1$, we finally have the following inequalities

$$p_i A^i(x,z,p) \ge \frac{\lambda}{2}(|p|^2 - 2\bar{b}\bar{z}^2)$$
$$|\bar{z}B(x,z,p)| \le \frac{\lambda}{2}\left(\epsilon|p|^2 + \frac{\bar{b}}{\epsilon}\bar{z}^2\right).$$

If we denote a^{ij} by a, then we can write $|A(x,z,p)| \le |a||p| + |bz| + |f|$. Also note that we can divide (32) by $\frac{2}{\lambda}$ to *finally* get the structural inequalities

$$|A(x,z,p)| \le |a||p| + 2\bar{b}^{1/2}\bar{z}$$
$$p \cdot A \ge |p|^2 - 2\bar{b}\bar{z}^2 \tag{34}$$
$$|\bar{z}B(x,z,p)| \le \epsilon|p|^2 + \frac{1}{\epsilon}\bar{b}\bar{z}^2$$

for some $\epsilon \in (0,1], \bar{z} = |z| + k, \bar{b} = \frac{1}{4}(|b|^2 + |c|^2 + k^{-2}|f|) + \frac{1}{2}(|d| + k^{-1}|g|)$. For the purposes of the following proof, we will let $k = k(R) = \frac{1}{2}(R^\delta\|f\|_q + R^{2\delta}\|g\|_{q/2})$, for $\delta = 1 - n/q$.

3.2.2 Moser Iteration

Before proving The Harnack Inequality, we must prove two theorems first.

Theorem 3.5. *Let L be uniformly elliptic with bounded coefficients. $f^i \in L^q(\Omega), g \in L^{q/2}(\Omega)$ for $q > n$. Let $u \in W^{1,2}(\Omega)$ be a subsolution in Ω. Then for any $B_{2R}(y) \subset \Omega$, and $p > 1$, we have*

$$\sup_{B_R(y)} u \le C\left(R^{-n/p}\|u^+\|_{L^p(B_{2R}(y))} + k(R)\right). \tag{35}$$

Theorem 3.6. *Let L be uniformly elliptic with bounded coefficients and suppose that $f^i \in L^q(\Omega), g \in L^{q/2}(\Omega)$ for $q > n$. Let $u \in W^{1,2}(\Omega)$ be a supersolution in Ω. If u is non-negative in $B_{4R}(y) \subset \Omega$ and $1 \le p < n/(n-2)$, we have*

$$R^{-n/p}\|u\|_{L^p(B_{2R}(y))} \le C\left(\inf_{B_R(y)} u + k(R)\right). \tag{36}$$

Proof. It is convenient to prove these two theorems conjointly in the case where u is a bounded non-negative subsolution. We begin by assuming that $R = 1, k > 0$. The general case will be obtained from transforming $x \mapsto x/R$ and letting $k \to 0$. Let $\beta \ne 0, \eta \in C_0^1(B_4)$ be non-negative. We define $v := \eta^2\bar{u}^\beta$. Recall that $\bar{u} = u + k$. Then we have that

$$Dv = 2\eta D\eta\bar{u}^\beta + \beta\eta^2\bar{u}^{\beta-1}Du.$$

Note that v is a valid test function. Then we plug this v into our definition of subsolutions

$$\int_\Omega \left(D_i v A^i(x,u,Du) - vB(x,u,Du)\right)\,dx \le 0$$

and obtain

$$\int_\Omega \left(2\eta D\eta\bar{u}^\beta A(x,u,Du) + \beta\eta^2\bar{u}^{\beta-1}DuA(x,u,Du)\right)\,dx \le \int_\Omega \eta^2\bar{u}^\beta B(x,u,Du)\,dx. \tag{37}$$

We will now attempt to apply our structural inequalities (34) into (37):

$$\begin{aligned}
\eta^2\bar{u}^{\beta-1}DuA(x,u,Du) &\ge \eta^2\bar{u}^{\beta-1}|Du|^2 - 2\eta^2\bar{b}\bar{u}^{\beta+1}\\
|\eta^2\bar{u}^\beta B(x,u,Du)| &= \eta^2\bar{u}^{\beta-1}|\bar{u}B(x,u,du)|\\
&\le \eta^2\bar{u}^{\beta-1}\left(\epsilon|Du|^2 + \frac{1}{\epsilon}\bar{b}\bar{u}^2\right)\\
&= \epsilon\eta^2|Du|^2u^{\beta-1} + \frac{1}{\epsilon}\bar{b}\eta^2\bar{u}^{\beta+1}.
\end{aligned}$$

The last structural inequality gives us this:

$$\begin{aligned}
|\eta D\eta\cdot A(x,u,Du)\bar{u}^\beta| &\le |a|\eta|D\eta|\bar{u}^\beta|Du| + 2\bar{b}^{1/2}\eta|D\eta|\bar{u}^{\beta+1}\\
&= |a|\eta|D\eta|\bar{u}^{\frac{\beta-1}{2}}\bar{u}^{\frac{\beta+1}{2}}|Du| + 2\bar{b}^{1/2}\eta|D\eta|\bar{u}^{\beta+1}\\
&\le \frac{\epsilon}{2}\eta^2\bar{u}^{\beta-1}|Du|^2 + \frac{|a|^2}{2\epsilon}|D\eta|^2\bar{u}^{\beta+1} + |D\eta|^2\bar{u}^{\beta+1} + \bar{b}\eta^2\bar{u}^{\beta+1}\\
&= \frac{\epsilon}{2}\eta^2\bar{u}^{\beta-1}|Du|^2 + \left(1 + \frac{|a|^2}{2\epsilon}\right)|D\eta|^2\bar{u}^{\beta+1} + \bar{b}\eta^2\bar{u}^{\beta+1}.
\end{aligned}$$

Applying this to (37) yields

$$\begin{aligned}
\int_\Omega \left(\beta\eta^2\bar{u}^{\beta-1} - 2\bar{b}\beta\eta^2\bar{u}^{\beta+1}\right)\,dx \le \int_\Omega \bigg\{&\epsilon\eta^2|Du|^2\bar{u}^{\beta-1} + \frac{1}{\epsilon}\bar{b}\eta^2\bar{u}^{\beta+1} + \epsilon\eta^2\bar{u}^{\beta-1}|Du|^2\\
&+ \left(2 + \frac{|a|^2}{\epsilon}\right)|D\eta|^2\bar{u}^{\beta+1} + 2\bar{b}\eta^2\bar{u}^{\beta+1}\bigg\}\,dx
\end{aligned}$$

Combining like terms yields

$$\int_\Omega (\beta - 2\epsilon)\eta^2|Du|^2\bar{u}^{\beta-1}\,dx \le \int_\Omega \left\{\left(2\beta + 2 + \frac{1}{\epsilon}\right)\bar{b}\eta^2 + \left(2 + \frac{|a|^2}{\epsilon}\right)|D\eta|^2\right\}\bar{u}^{\beta+1}\,dx.$$

Let $\epsilon = \min\left\{1, \frac{\beta}{4}\right\}$. Then we can further consolidate this into

$$\int_\Omega \eta^2 |Du|^2 \bar{u}^{\beta-1}\, dx \leq C(\beta) \int_\Omega \left(\bar{b}\eta^2 + \left(1+|a|^2\right)|D\eta|^2\right) \bar{u}^{\beta+1}\, dx. \tag{38}$$

We introduce the function

$$w := \begin{cases} \bar{u}^{(\beta+1)/2} & \text{if } \beta \neq -1 \\ \log \bar{u} & \text{if } \beta = -1 \end{cases}$$

so that

$$|\eta D w| = \begin{cases} \eta \frac{\beta+1}{2} \bar{u}^{(\beta-1)/2}|Du| & \text{if } \beta \neq -1 \\ \eta |Du| \bar{u}^{-1} & \text{if } \beta = -1 \end{cases}$$

and let $\gamma = \beta + 1$. Then we can rewrite (38) as

$$\int_\Omega |\eta D w|^2\, dx \leq \begin{cases} C(|\beta|)\gamma^2 \int_\Omega \left(\bar{b}\eta^2 + \left(1+|a|^2\right)|D\eta|^2\right) w^2\, dx & \text{if } \beta \neq -1 \\ C\int_\Omega \left(\bar{b}\eta^2 + \left(1+|a|^2\right)|D\eta|^2\right)\, dx & \text{if } \beta = 1. \end{cases} \tag{39}$$

Before we move any further, we need to introduce a few results from analysis of Sobolev spaces.

Lemma 3.7 (Interpolation Inequality)**.** *Let $p \leq q \leq r$. Then for $u \in L^r(\Omega)$, we have*

$$\|u\|_q \leq \epsilon \|u\|_r + \epsilon^{-\mu}\|u\|_p,$$

where

$$\mu = \frac{\frac{1}{p} - \frac{1}{q}}{\frac{1}{q} - \frac{1}{r}}.$$

From the Sobolev Inequality, we have

$$\|\eta w\|^2_{2\hat{n}/(\hat{n}-2)} \leq C \int_\Omega \left(|D\eta w|^2 + |\eta D w|^2\right)\, dx,$$

for $\hat{n} = n$ for $n > 2$, and $2 < \hat{2} < q$. Now we apply Hölder's Inequality and the Interpolation Inequality to get

$$\begin{aligned} \int_\Omega \bar{b}(\eta w)^2\, dx &\leq \|\bar{b}\|_{q/2}\|\eta w\|^2_{2q/(q-2)} \\ &\leq \|\bar{b}\|_{q/2}\left(\epsilon\|\eta w\|_{2\hat{n}/(\hat{n}-2)} + \epsilon^{-\sigma}\|\eta w\|_2\right)^2, \end{aligned}$$

where $\sigma = \hat{n}/(q - \hat{n})$. Now we attempt to plug in these estimates into (39). Let $\xi = \hat{n}/(\hat{n}-2)$. Now we add a factor of $\int_\Omega |wD\eta|^2\, dx$ and carry out some computations to get the following

