Subjective Expected Utility Theory

Mark Dean

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- In the first class we drew a distinction betweem
  - Circumstances of **Risk** (roulette wheels)
  - Circumstances of **Uncertainty** (horse races)
- So far we have been talking about roulette wheels
- Now horse races!

Risk vs Uncertainty

- Remember the key difference between the two
- Risk: Probabilities are **observable**
  - There are 38 slots on a roulette wheel
  - Someone who places a $10 bet on number 7 has a lottery with pays out $350 with probability 1/38 and zero otherwise
  - (Yes, this is not a fair bet)
- Uncertainty: Probabilities are **not observable**
  - Say there are 3 horses in a race
  - Someone who places a $10 bet on horse A does not necessarily have a 1/3 chance of winning
  - Maybe their horse only has three legs?

- If we want to model situations of uncertainty, we cannot think about preferences over **lotteries**
- Because we don't know the probabilities
- We need a different set up
- We are going to thing about **acts**
- What is an act?

States of the World

- First we need to define **states of the world**
- We will do this with an example
- Consider a race between three horses
  - A(rchibald)
  - B(yron
  - C(umberbach)
- What are the possible oucomes of this race?
  - Excluding ties

State	Ordering
1	A, B ,C
2	A, C, B
3	B, A, C
4	B, C, A
5	C, A, B
6	C, B, A

Acts

- This is what we mean by the states of the world
  - An exclusive and exhaustive list of all the possible outcomes in a scenario
- An **act** is then an action which is defined by the oucome it gives in each state of the world
- Here are two examples
  - Act f: A \$10 even money bet that Archibald will win
  - Act g: A \$10 bet at odds of 2 to 1 that Cumberbach will win

Acts

State	Ordering	Payoff Act f	Payoff Act g
1	A, B ,C	$10	-$10
2	A, C, B	$10	-$10
3	B, A, C	-$10	-$10
4	B, C, A	-$10	-$10
5	C, A, B	-$10	$20
6	C, B, A	-$10	$20

- So, how would you choose between acts f and g?
- SEU assumes the following:

1. Figure out the probability you would associate with each state of the world
2. Figure out the utility you would gain from each prize
3. Figure out the expected utility of each act according to those probabilities and utilities
4. Choose the act with the highest utility

- So, in the above example
- Utility from f :

$$\begin{aligned}&[\pi(ABC) + \pi(ACB)]\, u(10)\\&+ [\pi(BAC) + \pi(BCA)]\, u(-10)\\&+ [\pi(CBA) + \pi(CAB)]\, u(-10)\end{aligned}$$

  where π is the probability of each act
- Utility from g :

$$\begin{aligned}&[\pi(ABC) + \pi(ACB)]\, u(-10)\\&+ [\pi(BAC) + \pi(BCA)]\, u(-10)\\&+ [\pi(CBA) + \pi(CAB)]\, u(20)\end{aligned}$$

- Assuming utility is linear f is preferred to g if

$$\frac{[\pi(ABC) + \pi(ACB)]}{[\pi(CBA) + \pi(CAB)]} \geq \frac{3}{2}$$

- Or the probability of A winning is more than $3/2$ times the probability of C winning

Definition

Let X be a set of prizes, Ω be a (finite) set of states of the world and F be the resulting set of acts (i.e. F is the set of all functions $f : \Omega \rightarrow X$). We say that preferences $\succeq$ on the set of acts F has a subjective expected utility representation if there exists a utility function $u : X \rightarrow \mathbb{R}$ and probability function $\pi : \Omega \rightarrow [0, 1]$ such that $\sum_{\omega \in \Omega} \pi(\omega) = 1$ and

$$\begin{aligned} f &\succeq g \\ &\Leftrightarrow \sum_{\omega \in \Omega} \pi(\omega) u\left(f(\omega)\right) \geq \sum_{\omega \in \Omega} \pi(\omega) u\left(g(\omega)\right) \end{aligned}$$

- Notes
  - Notice that we now have **two** things to recover: Utility and preferences
  - Axioms beyond the scope of this course: has been done twice - first by Savage[1] and later (using a trick to make the process a lot simpler) by Anscombe and Aumann[2]
  - Utility pinned down to positive affine transform
  - Probabilities are unique

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[1] Savage, Leonard J. 1954. The Foundations of Statistics. New York, Wiley.

[2] Anscombe, F. J.; Aumann, R. J. A Definition of Subjective Probability. The Annals of Mathematical Statistics 34 (1963), no. 1, .

