Bounded Rationality Lecture 3

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The Story So Far.....

- Last time we introduced a general model of rational inattention
- Made only limited assumptions about the cost of attention
- Today we will introduce cost function based on the concept of Shannon Mutual Information
    - Most common cost function used in the rational inattention literature
- Discuss some of its properties
    - Relation to Logistic choice
    - Linear Quadratic Gaussian Case
    - Discrete Choice of Actions
- Introduce an application: Pricing with a rationally inattentive agent

Plan for Today

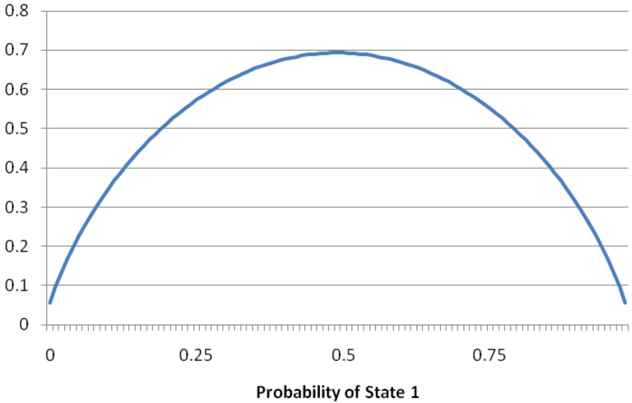
- Introduction to Shannon Entropy and Mutual Information
- Properties of Rational Inattention with Shannon Entropy
- Application [Martin 2012]

Shannon Entropy

- Shannon Entropy is a measure of how much 'missing information' there is in a probability distribution
- In other words - how much we do not know, or how much we would learn from resolving the uncertainty
- For a random variable X that takes the value x_i with probability $p(x_i)$ for $i = 1...n$, defined as

$$\begin{aligned} H(X) &= E(-\ln(p(x_i))) \\ &= -\sum_i p(x_i)\ln(p_i) \end{aligned}$$

Shannon Entropy



- Can think of it as how much we learn from result of experiment

Justification for Shannon Entropy

- Say we want our measure of entropy to have the following features
- Depends only on the probability distribution
  - $H(X) = H(p)$

Justification for Shannon Entropy

- Say we want our measure of entropy to have the following features
- Depends only on the probability distribution
- Maximized at a uniform probability distribution
  - $\max_{p \in \Delta^M} H(p) = H\left(\left\{\frac{1}{M}, \frac{1}{M}, ..., \frac{1}{M}\right\}\right)$

Justification for Shannon Entropy

- Say we want our measure of entropy to have the following features
- Depends only on the probability distribution
- Maximized at a uniform probability distribution
- Unaffected by adding zero probability state
  - $H(\{p_1....p_M\}) = H(\{p_1....p_M, 0\})$

Justification for Shannon Entropy

- Say we want our measure of entropy to have the following features
- Depends only on the probability distribution
- Maximized at a uniform probability distribution
- Unaffected by adding zero probability state
- Additive
  - $H(X, Y) = H(X) + \sum_x p(x)H(Y|x)$
  - (Most 'controversial' - other entropies relax this assumption)

Justification for Shannon Entropy

- Say we want our measure of entropy to have the following features
- Depends only on the probability distribution
- Maximized at a uniform probability distribution
- Unaffected by adding zero probability state
- Additive
- Then Entropy must be of the form (Khinchin 1957)

$$H(X) = -k \sum_{i} p(x_i) \ln(p_i)$$

Entropy and Information Costs

- Related to the notion of entropy is the notion of Mutual Information

$$I(X, Y) = \sum_x \sum_y p(x, y) \log \frac{p(x, y)}{p(x)p(y)}$$

- Measure of how much information one variable tells you about another
- Note that $I(X, Y) = 0$ if X and Y are independent