$$
\begin{aligned}
\|\eta w\|_{2\chi} &\le C\gamma^2 \int_\Omega \left(\bar{b}(\eta w)^2 + |wD\eta|^2 + |a|^2|wD\eta|^2\right)\, dx \\
&\le C\gamma^2 \|\bar{b}\|_{q/2}\left(\epsilon\|\eta w\|_{2\xi} + \epsilon^{-\sigma}\|\eta w\|_2\right)^2 + C\gamma^2\int_\Omega (1+|a|^2)|wD\eta|^2\, dx \\
&= C\gamma^2\left(\epsilon\|\eta w\|_{2\chi} + \epsilon^{-\sigma}\|\eta w\|_2\right)^2 + C\gamma^2\int_\Omega (1+|a|^2)|wD\eta|^2\, dx \\
&\le C\gamma^2\left(\epsilon^2\|\eta w\|_{2\chi}^2 + \epsilon^{1-\sigma}\|\eta w\|_{2\chi}\|\eta w\|_2 + \epsilon^{-2\sigma}\|\eta w\|_2^2 + \|wD\eta\|_2^2\right) \\
\|\eta w\|_{2\chi}^2(1 - C\gamma^2\epsilon^2) &\le C\gamma^2\left(\epsilon^{1-\sigma}\|\eta w\|_{2\chi}\|\eta w\|_2 + \epsilon^{-2\sigma}\|\eta w\|_2^2 + \|wD\eta\|_2^2\right) \\
&\le C\gamma^2\left(\frac{\epsilon^{1-\sigma}}{2}\|\eta w\|_{2\chi}^2 + \frac{\epsilon^{1-\sigma}}{2}\|\eta w\|_2^2 + \epsilon^{-2\sigma}\|\eta w\|_2^2 + \|wD\eta\|_2^2\right) \\
\|\eta w\|_{2\chi}^2\left(1 - C\gamma^2\left(\epsilon + \frac{\epsilon^{1-\sigma}}{2}\right)\right) &\le C\gamma^2\left(\frac{\epsilon^{1-\sigma}}{2} + \epsilon^{-2\sigma}\right)\left(\|\eta w\|_2^2 + \|wD\eta\|_2^2\right) \\
\|\eta w\|_{2\chi}^2 &\le \frac{C\gamma^2\left(\frac{\epsilon^{1-\sigma}}{2} + \epsilon^{-2\sigma}\right)}{1 - C\gamma^2\left(\frac{\epsilon^{1-\sigma}}{2} + \epsilon^{-2\sigma}\right)}\left(\|\eta w\|_2^2 + \|wD\eta\|_2^2\right) \\
&\le C(1+|\gamma|)^{\sigma+1}\|(\eta + |D\eta|)w\|_2^2
\end{aligned}
$$

Then we finally get

$$
\|\eta w\|_{2\chi} \le C(1+|\gamma|)^{\sigma+1}\|(\eta + |D\eta|)w\|_2, \tag{40}
$$

where $C = C(\hat{n}, \Lambda, \nu, q, |\beta|)$ is bounded when $|\beta|$ is bounded away from zero. We will now get a better cutoff function η. Let r_1, r_2 be such that $1 \le r_1 < r_2 \le 3, \eta \equiv 1$ in $B_{r_1}, \beta \equiv 0$ in $\Omega\backslash B_{r_2}$, with

$$
|D\eta| \le \frac{2}{r_2 - r_2}.
$$

Then we have from (40)

$$
\|w\|_{L^{2\chi}(B_{r_1})} \le \frac{C(1+|\gamma|)^{\sigma+1}}{r_2 - r_1}\|w\|_{L^2(B_{r_2})}. \tag{41}
$$

Before we move on, lets make a quick backtrack to functional spaces. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then I claim that if u is a measurable function on Ω such that $|u|^p \in L^1(\Omega)$ for some $p \in \mathbb{R}$, and define

$$
\phi_p(u) := \left(\frac{1}{|\Omega|}\int_\Omega |u|^p\, dx\right)^{1/p},
$$

then we have

$$
\lim_{p\to\infty} \phi_p(u) = \sup_\Omega |u|. \tag{42}
$$

To start, note that

$$
\phi_p(u) = \left(\frac{1}{|\Omega|}\int_\Omega |u|^p\, dx\right)^{1/p} \le \left(\frac{1}{|\Omega|}\int_\Omega \left(\sup_\Omega |u|\right)^p\, dx\right)^{1/p} = \sup_\Omega |u|.
$$

Taking the lim sup yields

$$
\limsup_{p\to\infty} \phi_p(u) \le \sup_\Omega |u|.
$$

Now fix ϵ and define $A = \{x \in \Omega | |u| \geq \sup_\Omega |u| - \epsilon\}$. Then we see that

$$\begin{aligned}\phi_p(u) &= \left(\frac{1}{|\Omega|}\int_\Omega |u|^p\,dx\right)^{1/p} \geq \left(\frac{1}{|\Omega|}\int_A |u|^p\,dx\right)^{1/p} \\ &\geq \left(\frac{1}{|\Omega|}\int_A \left(\sup_\Omega |u| - \epsilon\right)^p\,dx\right)^{1/p} \\ &= \frac{|A|}{|\Omega|}\left(\sup_\Omega |u| - \epsilon\right).\end{aligned}$$

Taking $\epsilon \to 0$ and then the lim inf yields

$$\liminf_{p\to\infty} \phi_p(u) \geq \sup_\Omega |u|.$$

This and our first inequality yield the claim (42). We go back to our proof. For $r < 4$, we define the function

$$\phi(p,r) := \left(\int_{B_r} |\bar{u}|^p\,dx\right)^{1/p}.$$

From what we just showed, we have that

$$\phi(\infty,r) = \lim_{p\to\infty}\phi(p,r) = \sup_{B_r}\bar{u}.$$

Now we can rewrite inequality (42) as

$$\phi(\chi\gamma,r_1) \leq \left(\frac{C(1+|\gamma|)^{\sigma+1}}{r_2-r_1}\right)^{2/|\gamma|}\phi(\gamma,r_2) \quad \text{if } \gamma > 0 \tag{43}$$

$$\phi(\gamma,r_2) \leq \left(\frac{C(1+|\gamma|)^{\sigma+1}}{r_2-r_1}\right)^{2/|\gamma|}\phi(\chi\gamma,r_1) \quad \text{if } \gamma < 0 \tag{44}$$

Now we are in the right position to start our iteration. Recall that when u is a subsolution, we have $\beta > 0$ and $\gamma > 1$. Taking $p > 1$ and setting $\gamma = \gamma_m = \chi^m p$ and $r_m = 1 + 2^{-m}$. Then plugging this into our inequality (43) gives us

$$\begin{aligned}\phi(\chi^m p,1) &\leq \left(\frac{C(1+|\chi^{m-1}p|)^{1+\sigma}}{r_2-1}\right)^{2/(\chi^{m-1}p)}\phi(\chi^{m-1}p,r_2) \\ &\leq \left(\frac{C(1+|\chi^{m-2}p|)^{1+\sigma}}{r_3-r_2}\right)^{2/(\chi^{m-2}p)}\left(\frac{C(1+|\chi^{m-1}p|)^{1+\sigma}}{r_2-1}\right)^{2/(\chi^{m-1}p)}\phi(\chi^{m-2},r_4) \leq \cdots \\ &\leq (C\chi)^{2(1+\sigma)\sum m\chi^{-m}}\phi(p,2) = C\phi(p,2).\end{aligned}$$

Letting $m \to \infty$ in the above yields

$$\sup_{B_1}|u| \leq C\|\bar{u}\|_{L^p(B_{r_2})}, \qquad C = C(\hat{n},\Lambda,\nu,q,p).$$

Transforming $x \mapsto x/R$ we have the desired estimate (35) if u is a subsolution. Now in the cases when u is a supersolution, we need to approach the problem a bit differently. Recall that when u is a supersolution, $\beta < 0$ and $\gamma < 1$. Then for any p, p_0 such that $0 < p_0 < p < \chi$, we have

$$\begin{aligned}\phi(p,2) &\leq C\phi(p_0,3) \\ \phi(-p_0,3) &\leq C\phi(-\infty,1).\end{aligned}$$

Then we will finish proving our theorem if we can show that $\phi(p_0,3) \leq C(-p_0,3)$. This is done in an intricate way and is left as an exercise. ☐

Putting our two theorems together give us the full Harnack Inequality:

Theorem 3.8. *Let L be uniformly elliptic and have bounded coefficients. Let $u \in W^{1,2}(\Omega)$ satisfy $u \geq 0$ in Ω and $Lu = 0$ in Ω. Then for any ball $B_{4R}(y) \subset \Omega$, we have*

$$\sup_{B_R(y)} u \leq C \inf_{B_R(y)} u,$$

where $C = C(n, \Lambda/\lambda, \nu R)$.

3.2.3 Applications of the Harnack Inequality

We can give a new proof of the strong maximum principle (instead of using Hopf's Lemma) now:

Theorem 3.9. *L uniformly elliptic, bounded coeff, $u \in W^{1,2}(\Omega), Lu \geq 0$ in Ω. Then if for some ball $B \subset\subset \Omega$ we have*

$$\sup_B u = \sup_\Omega u \geq 0,$$

then the function u must be constant in Ω.

Proof. We apply the Harnack inequality with $p = 1$ to the function $v = M - u$ and show that $M - u = 0$. □

Finally, we can use the Harnack Inequality to imply Hölder continuity.