The Ellsberg Paradox

- Unfortunately, while simple and intuitive, SEU theory has some problems when it comes to describing behavior
- These problems are most elegantly demostrated by the Ellsberg paradox
  - A version of which you have answered as a class
- This thought experiment has sparked a whole field of decision theory
- Fun fact: Danlel Ellsberg was the defence analysis who released the Pentagon papers (!)

The Ellsberg Paradox - A Reminder

- Choice 1: The 'risky bag'
  - Fill a bag with 20 red and 20 black tokens
  - Offer your subject the opportunity to place a $10 bet on the color of their choice
  - Then elicit the amount x such that the subject is indifferent between playing the gamble and receiving $\$x$ for sure.
- Choice 2: The 'ambiguous bag'
  - Repeat the above experiment, but provide the subject with no information about the number of red and black tokens
  - Then elicit the amount y such that the subject is indifferent between playing the gamble and receiving $\$y$ for sure.

The Ellsberg Paradox

- Typical finding
  - $x >> y$
  - People much prefer to bet on the risky bag
- This behavior cannot be explained by SEU?
- Why?

- What is the utility of betting on the risky bag?
- The probability of drawing a red ball is the same as the probability of drawing a black ball at 0.5
- So whichever act you choose to bet on, the utility of the gamble is

$$0.5u(\$10)$$

- What is the utility of betting on the ambiguous bag?
- Here we need to apply SEU
- What are the states of the world?
    - Red ball is drawn or black ball is drawn
- What are the acts?
    - Bet on red or bet on black

State	r	b
red	10	0
black	0	10

- How do we calculate the utility of these two acts?
  - Need to decide how likely each state is
  - Assign probabilities $\pi(r) = 1 - \pi(b)$
  - Note that these do **not** have to be 50%
  - Maybe you think I like red chips!

- Utility of betting on the red outcome is therefore

$$\pi(r)u(\$10)$$

- Utility of betting on the black outcome is

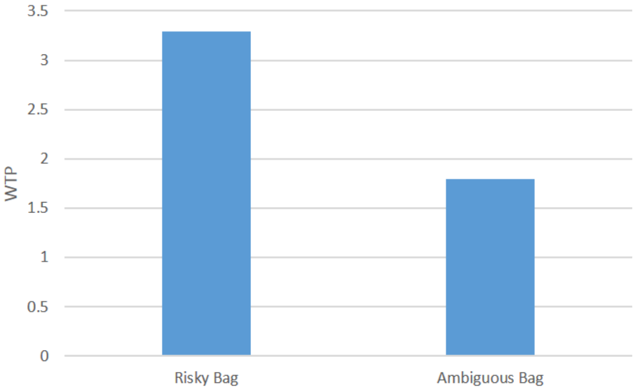
$$\pi(b)u(\$10) = (1 - \pi(r))u(\$10)$$

- Because you get to choose which color to bet on, the gamble on the ambiguous urn is

$$\max\{\pi(r)u(\$10), (1 - \pi(r))u(\$10)\}$$

  - is equal to $0.5u(\$10)$ if $\pi(r) = 0.5$
  - otherwise is **greater** than $0.5u(\$10)$
  - should always (weakly) prefer to bet on the ambiguous urn
  - intuition: if you can choose what to bet on, 0.5 is the worst probability

The Ellsberg Paradox



- 61% of you exhibit the Ellsberg paradox
- For more details see *Halevy, Yoram. "Ellsberg revisited: An experimental study." Econometrica 75.2 (2007): 503-536.*

- So, as usual, we are left needing a new model to explain behavior
- There have been many such attempts since the Ellsberg paradox was first described
- We will focus on 'Maxmin Expected Utility' by Gilboa and Schmeidler[3]

[3] Gilboa, Itzhak & Schmeidler, David, 1989. "Maxmin expected utility with non-unique prior," Journal of Mathematical Economics, Elsevier, vol. 18(2), pages 141-153, April.

