Probability Weighting

Mark Dean

Behavioral Economics
Spring 2017

- Let's think back to the Allais paradox
  - Prizes are \$0, \$16, \$18

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \succ \begin{pmatrix} 0.01 \\ 0.89 \\ 0.1 \end{pmatrix}$$

  - $$\begin{pmatrix} 0.89 \\ 0.11 \\ 0 \end{pmatrix} \prec \begin{pmatrix} 0.90 \\ 0 \\ 0.1 \end{pmatrix}$$

- What could be going wrong with the EU model?

- Many alternative models have been proposed in the literature
  - Disappointment: Gul, Faruk, 1991. "A Theory of Disappointment Aversion,"
  - Salience: Pedro Bordalo & Nicola Gennaioli & Andrei Shleifer, 2012. "Salience Theory of Choice Under Risk,"
- We are going to focus on one of the most widespread and straightforward:
  - Probability weighting

- Maybe the problem that the Allais paradox highlights is that people do not 'believe' the probabilities that are told to them
  - For example they treat a 1% probability of winning $0 as if it is more likely than that
    - 'I am unlucky, so the bad outcome is more likely to happen to me'
  - The difference between 0% and 1% seems bigger than the difference between 89% and 90%
- This is the idea behind the probability weighting model.

Simple Probability Weighting Model

- Approach 1: Simple probability weighting
- Let's start with expected utility

$$U(p) = \sum_{x \in X} p(x)u(x)$$

- And allow for probability weighting

$$V(p) = \sum_{x \in X} \pi(p(x))u(x)$$

  Where π is the probability weighting function
- This can explain the Allais paradox
  - For example if $\pi(0.01) = 0.05$

Simple Probability Weighting Model

- However, the simple probability weighting model is not popular
- For two reasons
  1. It leads to violations of stochastic dominance
  2. It doesn't really capture the idea of 'pessimism'

Simple Probability Weighting Model

- Violations of stochastic dominance
  - Let $F_p(x)$ be the probability of getting an outcome of x **or worse** according to p
  - e.g the cumulative distribution function of p

$$F_p(x) = \sum_{y \leq x} p(y)$$

  - We say that p (first order) stochastically dominates q if

$$F_p(x) \leq F_q(x)$$

    for every prize x
  - i.e, for any prize, the probability of getting something at least as bad is higher under q than under p

- E.g. for prizes $x_1 < x_2 < x_3$

$$\begin{pmatrix} 0.1 \\ 0.7 \\ 0.2 \end{pmatrix} \text{ stochastically dominates } \begin{pmatrix} 0.2 \\ 0.7 \\ 0.1 \end{pmatrix}$$

- But

$$\begin{pmatrix} 0.01 \\ 0.99 \\ 0 \end{pmatrix} \text{ does not stochastically dominates } \begin{pmatrix} 0.99 \\ 0 \\ 0.01 \end{pmatrix}$$

Simple Probability Weighting Model

- A property that we would generally like a model to have is that it obeys first order stochastic dominance
  - i.e. if p first order stochastically dominates q then $p \succ q$
- This is certainly the case for the expected utility model
- It turns out that this is not the case for the simple probability weighting model

Theorem

Unless π is the identity function, a decision maker who is behaving in line with the simple probability weighting model will violate stochastic dominance (i.e. we can find a p and a q such that p stochastically dominates q but $q \succ p$)

- Proof is beyond the scope of this course

- Think back to the Allais paradox

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \succ \begin{pmatrix} 0.01 \\ 0.89 \\ 0.1 \end{pmatrix}$$

- It seems as if the 1% probability of $0 is being overweighted
- Is this just because it is a 1% probability?
- Or is it because it is a 1% probability **of the worst prize**
- If it is the latter, this is something that the simple probability weighting model cannot capture
  - Weights are only based on probability

- Consider the following two examples

Example

Lottery p :49% chance of \$10, 49% of winning \$0, 2% chance of winning \$5

- Consider the following two examples

Example

Lottery p :49% chance of \$10, 49% of winning \$0, 2% chance of winning \$5

Example

Lottery p :49% chance of \$10, 49% of winning \$0, 2% chance of losing \$1000

- Consider the following two examples

Example

Lottery p :49% chance of \$10, 49% of winning \$0, 2% chance of winning \$5

Example

Lottery p :49% chance of \$10, 49% of winning \$0, 2% chance of losing \$1000

- Would you 'weigh' the 2% probability the same in each case?
  - Arguably not
  - If you were pessimistic then you might think that 2% is 'more likely' in the latter case than in the former
  - Can't be captured by the simple probability weighting model

- Because of these two concerns, the simple probability weighting model is rarely used
- Instead people tend to use **rank dependent utility** (sometimes also called cumulative probability weighting)
- Probability weighting depends on
  - The **probability** of a prize
  - Its **rank** in the lottery - i.e. how many prizes are better or worse than it
- In practice this is done by applying weights **cumulatively**
- Here comes the definition
  - It looks scary, but don't panic!