Theorem 3.10. *Let L be uniformly elliptic and have bounded coefficients. Then if $u \in W^{1,2}(\Omega)$ is a solution of*

$$Lu = g + D_i f^i,$$

then u is locally Hölder continuous in Ω, and for any ball $B_0 = B_{R_0}(y) \subset \Omega$ and $R \leq R_0$ we have

$$\text{osc}_{B_R(y)} u \leq CR^\alpha (R_0^{-\alpha} \sup_{B_0} |u| + k).$$

Proof. Start out by defining M_0, M_1, M_4, m_1, m_4. Then apply Harnack inequality with $p = 1$ to $M_4 - u$ and $u - m_4$ to end up with

$$\omega(R) \leq \gamma\omega(4R) + k(R).$$

Here $\omega(R) = \text{osc}_{B_R} u$. Before we go on, we have to prove something else first. Suppose ω is non-decreasing on $(0, R_0], R \leq R_0$ satisfies

$$\omega(\tau R_{\leq} \gamma\omega(R) + \sigma(R),$$

where σ is also non-decreasing, $0 < \gamma, \tau, < 1$. Then for any $\mu \in (0, 1)$, we have

$$\omega(R) \leq C \left(\left(\frac{R}{R_0} \right)^\alpha \omega(R)_0 + \sigma(R^\mu R_0^{1-\mu}) \right).$$

Of course the De Giorgi Lemma concludes the proof. □

4 Harnack Inequality for Non-Divergence Equations

The formulation of weak solutions to divergence equations relied heavily on the fact that the operator L was in divergence form. This allowed us to integrate by parts and so a weak solution u needs to be once weakly differentiable ($W^{1,2}$). A classical solution u must be at least second order continuously differentiable. In this section we will concern ourselves with the intermediate situation of *strong solutions*.

Definition 4.1. For operators of the form

$$Lu = a^{ij}u_{ij} + b^i u_i + c(x)u \tag{45}$$

with coefficients $a^{ij}, b^i, c(x)$ defined on a domain $\Omega \subset \mathbb{R}^n$ and a function f on Ω, a *strong solution* of $Lu = f$ is a function $u \in W^{2,p}(\Omega)$ that satisfies (45) almost everywhere. See [1][p. 185] for existence and uniqueness of equations of this type.

4.1 ABP Estimate

We begin by proving a maximum principle for strong solutions analogous to the classical one for classical solutions. It is called the Alexandrov-Bakelmann-Pucci Maximum Principle. In particular, we will prove this maximum principle for solutions in the space $W^{2,n}_{\text{loc}}(\Omega)$ as it is the natural environment for such equations. Recall that an operator L in the form (45) is said to be *elliptic* in the domain $\Omega \subset \mathbb{R}^n$ if the matrix (a^{ij}) is positive definite everywhere in Ω. Let $\mathscr{D} := \det(a^{ij})$ and $\mathscr{D}^\star = \mathscr{D}^{1/n}$ so that

$$0 < \lambda \leq \mathscr{D}^\star \leq \Lambda$$

where λ, Λ are the minimum and maximum eigenvalues of (a^{ij}). Assume $b = c = 0$ so that we have $Lu = a^{ij}u_{ij}$. Our condition on a^{ij} and f are now

$$f/\mathscr{D}^\star \in L^n(\Omega).$$

Theorem 4.2 (ABP Maximum Principle)**.** *Let $Lu \geq f$ in a bounded domain Ω and $u \in C^0(\overline{\Omega}) \cap W^{2,n}_{loc}(\Omega)$. Then*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} + \frac{d}{n\omega_n^{1/n}} \left\| \frac{f}{\mathscr{D}^\star} \right\|_{L^n(\Gamma^+)}$$

where $d = \operatorname{diam}\Omega$.

It is important to note that the Morrey's embedding theorem guarantees that $u \in W^{2,n}_{\text{loc}}(\Omega)$ will be at least continuous in Ω because whenever $kp > n$, $W^{k,p}$ is embedded in C^α for $0 \leq \alpha < k - \frac{n}{p}$. Before we begin the proof, we must first go through notions of contact sets and normal mappings.

Definition 4.3. Suppose u is an arbitrary function on Ω. The *upper contact set* Γ^+ or Γ^+_u is defined to be the subset of Ω such that the graph of u is below a support hyperplane in $\mathbb{R}^{n+1}$, i.e.

$$\Gamma^+ = \{y \in \Omega | u(x) \leq p(x-y) + u(y) \ \forall x \in \Omega, \text{ for some } p \in \mathbb{R}^n\}.$$

From the definition, u will be a concave function on Ω if and only if $\Gamma^+ = \Omega$. It is clear that $p = Du(y)$ if $u \in C^1(\Omega)$. Finally, if $u \in C^2(\Omega)$, the Hessian $D^2u \leq 0$ on Γ^+. This means that we can essentially think of Γ^+ as the subset of Ω where u is concave down. The upper contact set of u is closed in Ω.

Definition 4.4. Suppose $u \in C^0(\Omega)$ is arbitrary. We define the *normal mapping* $\chi(y) = \chi_u(y)$ for a point $y \in \Omega$ to be the set of slopes of supporting hyperplanes at y lying above the graph of u, i.e.

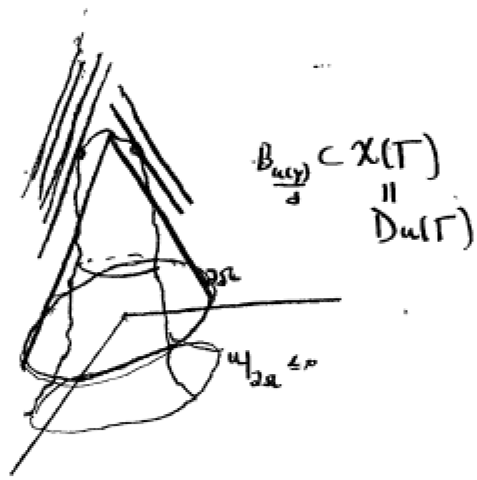
$$\chi(y) = \{p \in \mathbb{R}^n | u(x) \leq p(x-y) + u(y) \ \forall x \in \Omega\}.$$

Clearly $\chi(y)$ is nonempty if and only if $y \in \Gamma^+$. When $u \in C^1(\Omega)$, then $\chi_y = Du(y)$.

Proof of Theorem 4.2. Assume that $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$. Subtracting $\sup_{\partial\Omega} u$ from u yields the same differential inequality $L(u - \sup_{\partial\Omega}) \geq f$ and so we can assume $u \leq 0$ on the boundary. Note that we can assume $\sup_\Omega u \geq 0$ because if it was negative, there would be nothing to prove. This assumption implies that if $u(y) := \sup_\Omega u$, then y is in the interior of Ω because $u \leq 0$ on the boundary. Now I show that

$$B_{u(y)/d} \subset \chi(\Gamma) = Du(\Gamma). \tag{46}$$

This can be seen by sliding hyperplanes onto the graph of u. Consider the cone with vertex $u(y)$ with base $\partial\Omega$. Then the slope of the cone is $u(y)/d$ and $u \leq 0$ on $\partial\Omega$ implies that dropping down hyperplanes of slope $u(y)/d$ will eventually be tangent to $u(x)$ at a point $x \in \Gamma$. This can be seen very clearly in pictures:



It is a bit difficult to see, but in the picture, we have $u|_{\partial\Omega \leq 0}$. Now we compute

$$\begin{aligned} |Du(\Gamma)| &= \int_{Du(\Gamma)} 1 \\ &\leq \int_\Gamma |\det D^2 u|. \end{aligned} \tag{47}$$

Lets do a bit of linear algebra. I claim that given two positive matrices A, B, then

$$\det A \det B \leq \left(\frac{\operatorname{Tr} AB}{n}\right)^n.$$

To see this it suffices to show that

$$\det A \leq \left(\frac{\operatorname{Tr} A}{n}\right)^n.$$

Let $\lambda_1, \ldots, \lambda_n$ be A's eigenvalues. Then

$$\det A = \prod_{i=1}^n \lambda_i, \quad \operatorname{Tr} A = \sum_{i=1}^n \lambda_i.$$

Thus what we wish to show is equivalent to

$$\left(\prod \lambda_i\right)^{1/n} \leq \frac{1}{n} \sum \lambda_i,$$

which is exactly the inequality of arithmetic and geometric means. Taking $A = D^2u$,$B = (a^{ij})$ then **on** Γ

$$
\begin{aligned}
|\det D^2u| = \det -D^2u &= \frac{1}{\mathscr{D}}\det(a^{ij})\det -D^2u \\
&\leq \frac{1}{\mathscr{D}}\left(-\frac{f}{n}\right)^n.
\end{aligned} \tag{48}
$$

Combining (46),(47), and (48) yields

$$
\begin{aligned}
\omega_n\left(\frac{u(y)}{d}\right)^n &\leq \frac{1}{n^n\mathscr{D}}\|f^-\|^n_{L^n(\Gamma)} \\
u(y) &\leq \frac{d}{n\omega^{1/n}}\left\|\frac{f^-}{\mathscr{D}^\star}\right\|_{L^n(\Gamma)}.
\end{aligned}
$$

which is precisely the ABP estimate because we had replaced u with $u - \sup_{\partial\Omega} u$. □

The ABP maximum principle can be naturally extended to functions $u \in C^0(\overline{\Omega}) \cap W^{2.n}_{\mathrm{loc}}(\Omega)$ by approximating with smooth functions. It can also be extended for coefficients satisfying $|b|/\mathscr{D}^\star \in L^n(\Omega)$ and $c \leq 0$ in Ω, but these will be taken as a black box. Now we have the correct tools to begin proving the Harnack inequality for non-divergence equations. We say an operator of the form (45) is strictly elliptic when

$$
\frac{\Lambda}{\lambda} \leq \gamma, \quad \left(\frac{|b|}{\lambda}\right)^2, \frac{|c|}{\lambda} \leq \nu.
$$

We will assume throughout the rest of this section that the operator L given by (45) is uniformly elliptic.

Note that have the same ABP estimate from the other side: if $Lu \leq f$ and $u|_{\partial B_1} \geq 0$, then

$$
|\inf_{B_1} u| \leq C_n\left(\int_\Gamma f^n\right)^{1/n}
$$

where Γ is the convex envelop of u (contrary to the upper contact set Γ^+)

4.2 Measure theory and Harnack Inequality

In order to prove the Harnack inequality for non-divergence equations with measurable coefficients, we need a basic measure theoretic estimate that is true at "all scales." Our first main theorem is as follows

Theorem 4.5. *Let $u \in W^{2,n}$ and assume $a^{ij}u_{ij} \leq 0$ for some bounded measurable and uniformly elliptic a^{ij} in B_2. If $u \geq 0$ and $u(x) \leq 1$ for some $x \in \partial B_1$, then*

$$
|\{u \leq M\} \cap B_{1/2}| \geq \mu \tag{49}
$$

for some M large and μ small universal.