Maxmin Expected Utility

- Maxmin expected utility has a very natural interpretation....
- The world is out to get you!
  - Imagine that in the Ellsberg experiment was run by an evil and sneaky experimenter
  - After you have chosen whether to bet on red or black, they will increase your chances of losing
  - They will sneak some chips into the bag of the **opposite** color to the one you bet on
  - So if you bet on red they will put black chips in and visa versa

Maxmin Expected Utility

- How should we think about this?
- Rather than their being a single probability distribution, there is a **range** of possible distributions
- After you chose your act, you evaluate it using the **worst** of these distributions
- This is maxmin expected utility
  - you **maximize** the **minimum** utility that you can get across different probability distributions
- Has links to robust control theory in engineering
  - This is basically how you design aircraft

Definition

Let X be a set of prizes, Ω be a (finite) set of states of the world and F be the resulting set of acts (i.e. F is the set of all functions $f : \Omega \to X$). We say that preferences $\succeq$ on the set of acts F has a Maxmin expected utility representation if there exists a utility function $u : X \to \mathbb{R}$ and convex set of probability functions Π and

$$\begin{aligned} f &\succeq g \\ &\Leftrightarrow \min_{\pi \in \Pi} \sum_{\omega \in \Omega} \pi(\omega) f(\omega) \geq \min_{\pi \in \Pi} \sum_{\omega \in \Omega} \pi(\omega) g(\omega) \end{aligned}$$

Maxmin Expected Utility

- Maxmin expected utility can explain the Ellsberg paradox
  - Assume that $u(x) = x$
  - Assume that you think $\pi(r)$ is between 0.25 and 0.75
  - Utility of betting on the risky bag is $0.5u(x) = 5$
  - What is the utility of betting on red from the ambiguous bag?

$$\min_{\pi(r) \in [0.25, 0.75]} \pi(r) u(\$10) = 0.25u(\$10) = 2.5$$

  - Similary, the utility from betting on black is

$$\min_{\pi(r) \in [0.25, 0.75]} (1 - \pi(r))\, u(\$10) = 0.25u(\$10) = 2.5$$

  - Maximal utility from betting on the ambiguous bag is lower than that from the risky bag

Maxmin Expected Utility and No Trade Regions

- Models of ambiguity aversion have been used to explain a number of phenomena in economics and finance
- One example: the existence of a 'no trade' region in asset prices[4]
  - Imagine that there is a financial asset that pays $10 if a company is a success, and $0 otherwise.
  - The price of the asset is p.
  - As an investor, you are can buy 1 unit of this asset, or you can short sell 1 unit of the asset.
  - If you buy the asset you pay p and receive $10 if the company is a success.
  - If you short sell the asset, then you have receive p for sure, but have to pay $10 if the company does well.

[4] Dow, James & Werlang, Sergio Ribeiro da Costa, 1992. "Uncertainty Aversion, Risk Aversion, and the Optimal Choice of Portfolio," Econometrica, Econometric Society, vol. 60(1), pages 197-204, January.

- How would an SEU person decide what to do?
- Let $\pi(g)$ be the probability they assign to the company doing well
- Assume utility is linear
- Utility from buying the asset is

$$\pi(g)\,(10-p)+(1-\pi(g))(-p)$$

- Utility from selling the asset is

$$\pi(g)\,(p-10)+(1-\pi(g))(p)$$

- Utility from doing neither is 0

Maxmin Expected Utility and No Trade Regions

- So, if

  $$p < 10\pi(g)$$

  Then the best option is to buy, whereas if

  $$p > 10\pi(g)$$

  the best option is to short sell
- Key point: they would like to trade at any p
  - At $p = 10\pi(good)$ they will be indifferent

Maxmin Expected Utility and No Trade Regions

- What about a Maxmin expected utility person?
- Let's say they have a range of possible probabilities of the firm doing well
  - $\pi^*(g)$ is the highest
  - $\pi_*(g)$ is the lowest

  with $\pi^*(g) > \pi_*(g)$

Maxmin Expected Utility and No Trade Regions

- Which probability will they use to assess buying the asset?
  - The value of the asset is increasing in $\pi(g)$,
  - Will use the **lowest** value $\pi_*(g)$
  - So the value of buying the asset is

$$\pi_*(g)\,(10 - p) + (1 - \pi_*(g))(-p)$$

  - will buy if

$$p < 10\pi_*(g)$$

Maxmin Expected Utility and No Trade Regions

- Which probability will they use to assess short selling the asset?
  - The value of the short selling the asset is decreasing in $\pi(g)$,
  - Will use the **highest** value $\pi^*(g)$
  - So the value of buying the asset is

$$\pi^*(g)\,(10 - p) + (1 - \pi^*(g))(-p)$$

  - will buy if

$$p > 10\pi^*(g)$$

Maxmin Expected Utility and No Trade Regions

- Unlike for the SEU guy there is a **no trade region** for prices
- If we have
$$10\pi_*(g) < p < 10\pi^*(g)$$
- Then the DM will not want to sell or buy the asset
- This is because they use different probabilities to assess each case