Definition

A decision maker's preferences $\succeq$ over $\Delta(X)$ can be represented by a rank dependant utility model if there exists a utility function $u : X \to \mathbb{R}$ and a cumulative probability weighting function $\psi : [0,1] \to [0,1]$ such that $\psi(0) = 0$ and $\psi(1) = 1$, such that the function $U : \Delta(X) \to \mathbb{R}$ represents $\succeq$, where $U(p)$ is constructed in the following way:

1. The prizes of p are ranked $x_1, x_2, \ldots, x_n$ such that $x_1 \succ x_2 \cdots \succ x_n$
2. $U(p)$ is determined as

$$U(p) = \psi(p_1)u(x_1) + \sum_{i=2}^{n} \left(\psi\left(\sum_{j=1}^{i} p_j \right) - \psi\left(\sum_{k=1}^{i-1} p_k \right) \right) u(x_i)$$

- Let's go through an example: for prizes $10 > 5 > 0$ let p be equal to

$$\begin{pmatrix} 0.1 \\ 0.7 \\ 0.2 \end{pmatrix}$$

- How do we apply RDU?

- Well, first note that there are three prizes, so we can rewrite the expression above as

$$\begin{aligned} U(p) &= \psi(p_1)u(x_1) \\ &+ (\psi(p_1+p_2) - \psi(p_1))\, u(x_2) \\ &+ (\psi(p_1+p_2+p_3) - \psi(p_1+p_2))\, u(x_3) \end{aligned}$$

- The weight attached to the best prize is the weight of p_1
- The weight attached to the second best prize is the weight on the probability of
  - Getting something at least as good as the second prize
  - Minus the probability of getting something better than the second prize
  - And so on
- Notice that if ψ is the identity function this is just expected utility

- In this specific case

$$
\begin{aligned}
U(p) &= \psi(p_1)u(x_1) \\
&+ (\psi(p_1 + p_2) - \psi(p_1))\, u(x_2) \\
&+ (\psi(p_1 + p_2 + p_3) - \psi(p_1 + p_2))\, u(x_3)
\end{aligned}
$$

- Becomes

$$
\begin{aligned}
U(p) &= \psi(0.1)u(10) \\
&+ (\psi(0.8) - \psi(0.1))\, u(5) \\
&+ (\psi(1) - \psi(0.8))\, u(0)
\end{aligned}
$$

Rank Dependent Utility and the Allais Paradox

- We will now show how RDU can lead to the Allais paradox.
- In order to do so, we will think of a slight modification of the previous experiment

Question 1 What is the amount of money x that would make the DM indifferent between

$$1,000,000 \text{ for sure}$$

and

$$1\% \text{ chance of } 0$$
$$89\% \text{ chance of } 1,000,00$$
$$10\% \text{ chance of } x$$

Question 2 What is the amount of money z that would make the DM indifferent between

$$11\% \text{ chance of } 1,000,000 \text{ and } 89\% \text{ chance of } 0$$

and

$$10\% \text{ chance of } z \text{ and } 90\% \text{ chance of } 0$$

Rank Dependent Utility and the Allais Paradox

- Expected utility: $z = x$ (check that you understand why this is the case)
- Allais-type behavior: $x > z$
- What about RDU?
- For simplicity, assume $u(x) = x$

- What is the RDU of

$$1,000,000 \text{ for sure}$$

Rank Dependent Utility and the Allais Paradox

- What is the RDU of

$$1,000,000 \text{ for sure}$$

$$\psi(1)u(1,000,000) = 1,000,000$$

- What is the RDU of

$$1\% \text{ chance of } 0$$
$$89\% \text{ chance of } 1,000,00$$
$$10\% \text{ chance of } x$$

- What is the RDU of

1% chance of 0

89% chance of $1,000,00$

10% chance of x

$$\begin{aligned}&\psi(0.1)x\\&+(\psi(0.99)-\psi(0.1))\,1,000,000\\&+(\psi(1)-\psi(0.99))\,0\end{aligned}$$

(assuming $x > 1,000,000$)

Rank Dependent Utility and the Allais Paradox

- What is the RDU of

$$11\% \text{ chance of } 1,000,000 \text{ and } 89\% \text{ chance of } 0$$

- What is the RDU of

11% chance of 1, 000, 000 and 89% chance of 0

$$\psi(0.11)1,000,000 + (1 - \psi(0.11))\,0$$

Rank Dependent Utility and the Allais Paradox

- What is the RDU of

  10% chance of z and 90% chance of 0

- What is the RDU of

10% chance of z and 90% chance of 0

$$\psi(0.10)z + (1 - \psi(0.10))\, 0$$

Rank Dependent Utility and the Allais Paradox

- So, if the first two lotteries are indifferent we have

$$\begin{aligned} 1{,}000{,}000 = \quad & \psi(0.1)x \\ & + (\psi(0.99) - \psi(0.1))\, 1{,}000{,}000 \\ & + (\psi(1) - \psi(0.99))\, 0 \end{aligned}$$