Heuristically, this is saying that if u goes below 1 at a certain point on the boundary of a ball of radius one, then it can't get much bigger in a ball of radius two. That is, the ABP estimate will tell us that the size of u can be controlled.

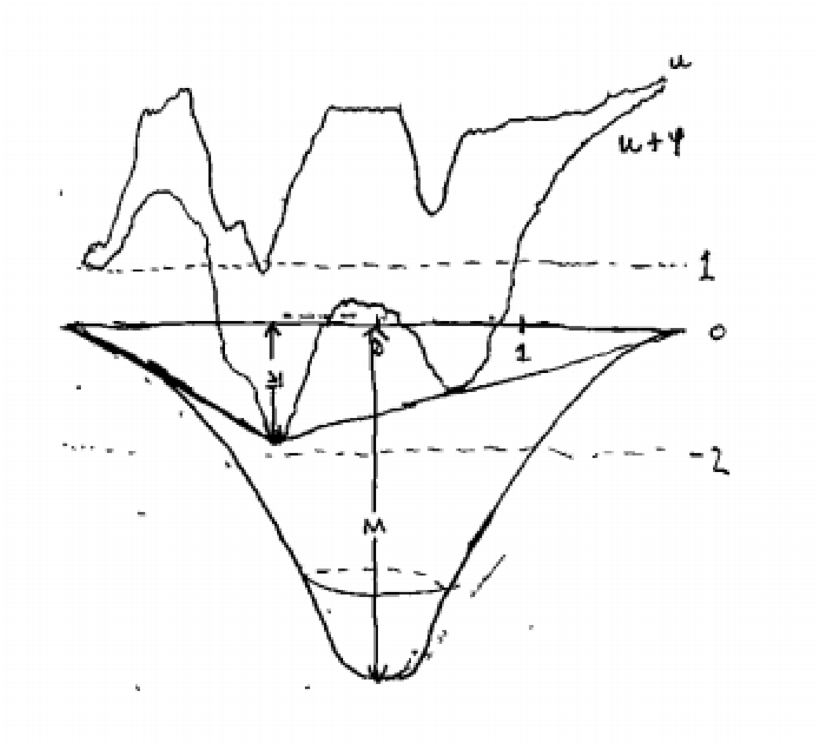
Proof. Define $\alpha = \max\{1, (n-1)\Lambda/\lambda - 1\}$ and take a barrier function φ of the form $\varphi(x) = M_1 - M_2|x|^\alpha$ in $B_2 - B_{1/2}$ with M_1 and M_2 chosen such that $\varphi_{\partial B_2} = 0$ and $\varphi|_{\partial B_1} = -2$. Note that we can cap off in $B_{1/2}$ smoothly so that the constants still depend only on n, λ, Λ (we say cap off to smooth out the inner radius with a paraboloid of radius 2; it'll be made clear with the picture). Suppose that $r \geq 1/2$ at the point $(r, 0, \ldots, 0)$. Then we compute some derivatives at this point:

$$\begin{aligned} D_{ij}\varphi &= 0 \quad \text{for } i \neq j \\ D_{11}\varphi &= -M_2\alpha(1+\alpha)r^{-\alpha-2}r^{-\alpha-2} \\ D_{ii}\varphi &= M_2\alpha r^{-\alpha-2}. \end{aligned}$$

By rotational symmetry of our function and uniform ellipticity, we have for $|x| \geq 1/4$,

$$\begin{aligned} a^{ij}\varphi_{ij}|_x &= M_2(\Lambda(n-1)\alpha|x|^{-\alpha-2} - \lambda\alpha(1+\alpha)|x|^{-\alpha-2}) \\ &= M_2\alpha|x|^{-\alpha-2}(\Lambda(n-1) - \lambda(1+\alpha)) \leq 0 \end{aligned} \tag{50}$$

by our choice of α. However, for $|x| \leq 1/4$, $a^{ij}u_{ij} \leq C = C(n, \lambda, \Lambda)$. This observation along with (50) shows that $a^{ij}\varphi_{ij} \leq C\eta$ for some C universal with $\eta \equiv 1$ in $B_{1/4}$ and $\eta \equiv 0$ outside of $B_{1/2}$, i.e. $\eta \in C_0^\infty$ (it is smooth because of the smoothness of $|x|^\alpha$ in this domain). Finally note that $|\varphi| \leq M$ for some M universal given by the distance of the vertex of the parabola that we capped off by and the origin.



By our assumption that $u \leq 1$ somewhere on ∂B_1 we have that $w = u + \varphi \leq -1$ somewhere on ∂B_1. Then ABP given gives us

$$1 \leq C \int_{\Gamma_w \cap B_1\}} \eta^n \leq C|\{\Gamma_w\} \cap B_{1/2}|.$$

Now the crucial part of the proof is to notice that the convex envelop of w is in the set that $w \leq 0 \Longrightarrow u + \varphi \leq 0 \Longrightarrow u \leq -\varphi \leq M$. Hence our estimate just gave us $1 \leq C|\{u \leq M\} \cap B_{1/2}|$. This concludes the proof with $\mu = 1/C$. ☐

Corollary 4.6. *We can show that this is scaling invariant. That is, if u is defined in B_{2r} and $u \leq \alpha$ somewhere on ∂B_r, then $|\{u \leq M\alpha\} \cap B_{r/2}| \geq \mu$.*

Lemma 4.7. *Suppose we have the same u as in Theorem 3.5. Then*

$$|\{u \geq M^k\} \cap B_{1/2}| \leq (1-\mu)^k$$

for $k = 1, 2, \ldots,$ and M, μ are as in Theorem 3.5.

This implies that

$$|\{u \geq t\} \cap Q_1| \leq dt^{-\epsilon} \tag{51}$$

for any $t > 0$ where d, ϵ are positive universal constants.

Proof. This proof will follow from Calderon-Zygmund decomposition and induction. For $k = 1$, this is the statement of Theorem 4.5. Now suppose it holds for $k-1$. We introduce the classical decomposition: if Q is a dyadic cube different from Q_1, we say $\tilde{Q}$ is the predecessor of Q if Q is one of the 2^n cubes obtained from dividing $\tilde{Q}$. Then the decomposition states the following: if $A \subset B \subset Q_1$ are measurable sets and $0 < \delta < 1$ such that

(1) $|A| \leq \delta$, and

(2) if Q is a dyadic cube such that $|A \cap Q| > \delta|Q|$, then $\tilde{Q} \subset B$,

then $|A| \leq \delta|B|$.

Recall that we assume the Lemma holds for $k-1$. For convenience, we use cubes because Theorem 4.5 still holds for them. Let $A = \{u > M^k\} \cap Q_1$ and $B = \{u > M\} \cap Q_1$. We'll be done with our lemma if we show that $|A| \leq (1-\mu)|B|$ by our inductive hypothesis. This is where we'll use the decomposition. Clearly $A \subset B \subset Q_1$ and $|a| \leq |\{u > M\} \cap Q_1| \leq 1 - \mu$ by Theorem 4.5. Now we need to show (2) also holds.

(2) will follow if we show that if $Q = Q_{1/2^i}(x_0)$ is a dyadic cube such that $|A \cap Q| > (1-\mu)|B|$, then $\tilde{Q} \subset B$. Suppose it isn't and take $\tilde{x} \in \tilde{Q}$ such that $u(\tilde{x}) \leq M^{k-1}$. Now consider the transformation $x = x_0 + 1/2^i y$ for $y \in Q_1$ and $x \in Q = Q_{1/2^i}(x_0)$ and the function

$$\tilde{u}(y) = u(x)/M^{k-1}.$$

Then I claim that $\tilde{u}$ is under the hypothesis of Theorem 4.5. By (49), it follows that

$$\mu < |\{\tilde{u}(y) \leq M\} \cap Q_1| = 2^{in}|\{u(x) \leq M^k\} \cap Q|.$$

We see that $|Q - A| > \mu|Q|$, which contradicts our assumption that $|A \cap Q| > (1-\mu)|Q|$. Showing that $\tilde{u}$ satisfies the conditions of Theorem 4.5 will be left as an exercise. As a hint, the following property of dyadic cubes: if Q is a dyadic cube $Q = Q_{1/2^i}(x_0)$ for some $i \geq 0$ and $x_0 \in Q_1$, then

$$Q_{4\sqrt{n}/2^i}(x_0) \subset Q_{4\sqrt{n}},$$

$$i \geq 1 \Longrightarrow \tilde{Q} \subset Q_{3/2^i}(x_0).$$

(51) follows from the results above by taking $d = (1-\mu)^{-1}$ and ϵ such that $1 - \mu = M^{-\epsilon}$. □

Now we need yet another measure theoretic lemma. Let f be a measurable function on a domain Ω in $\mathbb{R}^n$. Define the distribution function $\mu(t) = |\{f \geq t\}|$ for $t > 0$. This measures the "relative" size of f. Note that μ is a decreasing function on the positive real line.

Lemma 4.8. *For any $p > 0$ and $|f|^p \in L^1(\Omega)$,*

$$\int_\Omega |f|^p = p \int_0^\infty t^{p-1}\mu(t)dt.$$

Proof. This is just computations. Suppose $f \in L^1$. Then

$$\int_\Omega |f| = \int_\Omega \int_0^{|f(x)|} dtdx$$
$$= \int_0^\infty \mu(t)$$

and the lemma will hold for general p after change of variables. □

Now we are finally able to prove the weak Harnack inequality:

Theorem 4.9 (Weak Harnack Inequality)**.** *Suppose $u \in W^{2,n}(Q_1)$ satisfies $a^{ij}u_{ij} \leq 0$ in Q_1 and u is non-negative. Then*

$$\|u\|_{L^p(Q_{1/4})} \leq C \inf_{Q_{1/2}} u$$

where $p > 0$ and C universal.