- Note also that mutual information can be rewritten in the following way

$$
\begin{aligned}
I(X,Y) &= \sum_x \sum_y p(x,y) \log \frac{p(x,y)}{p(x)p(y)} \\
&= \sum_x \sum_y p(x,y) \log \frac{p(x|y)}{p(x)} \\
&= \sum_y \sum_x p(x,y) \ln P(x|y) - \sum_x \sum_y p(x,y) \ln p(x) \\
&= \sum_y p(y) \sum_x p(x|y) \ln P(x|y) - \sum_y p(x) \ln p(x) \\
&= H(X) - H(X|Y)
\end{aligned}
$$

- Difference between entropy of X and the expected entropy of X once Y is known

- Most papers assume that information costs are linear in the mutual information of the prior and the posterior

$$
\begin{aligned}
K(\beta, \lambda) &= k \sum_m \sum_{t \in T(\lambda)} \beta_m \lambda_m(t) \ln \frac{\lambda_m(t)}{P(t)} \\
&= k \sum_{t \in T(\lambda)} P(t) \sum_m t_m \ln t_m - \sum_m \beta_m \ln \beta_m
\end{aligned}
$$

- Key feature: Entropy is strictly*concave*
- So negative of entropy is strictly convex
- Say we choose a signal structure with two posteriors t and t'
- It must be that

$$p(t)t + p(t')t' = \beta$$

- so

$$\begin{aligned} p(t)H(t) + p(t')H(t') &> H(p(t)t + p(t')t') \\ &= H(\beta) \end{aligned}$$

- So the cost of ‘learning something’ is always positive

Solving Rational Inattention Models

- Solving Rational Attention Models can be difficult analytically
- General approach - ignore choice of information structure, instead focus on joint distribution of choice variable and state
  - i.e. choose state dependent stochastic choice directly
- Example (Matejka and McKay 2011) - continuous state space, finite action space

Solving Rational Inattention Models

- $\mathcal{D}$ set of all state contingent stochastic choice functions for some state space Ω and set of acts A
- Remember $D_\omega(f)$ is the probability of choosing f in state ω
- Remember that , for $D \in \mathcal{D}$, the mutual information between choices f and objective state ω is given by

$$I(D, \omega) = H(f) - H(f|\omega)$$

Solving Rational Inattention Models

- Decision problem of agent is to choose $D \in \mathcal{D}$ to maximize

$$\sum_{f \in A} \int_{\omega} u(f(\omega)) D_{\omega}(f) G(d\omega)$$
$$-\lambda \left[\sum_{f \in A} \int_{\omega} D_{\omega}(f) \ln D_{\omega}(f) G(d\omega) + \sum_{f \in A} D(f) \ln D(f) \right]$$

- Subject to

$$\sum_{f \in A} D_{\omega}(f) = 1 \text{ Almost surely}$$

- Where $D(f)$ is the unconditional probability of choosing f

$$
\begin{aligned}
L(D) &= \sum_{f\in A}\int_{\omega} u(f(\omega))D_{\omega}(f)G(d\omega) \\
&\quad -\lambda\left[\sum_{f\in A}\int_{\omega} D_{\omega}(f)\ln D_{\omega}(f)G(d\omega)+\sum_{f\in A} D(f)\ln D(f)\right] \\
&\quad -\int_{\omega}\mu(\omega)\left[\sum_{f\in A} D_{\omega}(f)-1\right]G(d\omega)
\end{aligned}
$$

- FOC WRT $D_{\omega}(f)$ (assuming $>$0)

$$
u(f(\omega)) - \mu(\omega) + \lambda[\ln D(f) + 1 - \ln D_{\omega}(f) - 1] = 0
$$

- Note that this is a convex problem

Solution

- FOC WRT $D_\omega(f)$ (assuming >0)

$$u(f(\omega)) - \mu(\omega) + \lambda[\ln D(f) + 1 - \ln D_\omega(f) - 1] = 0$$

- Which gives

$$D_\omega(f) = D(f) \exp^{\frac{u(f(\omega)) - \mu(\omega)}{\lambda}}$$

- Plug this into

$$\begin{aligned} \sum_{f \in A} D_\omega(f) &= 1 \\ \Rightarrow \quad e^{\frac{\mu(v)}{\lambda}} &= \sum_{f \in A} D(f) e^{\frac{u(f(\omega))}{\lambda}} \end{aligned}$$

- Which in turn gives...