- Which implies

$$x = \frac{1 - (\psi(0.99) - \psi(0.1))}{\psi(0.1)} 1{,}000{,}000$$

- If the second two lotteries are indifferent we get

$$
\begin{aligned}
& \psi(0.11)1,000,000 + (1 - \psi(0.11))\, 0 \\
= \;& \psi(0.1)z + (1 - \psi(0.1))\, 0 \\
\Rightarrow \;& z = \frac{\psi(0.11)}{\psi(0.1)} 1,000,000
\end{aligned}
$$

Rank Dependent Utility and the Allais Paradox

- So we get Allais type effects if

$$\frac{1-(\psi(0.99)-\psi(0.1))}{\psi(0.1)} > \frac{\psi(0.11)}{\psi(0.1)}$$

- Or

$$\psi(1)-\psi(0.99) > \psi(0.1)-\psi(0.11)$$

- i.e. the weight of going from certainty to 99% is bigger than the weight of going from 11% to 10%

Rank Dependent Utility and the Allais Paradox

- So we get Allais type effects if

$$\frac{1-(\psi(0.99)-\psi(0.1))}{\psi(0.1)} > \frac{\psi(0.11)}{\psi(0.1)}$$

- Or

$$\psi(1)-\psi(0.99) > \psi(0.1)-\psi(0.11)$$

i.e. the weight of going from certainty to 99% is bigger than the weight of going from 11% to 10%

Rank Dependent Utility and the Allais Paradox

- Is this always going to be the case?
- To explore, let's assume a particular form for probability weighting

$$\psi(x) = x^m$$

- And plug in some values for m

Rank Dependent Utility and the Allais Paradox

- $m = 2$

$$1^2 - 0.99^2 \approx 0.020 > 0.0020 \approx 1^2 - 0.11^2$$

  - Allais paradox

Rank Dependent Utility and the Allais Paradox

- $m = 2$

$$1^2 - 0.99^2 \approx 0.020 > 0.0020 \approx 1^2 - 0.11^2$$

  - Allais paradox
- $m = 1$

$$1^1 - 0.99^1 \approx 0.01 = 0.01 \approx 1^1 - 0.11^1$$

  - No Allais paradox

Rank Dependent Utility and the Allais Paradox

- $m = 2$

$$1^2 - 0.99^2 \approx 0.020 > 0.0020 \approx 1^2 - 0.11^2$$

  - Allais paradox

- $m = 1$

$$1^1 - 0.99^1 \approx 0.01 = 0.01 \approx 1^1 - 0.11^1$$

  - No Allais paradox

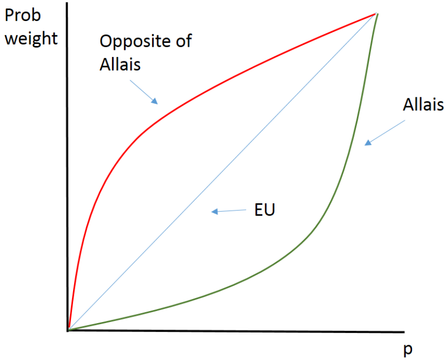
- $m = 0.5$

$$1^{0.5} - 0.99^{0.5} \approx 0.005 < 0.015 \approx 1^1 - 0.11^1$$

  - **Opposite** of Allais paradox

Rank Dependent Utility and the Allais Paradox

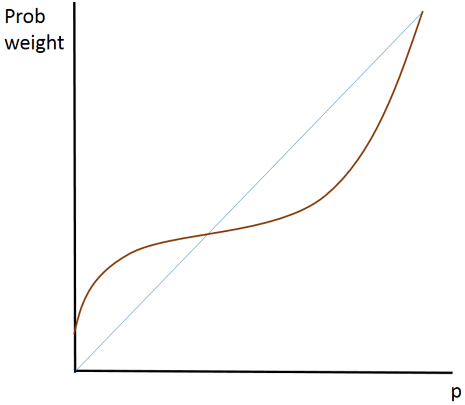
- Turns out we get the common consequence effect if and only if the prob weighting function is convex



- There is a sense in which this is a 'pessimistic' probability weighting function

S Shaped Probability Weighting

- Convex probability weighting functions are not particularly popular
- Usually data is best fit by an ‘s shaped’ probability weighting function



S Shaped Probability Weighting

- For example, from Prelec [1998]

$$\psi(x) = \exp(-(-\ln(x)^{\alpha}))$$

- Why?
- Overweights small probability gains, as well as small probability losses
- Explains why people buy Lottery tickets
- Estimates from Gonzales and Wu [1999]

S Shaped Probability Weighting