Proof. We already have most of the machinery for the proof. This follows from (51) in Lemma 4.7 because if we suppose u is under the assumptions of the emma, $|\{u \geq t\} \cap Q_1| \leq dt^{-\epsilon}$ and Lemma 4.8 implies for $p = \epsilon/2$

$$\begin{aligned}
\int_{Q_1} u^p &= p\int_0^\infty t^{p-1}|\{u \geq t\} \cap Q_1|dt \\
&\leq p\int_0^\infty t^{p-1} \cdot d \cdot t^{-\epsilon}dt \\
&= d \cdot p\int_1^\infty t^{p-1}t^{-2p}dt \quad (\inf u \geq 1) \\
&= d \cdot p\int_1^\infty t^{-p-1}dt \\
&= d \cdot p\frac{t^{-p}}{p}\Big|_0^\infty \\
&= d.
\end{aligned}$$

hence $\|u\|_{L^p(Q_1)} \leq C$ and we re-scale u away from the assumptions of 4.7 to get the result of the weak HI. □

The second piece of HI is called the local maximum principle

Theorem 4.10. *Let $u \in W^{2,n}(Q_1)$ and let $a^{ij}u_{ij} \geq 0$. Then for any $0 < p \leq n$, we have*

$$\sup_{Q_{1/2}} u \leq C_p\|u^+\|_{L^p(Q_{3/4})}.$$

for C universal depending on p.

Proof. See [1][p. 244]. □

Corollary 4.11. *Taking $p = 1$ in Theorem 4.10, we obtain an extension of the mean value inequality for non-negative subharmonic functions:*

$$u(y) \leq \frac{C}{R^n}\int_{B_R} u$$

when $Lu \geq 0, C = C(n, \gamma, \nu R^2)$.

5 Curved $C^{1,\alpha}$ Domains

5.1 Estimate for Laplacian

We begin with a simple definition.

Definition 5.1. A continuous function u on $\mathbb{R}^n$ is said to be $C^{2,\alpha}$ at x_0 if there exists a quadratic polynomial P and constants C, ρ such that

$$\|u - P\|_{L^\infty(B_r)} \le Cr^{2+\alpha}, \quad \forall r \le \rho. \tag{52}$$

We say $u \in C^{2,\alpha}(B_1)$ if u is $C^{2,\alpha}$ at all $x \in B_1$.

Now we develop a tool that helps us determine when a function is $C^{2,\alpha}$.

Proposition 5.2. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Suppose we can find a sequence of paraboloids $P_k = a_k + b_k \cdot x + \frac{1}{2}x^T c_k x$ and an $r < 1$ such that*

$$\|u - P_k\|_{L^\infty(B_{r^k})} \le Cr^{k(2+\alpha)}$$

where $B_{r^k} \subset\subset \Omega$. Then $u \in C^{2,\alpha}(0)$ where $0 < \alpha < 1$.

Proof. Consider the quadratic scaling of a function $f_r(x) = \frac{1}{r^2}f(rx)$. By our hypothesis we have

$$\begin{aligned} \|P_{k+1} - P_k\|_{L^\infty(B_{r^{k+1}})} &\le \|u - P_{k+1}\|_{L^\infty(B_{r^{k+1}})} + \|u - P_k\|_{L^\infty(B_{r^{k+1}})} \\ &\le Kr^{(k+1)(2+\alpha)} + Kr^{k(2+\alpha)} \\ &\le 2Kr^{k(2+\alpha)}. \end{aligned}$$

Quadratic rescaling gives us

$$\|P_{k+1,r^k} - P_{k,r^k}\|_{L^\infty(B_1)} \le 2Kr^{k\alpha}.$$

Since the coefficients of polynomials on B_1 are controlled by the L^∞ norm, we have

$$|a_{k+1} - a_k| \le Kr^{k(2+\alpha)}, \quad |b_{k+1} - b_k| \le Kr^{k(\alpha+1)}, \quad |c_{k+1} - c_k| \le Kr^{k\alpha}. \tag{53}$$

Thus we have that the P_k converges to a polynomial $P = a + bx + x^T cx$. Putting these together yield

$$\begin{aligned} \|u - P\|_{L^\infty(B_{r^k})} &\le \|u - P_k\|_{L^\infty(B_{r^k})} + |a_k - a| + r^k|b_r - b| + r^{2k}|c_k - c| \\ &\le CKr^{k(2+\alpha)} \end{aligned}$$

and so we have that $u \in C^{2,\alpha}$. □

It is important to realize why we can't have $\alpha = 0, 1$. Notice that for $\alpha = 0$, the proof breaks down in (53) because we'd get $|c_{k+1} - c_k| \le K$, so the sequence of polynomials doesn't converge. The case for $\alpha = 1$ is left as an exercise. We utilize this proposition to prove the following theorem.

Theorem 5.3. *Suppose $\Delta u = f$ in B_1 where $f \in C^\alpha(B_1)$. Then $u \in C^{2,\alpha}(B_{1/2})$ and*

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \le C_{n,\alpha}\left(\|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(B_1)}\right).$$

Proof. By subtracting $\frac{1}{2n}f(0)|x|^2$ we assume $f(0) = 0$. Dividing by $\|u\|_{L^\infty(B_1)} + \frac{1}{\epsilon}\|f\|_{C^\alpha(B_1)}$ for some $\epsilon > 0$ to be chosen later, we can assume that $\|f\|_{C^\alpha(B_1)} \le \epsilon$ and $|u| \le 1$. Let w be the harmonic function that agrees with u on ∂B_1. Then $\Delta(u - w) = f$ and again $\|f\|_{C^\alpha(B_1)} \le \epsilon$. By the maximum principle,

$$\|u - w\|_{L^\infty(B_1)} \le C\epsilon.$$

Let P_1 be the harmonic quadratic approximation to w that satisfies

$$\|w - P_1\|_{L^\infty(B_r)} \le Cr^3.$$

The constant C that is above is in terms of n because $|w| \le 1$. Putting these together we have

$$\|u - P_1\|_{L^\infty(B_r)} \le C(r^3 + \epsilon).$$

Now here we use the fact that $0 < \alpha < 1$ so choose r so small that $2Cr^3 \le r^{2+\alpha}$ and choose ϵ so small that $2C\epsilon \le r^{2+\alpha}$. This gives

$$\|u - P_1\|_{L^\infty(B_r)} \le r^{2+\alpha}.$$

Now take the $2+\alpha$ rescaling $v(x) = \frac{1}{r^{2+\alpha}}(u - P_1)(rx)$. Notice that $|v| \le 1$ and since P_1 is harmonic we have that $\Delta v = \frac{1}{r^\alpha}\Delta u(rx) = \frac{1}{r^\alpha} f(rx) = g(x)$. The right side *again* satisfies $|g| \le \epsilon$ so we repeat our process again to find a harmonic polynomial P_2 such that

$$\|v - P_2\|_{L^\infty(B_r)} \le r^{2+\alpha}$$

and so scaling back this will become

$$\|u(x) - P_1(x) - r^{2+\alpha}P_2(x/r)\|_{L^\infty(B_{r^2})} \le r^{2(2+\alpha)}.$$

Iterating this will give us a sequence of polynomials approximating u. Thus we are in the position of Proposition 1.2, and we are finished. □

Lets shift gears a little bit and develop tools for boundary estimates. We say that a Ω is a Schauder domain, or a $C^{k,\alpha}$ – domain if it is a domain in Euclidean space with sufficiently regular boundary i.e., $\partial\Omega$ can locally be viewed as the graph of a $C^{k,\alpha}$ function. For this reason, we will assume that $\partial\Omega = \{(x', g(x'))\}$ where $g \in C^{2,\alpha}, x' \in \mathbb{R}^{n-1}$. Recall that the first step in the iteration for Theorem 5.3 required us to obtain the estimate $|u - w| \le C\epsilon$ in B_1. Since this sort of iteration process is all we can do at the moment, I will now attempt to show the first step:

Lemma 5.4. *Let B_1 be the ball centered at the origin, $f \in L^\infty$, and suppose that u solves*

$$\begin{cases} \Delta u = f & \text{in } \Omega \cap B_1 \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $|u| \le 1$ and $|f| \le \delta$. Suppose the domain is as flat as

$$B_1 \cap \{x_n \ge \epsilon\} \subset B_1 \cap \Omega.$$

Then there is a harmonic function w in $B_{1/4}$ with $w(x', 0) = 0$ so that

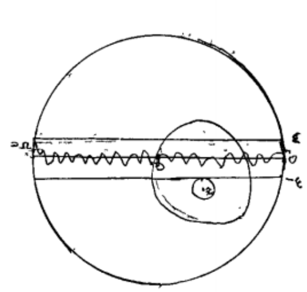
$$|u - w| \le C(\epsilon + \delta)$$

on $B_{\frac{1}{4}}(0) \cap \overline{\Omega}$.

Proof. We first consider the case where $f = 0$, so u is harmonic on $\Omega \cap B_1$. Let $\Gamma(x)$ be the fundamental solution to Laplace's equation, and define the following barrier function as follows

$$G(x) := \frac{\Gamma(x) - \Gamma(r)}{\Gamma(R) - \Gamma(r)}$$

for $0 < r < R$. Notice that $0 \le G, G(R) = 1, G(r) = 0$, and that G is harmonic in the annulus of little radius r and big radius R. Consider the lines $x_n = \pm\epsilon$ and let $x_0 = (x_0', -\epsilon) \in \mathbb{R}^n$ be so small that it is in B_1. Lets look at G the barrier function with smaller radius tangent to $x_n = -\epsilon$, and see that it is centered at $(x_0', -\epsilon - r) =: \tilde{x}_0$.



Now I will show that $|u| \le G$ on the common domain of influence. First note that $G \ge |u|$ on $\partial\Omega$ because $u = 0$ on $\partial\Omega$. Recall that we assumed $|u| \le 1$ in Ω, and since $G = 1$ on $\partial B(\tilde{x}_0, R)$, we see that $|u| \le G$ on $\Omega \cap \partial B(\tilde{x}_0, R)$. Putting these two together yields $|u| \le G$ on $\partial\left(\Omega \cap B(\tilde{x}_0, R)\right)$, and the maximum principle (which applies because these two are harmonic on $\Omega \cap B(\tilde{x}_0, R)$) tells us that $|u| \le G$ on the entire common domain of influence.

Lets restrict ourselves for a moment on rays starting from $\tilde{x}_0$ and moving in the x_n direction. Notice that since G is Lipschitz, we have

$$|G(0, \dots, 0, t) - G(0, \dots, 0, s)| \le C|t - s|.$$

Let $t \in \mathbb{R}$ be such that $\tilde{x}_0 + te_n \in \Omega$. We now use the fact that G is zero on the smaller ball of radius r:

$$\begin{aligned} |u(\tilde{x}_0 + te_n)| &\le G(\tilde{x}_0 + te_n) \\ &= G(\tilde{x}_0 + te_n) - G(\tilde{x}_0 + re_n) \\ &\le C|t - r| = C|x_n + \epsilon|. \end{aligned}$$

Observe that $\tilde{x}_0$ can be moved horizontally as long as the outer ball of radius R is in B_1, and the inner ball of r is below $-\epsilon$, and so,

$$|u| \le C|x_n + \epsilon| \quad \text{on all of } B_{\frac{1}{4}}(0) \cap \overline{\Omega} \tag{54}$$

Now that we have this estimate, lets find the corresponding harmonic polynomial that gives us the desired result. Let w be the harmonic polynomial such that

$$\begin{cases} w = u & \text{on } \partial(B^+_{1/4}) \cap \Omega \cap \{x_n \ge \epsilon\} \\ w = 0 & \text{on } \partial(B^+_{1/4}) \cap \{0 \le x_n < \epsilon\} \\ w = 0 & \text{on } \{x_n = 0\}. \end{cases}$$

There is the possibility that w is not continuous on $\partial(B^+_{1/4}) \cap \Omega \cap \{x_n = \epsilon\}$ because on one side, $w = u$ and on the other $w = 0$. The way to fix this is the following: move $\delta > 0$ above ϵ and consider a C^∞ cutoff function v_δ that satisfies

$$\begin{cases} 0 \le v_\delta \le 1 & W \subset\subset \partial(B^+_{1/4}) \cap \Omega \cap \{\epsilon \le x_n \le \epsilon + \delta\} \\ v_\delta \equiv 1 & V \subset\subset W \end{cases}$$

where v_δ decreases monotonically to zero. Now define the new harmonic function

$$w_\delta := \begin{cases} w = u & \text{on } \partial(B^+_{1/4}) \cap \Omega \cap \{x_n \ge \epsilon + \delta\} \\ w = v_\delta^2 u & \text{on } \partial(B^+_{1/4}) \cap \Omega \cap \{\epsilon \le x_n \le \epsilon + \delta\} \\ w = 0 & \text{on } \partial(B^+_{1/4}) \cap \{0 \le x_n < \epsilon\} \\ w = 0 & \text{on } \{x_n = 0\}. \end{cases}$$

Then since

(i) $|w_\delta| = |u| \leq G$ on $\partial(B^+_{1/4}) \cap \Omega \cap \{x_n \geq \epsilon + \delta\}$

(ii) $|w_\delta| = |v_\delta^2 u| \leq |u| \leq G$ on $\partial(B^+_{1/4}) \cap \Omega \cap \{\epsilon \leq x_n \leq \epsilon + \delta\}$

(iii) $G \geq 0 = w_\delta$ on $\partial(B^+_{1/4}) \cap \{0 \leq x_n < \epsilon\}$

(iv) $G \geq 0 = w_\delta$ on $\{x_n = 0\}$,

putting (i) - (iv) together yields $|w_\delta| \leq G$ on $\partial \left(\Omega \cap B^+_{1/4}\right)$. Thus the maximum principle then implies

$$|w_\delta| \leq C|x_n + \epsilon| \quad \text{on all of } B^+_{1/4} \cap \overline{\Omega}. \tag{55}$$

Similarly on $B^-_{1/4} \cap \overline{\Omega}$, one could extend oddly by $\tilde{w}_\delta(x', x_n) = -w_\delta(x', -x_n)$ on $B^-_{1/4}$, and comparing $\tilde{w}$ with $-u$, one achieves the same bound as (55) on $B^-_{1/4} \cap \overline{\Omega}$. Define the new oddly reflected function

$$w_{\text{odd}} = \begin{cases} w_\delta & \text{on } B^+_{1/4} \\ \tilde{w}_\delta & \text{on } B^-_{1/4} \end{cases}$$

which satisfies

$$|w_{\text{odd}}| \leq C|x_n + \epsilon| \quad \text{on } \textit{all of } B_{1/4} \cap \overline{\Omega}. \tag{56}$$

Putting (54) and (56) together finally proves the Lemma for u harmonic:

$$|u - w_{\text{odd}}| \leq C\epsilon$$

on $B_{1/4} \cap \overline{\Omega}$. Now lets consider the general case $\Delta u = f$ where $f \in L^\infty$ and $|f| \leq \delta$. Here, apply the above to $\tilde{u} = u \pm \frac{\delta}{2n}|x|^2$ and comparison principles allow us to say

$$|u - w_{\text{odd}}| \leq C(\epsilon + \delta).$$

□

Corollary 5.5. *Suppose u satisfies*

$$\begin{cases} \Delta u = f & \textit{in } \Omega \\ u \leq \epsilon & \textit{on } \partial\Omega \end{cases} \tag{57}$$

with again the same flatness assumption on Ω ($B_1 \cap \{x_n \geq \epsilon\} \subset B_1 \cap \Omega$). Then the same estimate as Lemma 5.4 holds.

Proof. In order to see why this is true, we must examine where we used $u = 0$ on $\partial\Omega$ in the proof of Lemma 5.4. We were comparing trying to show that $|u| \leq G$ on the common domain of influence and used the fact that $u = 0 \leq G$ on $\partial\Omega$. In order to get the same estimates of the Lemma for (57), we simply move the point $\tilde{x}_0 = (x_0', -\epsilon - r)$ to $\tilde{x}_0 = x_0', -\epsilon - r - 2\epsilon'$. The reason for this is that the barrier function G grows at the rate

$$|DG(r)| \leq Cr^{1-n},$$

and so by moving down a sufficiently small amount, G will be bigger than ϵ, and in turn bigger than u on $\partial\Omega$ □

We are now ready to prove $C^{1,\alpha}$ estimates at the boundary:

Theorem 5.6. *Suppose u satisfies*

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $f(0) = g(0) = \nabla g(0) = 0, |u| \le 1, |f| \le \delta$. Suppose Ω is a $C^{1,\alpha}$ domain, that is $\partial\Omega = \{(x', g(x'))\}$ for $g \in C^{1,\alpha}$ that satisfies $|g| \le \delta r^{1+\alpha}$. Then $u \in C^{1,\alpha}(B_{r/2} \cap \overline{\Omega})$ with the estimate

$$\|u\|_{C^{1,\alpha}(B_{r/2}\cap\overline{\Omega})} \le C_{n,\alpha}(\|u\|_{L^\infty} + \|f\|_{L^\infty} + \|g\|_{C^{1,\alpha}(|x'|)})$$

Proof. Consider $B_r \cap \overline{\Omega}$ we want to show that there exists a linear function l such that

$$\|u - l|_{L^\infty(B_r\cap\overline{\Omega})} \le r^{1+\alpha}$$

and WLOG assume $r \le 1$. Then since $u = 0$ along the boundary, all the tangential derivatives are then zero and hence we may reduce this to

$$\|u - ax_n\|_{L^\infty(B_r\cap\overline{\Omega})} \le r^{1+\alpha} \tag{58}$$

and we may also assume $|a| \le 1$. In order to achieve this by some sort of iteration used in Theorem 5.3, we need to conclude

$$\|u - \tilde{a}x_n\|_{L^\infty(B_{\rho r}\cap\overline{\Omega})} \le (\rho r)^{1+\alpha}$$

for some $\tilde{a}$. The first thing we do is to re-scale the ball B_r into B_1 (and hence diluting Ω to $\tilde{\Omega}$) and it is clear that we need to define some $\tilde{u}$ that satisfies

$$u(x) - ax_n = r^{1+\alpha}\tilde{u}(x/r)$$

because $\tilde{u}$'s domain of influence is B_1. Our hypothesis then becomes $|\tilde{u}| \le 1$ and note that we can rewrite it as

$$\tilde{u}(x) = \frac{u(rx) - a(rx_n)}{r^{1+\alpha}}.$$

Observe that diluting Ω to $\tilde{\Omega}$ makes the latter the graph of $\tilde{g}(x') = \frac{1}{r}g(r\tilde{x}')$ which satisfies $|\tilde{g}| \le \delta r^\alpha$. Now we need to figure out what equation $\tilde{u}$ satisfies: $\Delta\tilde{u} = \frac{r^2\Delta u(rx)}{r^{1+\alpha}} = r^{1-\alpha}f(rx) =: \tilde{f}$. The bound that $\tilde{f}$ satisfies is $|\tilde{f}| \le \delta r^{1-\alpha} \le \delta$, where we used that $r \le 1$. Now lets figure out the boundary conditions that $\tilde{u}$ satisfies:

$$\begin{aligned} \left|\tilde{u}|_{\partial\tilde{\Omega}}\right| &= \left| \left| \frac{u(rx) - a(rx_n)}{r^{1+\alpha}} \right|_{\partial\tilde{\Omega}} \right| \\ &= \frac{|ax_n r|}{r^{\alpha+1}} = \frac{|ax_n|}{r^\alpha} \\ &\le \delta \end{aligned}$$

where we used the fact that $u(x)$ vanishes on $\partial\Omega \implies u(rx)$ vanishes on $\partial\tilde{\Omega}$. Putting everything together gives us

$$\begin{aligned} &|\tilde{u}| \le 1 \\ &|\tilde{g}| \le \delta r^\alpha \\ &\Delta\tilde{u} = \tilde{f}, |\tilde{f}| \le \delta \\ &|\tilde{u}| \le \delta \text{ on } \partial\tilde{\Omega}. \end{aligned} \tag{59}$$

Now we are exactly in the position of Corollary 5.5 because the flatness condition is precisely met by (59) (after replacing the ϵ in the Lemma with δ because $\partial\tilde{\Omega}$ is the graph of $\tilde{g}$, and so its height in B_1 is at most δ by (59)). Applying the Lemma gives us $|\tilde{u} - w| \le C\delta$ in $B_{\frac{1}{4}} \cap \tilde{\Omega}$. Recall that w is the solution that vanishes on $x' \Longrightarrow |w - bx_n|_{B_\rho} \leq C\rho^2$. Now noticing that the radius of $B_{\frac{1}{4}}$ doesn't have to be $\frac{1}{4}$ (as long as it is small enough for the proof of Lemma 5.4 to hold) we reach

$$|\tilde{u} - bx_n|_{B_{\rho \cap \bar{\tilde{\Omega}}}} \leq C(\delta + \rho^2).$$

By picking δ and ρ small enough, and using the fact that $0 < \alpha < 1$ yields

$$|\tilde{u} - bx_n|_{B_{\rho \cap \bar{\tilde{\Omega}}}} \leq C\rho^{1+\alpha}.$$

Rescaling back gives us

$$|u - \tilde{a}x_n|_{B_{\rho r \cap \bar{\Omega}}} \leq C(\rho r)^{1+\alpha}$$

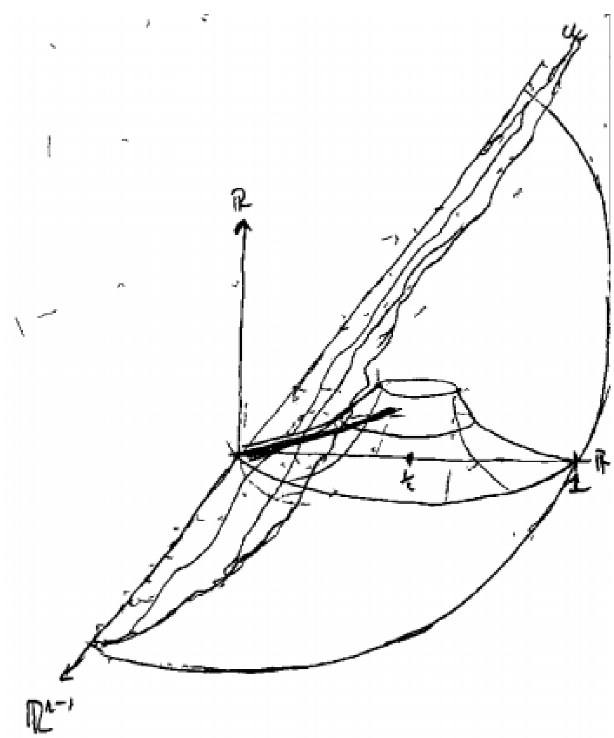
with $\tilde{a} = a + r^\alpha b$. We can now iterate just like in the last step of Theorem 5.3 to find that $u \in C^{1,\alpha}$. □

5.2 General Elliptic Operators

The goal of this section is to generalize Krylov's estimate for flat domains to general $C^{1,\alpha}$ domains. First we state and prove his estimate

Theorem 5.7. *Assume $u \in W^{2,n}(B^+) \cap C^0(\bar{B}^+)$ is non-negative and $u(e_n/2) \geq 1$. If u satisfies the equation $a^{ij}u_{ij} = 0$ in the half ball $B^+ = B_1 \cap \mathbb{R}^n_+$ with $u = 0$ on $T = B_1 \cap \partial\mathbb{R}^n_+$, and the a^{ij} measurable bounded and uniformly elliptic. Then $u \in C^{1,\alpha}(B^+_{1/2})$.*

Proof. First we note that if $u \geq 1$ at $e_n/2$, we can apply the Harnack inequality to find that $u \geq c$ on $B_{1/10}(e_n/2)$. Now look at the barrier function $\phi(x) = C_1|x - \frac{e_n}{2}|^\alpha - C_1$ with C_1 and C_2 chosen such that $\phi \equiv 0$ on $\partial B_{1/2}(e_n/2)$ and $\phi = c$ on $\partial B_{1/10}(e_n/2)$. Here $\alpha = \alpha(n, \Lambda, \lambda)$. By an almost identical computation to the barrier function constructed in the HI for non-divergence equations, one can see that $a^{ij}\phi_{ij} \geq 0$ on $B_{1/2}(e_n/2) - B_{1/10}(e_n/2)$. Now we can use the ABP maximum principle to deduce that $u \geq \phi$ on the common domain of influence. Then by Hopf lemma we know that this barrier is going to start at an angle, and so we can actually conclude that $u(x) \geq \delta' x_n$.



Note that this was just in this particular domain $B_{1/2}(e_n)$ but we want to establish the proof in all of $B^+_{1/2}$. In order to remedy, we have to slide our barrier to the left and right by x_0 such that outer ball where ϕ vanishes is tangent to B^+. The center of these two extreme balls is given by $\frac{1}{2}(1, \dots, 1)$ and $\frac{1}{2}(-1, \dots, -1, 1)$. So how does this help us? We can actually apply the above estimate at each of the translated barriers because we can iterate HI to get a thin strip of domain where we know $u \geq C$ and so we can construct similar barriers on every point along this strip. Then at the end of these iterations take the smallest angle of the planes generated by each barrier function and trap all of $u \geq cx_n$ in $B_{1/2}$. Since u has this flatness associated to is, it is $C^{1,\alpha}$, as will be seen in the proof of the next theorem. □

We now introduce the final theorem of this thesis. We first state it and prove an important lemma that will be useful.

Theorem 5.8. *Let $u \in W^{2,n}(\Omega) \cap C^0(\bar{\Omega})$ where Ω is a $C^{1,\alpha}$ domain such that $0 \in \partial\Omega$ and near zero $\partial\Omega = \{(x', g(x'))\}$ with $|g(x')| \leq \delta|x'|^{1+\alpha}$. Suppose $Lu = a^{ij}(x)u_{ij} = f(x)$ strongly on Ω with $\lambda I \leq a^{ij} \leq \Lambda I, |f| \leq \delta$ in $B_1 \cap \Omega$ and $u \leq \delta|x'|^{1+\alpha}$ on $\partial\Omega$. Then $u \in C^{1,\alpha}(0)$.*

Lemma 5.9. *Assume u satisfies*

$$a_0^- x_n - M\delta \leq u \leq a_0^+ x_n + M\delta$$

in B_1 with $|a_0^+ - a_0^-| = 1$ and $a_0^+, a_0^- \in [-10, 10]$. Then, in B_{r_0} with $r_0 \leq 1$, u satisfies

$$a_1^- x_n - M\delta r_0^{1+\alpha} \leq u \leq a_1^+ x_n + M\delta r_0^{1+\alpha}$$

with $a_1^+ \leq a_0^+ + C\delta,\ a_0^- - C\delta \leq a_1^-$, and $a_1^+ - a_1^- \leq (1 - \eta)$ for small η.

Proof. First we look at $u(1/2e_n)$ and notice that

$$\frac{1}{2}a_0^- - M\delta \leq u(1/2e_n) \leq \frac{1}{2}a_0^+ + M\delta.$$

Now notice that $u(1/2e_n)$ is either closer to the plane on top or the plane on the bottom: since $1/2|a_0^+ - a_0^-| = 1/2$ and the $M\delta$ is common in both hands of the inequality, we have either

$$u(1/2e_n) - \frac{1}{2}a_0^- \geq \frac{1}{4} \quad \text{or} \quad \frac{1}{2}a_0^+ - u(1/2e_n) \geq \frac{1}{4}.$$

Define $\bar{u} := u - a_0^- x_n$. The hypothesis of the lemma becomes

$$-M\delta \leq \bar{u} \leq (a_0^+ - a_0^-)x_n + M\delta = x_n + M\delta.$$

Notice that $a_0^+, a_0^- \in [-10, 10]$ implies

$$|\bar{u}| \leq |u| + |a_0^- x_n| \leq \delta|x'|^{1+\delta} + 10|x_n| = \delta|x'|^{1+\alpha} + 10|g(x')| \leq 11\delta|x'|^{1+\alpha}$$

on $\partial\Omega$ near zero. It is also important to notice that $\bar{u}$ solves the same equation as u. Since we assumed that u was closer to the lower plane, we have that $\bar{u}(1/2e_n) \geq \frac{1}{4} - M\delta$. The Harnack Inequality tells us that $\bar{u} \geq C_0 - M\delta$ (with C_0 independent of δ) on $B_{1/10}(1/2e_n)$.

Now consider the barrier $w(x) = M_1|x|^{-\gamma} - M_2$ with M_1, M_2 chosen such that $w \equiv 1$ on $\partial B_{1/10}(1/2e_n)$ and $w \equiv 0$ on $\partial B_{1/2}(1/2e_n)$. We compute the derivatives of w as

$$w_{ij}(x) = \begin{cases} M_1\gamma(x_1^2 + \cdots + x_n^2)^{\gamma/2-1} + M_1\gamma(\gamma/2 - 1)(x_1^2 + \cdots + x_n^2)(2x_i)(x_i) & i = j \\ M_1\gamma(\gamma/2 - 1)(x_1^2 + \cdots + x_n^2)^{\gamma/2-2}x_i(2x_j) & i \neq j. \end{cases}$$

Thus, looking in the radial direction $(0, \dots, 0, r)$ we have

$$D^2w = \begin{pmatrix} \frac{1}{r}w'(r) & & & \\ & \ddots & & \\ & & \frac{1}{r}w'(r) & \\ & & & w''(r) \end{pmatrix}$$

and so $w'' > 0, w' < 0$ implies that

$$a^{ij}w_{ij} \geq \lambda w''(r) + \Lambda \frac{n-1}{r} w' = \gamma M_1 r^{\gamma-2}(\lambda(\gamma-1) + \Lambda(n-1)) \geq C_1$$

by choosing γ. Now define

$$W := \frac{C_0 w}{2} - 100\delta.$$

We want to do our estimates on $\Omega \cap B_1 - B_{1/10}(1/2e_n)$ and so we examine first $\partial B_1 \cap \Omega$. Here we know W is very negative from its construction (more so than $\bar{u} \geq -M\delta$) and if it isn't, we simply move our computations from $1/2e_n \mapsto 1/4e_n$. This gives us $\bar{u} \geq W$ on $\Omega \cap \partial B_1$. From the bounds on g, we know that the first points of contact between ∂B_1 and $\partial\Omega$ will be at most size δ. But from our estimates of $\bar{u}$ near zero, we have

$$\bar{u} \geq -11\delta \geq \max_{|x_n| \leq \delta} W \geq \max_{\partial\Omega \cap \{|x_n| \leq \delta\}} W.$$

Note that $W \equiv C_0/2 - 100\delta$ on $\partial B_{1/10}(1/2e_n)$ and $\bar{u} \geq C_0 - M\delta$ on $B_{1/10}(1/2)$ by the harnack inequality. Hence $\bar{u} \geq W$ on $\partial B_{1/10}(1/2e_n)$. If δ is small enough, which can be done by multiplying f by a constant since it doesn't change any of our inequalities, we have $LW = C_0 C_1 \geq L\bar{u} \sim \delta$ and so the maximum principle dictates that $W \leq \bar{u}$ on $\Omega \cap B_1 - B_{1/10}(1/2e_n)$.

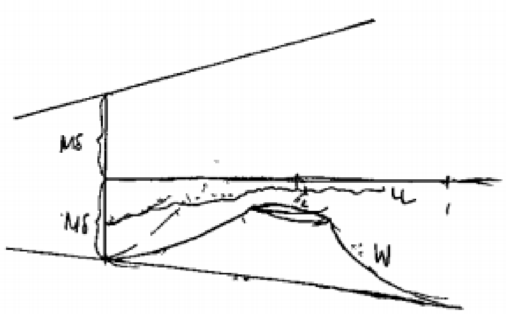
Observe that in the radial direction $\{x' = 0\}$, W satisfies

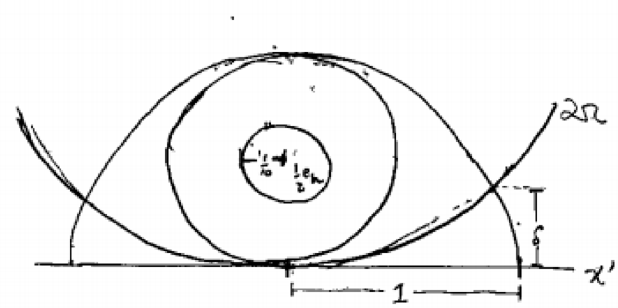
$$\begin{aligned} W(0, x_n) &= \frac{C_0}{2} w(0, x_n) - 100\delta \\ &\geq C_0 C_3 x_n - 100\delta \end{aligned}$$

because of how $|x|^{-\gamma}$ grows. Also notice that $x' = 0$ was not specific and we could have taken any $x' \in \{|x_n| \leq \delta\} \cap \partial\Omega$ as we look for planes in the radial x_n direction starting from $\partial\Omega$. Taking the maximum C_3 of all such planes we obtain

$$\begin{aligned} u - a_0^- x_n &= \bar{u} \geq W \\ &\geq C_0 C_3 x_n - 100\delta \\ u &\geq (C_0 C_3 + a_0^-) - 100\delta \end{aligned}$$

and $C_0 C_3 + a_0^- =: a_1^-$.





Now assume that u is closer to the upper plane than the lower plane. In this case then u satisfies

$$\frac{1}{2}a_0^+ - u(1/2e_n) \geq 1/4.$$

We define $\underline{\mathrm{u}} = a_0^+ x_n - u$ and clearly we are looking for a lower bound for $\underline{\mathrm{u}}$. The hypothesis of the Lemma becomes

$$\begin{aligned} -a_0^- x_n + M\delta \geq -u \geq -a_0^+ x_n - M\delta \\ (a_0^+ - a_0^-)x_n + M\delta \geq \underline{\mathrm{u}} \geq -M\delta. \\ x_n + M\delta \geq \underline{\mathrm{u}} \geq -M\delta. \end{aligned}$$

Near zero $\underline{\mathrm{u}}$ satisfies the estimate

$$|\underline{\mathrm{u}}| \leq |a_0^+ x_n| + |u| \leq 10|x_n| + \delta|x'|^{1+\alpha} = 10|g(x')| + \delta|x'|^{1+\alpha} \leq 11\delta|x'|^{1+\alpha}.$$

Also since u was closer to the upper plane, $\underline{\mathrm{u}}(1/2e_n) \geq 1/4 - M\delta$. The Harnack Inequality tells us that $\underline{\mathrm{u}} \geq C_0 - M\delta$. As one can probably tell, the proof is going to be identical to the one for $\bar{u}$.

Now consider the barrier $w(x) = M_1|x|^{-\gamma} - M_2$ with M_1, M_2 chosen such that $w \equiv 1$ on $\partial B_{1/10}(1/2e_n)$ and $w \equiv 0$ on $\partial B_{1/2}(1/2e_n)$. In a very similar sense as the lower bound by the barrier in the previous case, we have $a^{ij}w_{ij} \geq C_1 = C_1(n, \lambda, \Lambda, \gamma)$. Define W by

$$W := \frac{C_0 w}{2} - 100\delta.$$

We want to do our estimates on $\Omega \cap B_1 - B_{1/10}(1/2e_n)$ and so we examine first $\partial B_1 \cap \Omega$. Here we know W is very negative from its construction (more so than $\underline{\mathrm{u}} \geq -M\delta$) and if it isn't, we simply move our computations from $1/2e_n \mapsto 1/4e_n$. This gives us $\underline{\mathrm{u}} \geq W$ on $\Omega \cap \partial B_1$. From the bounds on g, we know that the first points of contact between ∂B_1 and $\partial\Omega$ will be at most size δ. But from our estimates of $\underline{\mathrm{u}}$ near zero, we have

$$\underline{\mathrm{u}} \geq -11\delta \geq \max_{|x_n| \leq \delta} W \geq \max_{\partial\Omega \cap \{|x_n| \leq \delta\}} W.$$

Note that $W \equiv C_0/2 - 100\delta$ on $\partial B_{1/10}(1/2e_n)$ and $\underline{\mathrm{u}} \geq C_0 - M\delta$ on $B_{1/10}(1/2)$ by the harnack inequality. Hence $\underline{\mathrm{u}} \geq W$ on $\partial B_{1/10}(1/2e_n)$. If δ is small enough, which can be done by multiplying f by a constant since it doesn't change any of our inequalities, we have $LW = C_0 C_1 \geq L\underline{\mathrm{u}} \sim \delta$ and so the maximum principle dictates that $W \leq \underline{\mathrm{u}}$ on $\Omega \cap B_1 - B_{1/10}(1/2e_n)$.

Observe that in the radial direction $\{x' = 0\}$, W satisfies

$$\begin{aligned} W(0, x_n) &= \frac{C_0}{2} w(0, x_n) - 100\delta \\ &\geq C_0 C_3 x_n - 100\delta \end{aligned}$$

because of how $|x|^{-\gamma}$ grows. Also notice that $x' = 0$ was not specific and we could have taken any $x' \in \{|x_n| \leq \delta\} \cap \partial\Omega$ as we look for planes in the radial x_n direction starting from $\partial\Omega$. Taking the maximum C_3 of all such planes we obtain

$$\begin{aligned} a_0^+ x_n - u = \underline{\mathrm{u}} &\geq W \\ &\geq C_0 C_3 x_n - 100\delta \\ (a_0^+ - C_0 C_3)x_n + 100\delta &\geq u \end{aligned}$$

and $a_0^+ + C_0 C_3 =: a_1^+$. Thus we can see that even though the distance from the origin increases a little bit, the slope decreases a lot more. □

Proof of Theorem. We rescale u by defining

$$\tilde{u}(x) = \frac{u(r_0 x)}{r_0}.$$

Then by Lemma 5.9, in B_{r_0} we have

$$a_1^- x_n - M\delta r_0^\alpha \le \tilde{u} \le a_1^+ x_n + M\delta r_0^\alpha$$

with $|a_1^+ - a_1^-| = 1 - \eta$ by direct computations. We also rescale the boundary by $\tilde{g}(x') = g(r_0 x')/r_0 \implies |\tilde{g}| \le \delta r_0^\alpha |x'|^{1+\alpha}$. We can also compute

$$\tilde{L}\tilde{u} = \tilde{a}^{ij}(x)\tilde{u}_{ij}(x) = \frac{a^{ij}(r_0 x)}{r_0} \frac{r_0^2 u_{ij}(r_0 x)}{r_0} = f(r_0 x).$$

This implies that we can iterate the conclusion of the Lemma by rescaling and so iterating k times with $r = r_0^k$, we have

$$a_k^- x_n - M\delta r^{1+\alpha} \le u \le a_k^+ x_n + M\delta r^{1+\alpha}$$

in $B_r = B_{r_0^k}$ with $|a_k^+ - a_k^-| = (1-\eta)^k \approx r^\beta$ for some β. This implies convergence of the $a_k^\pm$ to an a_∞ in the sense that $|a_k^+ - a_\infty| \le Cr^\beta, |a_k^- - a_\infty| \le Cr^\beta$. Then

$$a_k^- x_n - M\delta r^{1+\alpha} \le u \le a_k^+ x_n + M\delta r^{1+\alpha}$$
$$(a_k^- - a_\infty)x_n - M\delta r^{1+\alpha} \le u - a_\infty x_n \le (a_k^+ - a_\infty)x_n + M\delta r^{1+\alpha}.$$

Our analysis finally gives

$$|u - a_\infty x_n| \le Cr^{1+\beta} + M\delta r^{1+\alpha}.$$

This proves that if $\beta < \alpha$, then $u \in C^{1,\beta}(0)$. If $\beta \ge \alpha$ then $u \in C^{1,\alpha}(0)$.

□

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[2] Caffarelli, L., Cabré, X. *Fully Nonlinear Elliptic Equations,* AMS,1995