Comments

$$D_\omega(f) = \frac{D(f)\exp^{\frac{u(f(\omega))}{\lambda}}}{\sum_{f\in A} D(f)e^{\frac{u(f(\omega))}{\lambda}}}$$

- Similar in form to logistic random choice
- If alternatives are ex ante identical, this *is* logistic choice
- Otherwise choice probabilities are 'warped' by $D(f)$ - which contains information on the prior value of each option
- As costs go to zero, deterministically pick best option in that state
- As costs go to infinity, deterministically pick the best option ex ante

Comments

- The above is not a complete solution
- Does not solve for $D(f)$
- One can completely characterize solution in closed form if one knows what acts are chosen in what states
- Checking which acts are chosen is a hard problem
- There are algorithms that can solve these problems
  - Blahut-Arimoto Algorithm
  - See Cover and Thomas [1991] for more details
- May be better to tackle choice of posteriors directly

Choosing Posteriors Directly

- Consider the case of two state and two acts

$$\begin{array}{ccc} & \omega_1 & \omega_2 \\ f & u_1^f & u_2^f \\ g & u_1^g & u_1^g \end{array}$$

- And the problem of choosing posterior states t and s (where number is probability of state 1 in that posterior)

- Optimization problem (assuming that f is chosen at t)

$$P(t)\left[tu_1^f + (1-t)u_2^f\right] + (1-P(t))\left[su_1^g + (1-s)u_2^g\right]$$
$$-kP(t)\left(\left[t\ln t + (1-t)\ln(1-t)\right] + (1-P(t)\left[s\ln s + (1-s)\right.\right.$$

- subject to

$$P(t)t + (1-P(t))s = \beta$$

$$\left[u_1^f - u_2^f - k \ln \frac{t}{1-t}\right] = \mu$$
$$\left[u_1^g - u_2^g - k \ln \frac{s}{1-s}\right] = \mu$$

$$\begin{aligned} & \left[tu_1^f + (1-t)u_2^f\right] - [su_1^g + (1-s)u_2^g] \\ & -k\left(t \ln t + (1-t)\ln(1-t) - s \ln s - (1-s)\ln s\right) \\ = \; & \mu(t-s) \end{aligned}$$

$$\begin{aligned}
& \left[tu_1^f + (1-t)u_2^f\right] - \left[su_1^g + (1-s)u_2^g\right] \\
& -k\left(t\ln t - (1-t)\ln(1-t) + s\ln s + (1-s)\ln s\right) \\
= & \left[u_1^f - u_2^f - k\left(\ln t - \ln(1-t)\right)\right](t-s)
\end{aligned}$$

$$\begin{aligned}
& s\left[u_1^f - u_1^g - k\left(\ln t + \ln s\right)\right] + \\
& (1-s)\left[u_2^f - u_2^g - k\left(\ln(1-t) + \ln(1-s)\right)\right] \\
= & \ 0
\end{aligned}$$

$$\begin{aligned}
\Rightarrow & \left[u_1^f - u_1^g - k\left(\ln t - \ln s\right)\right] \\
= & \left[u_2^f - u_2^g - k\left(\ln(1-t) - \ln(1-s)\right)\right] \\
= & \ 0
\end{aligned}$$

Implies

$$\frac{u_1^f - u_1^g}{\ln t - \ln s} = \frac{u_2^f - u_2^g}{\ln(1-t) - \ln(1-s)} = k$$

- This tells us
  1. $\frac{u_m^f - u_m^g}{\ln t_m - \ln s_m}$ is a constant
  2. Posterior beliefs do not depend on priors
- Both of these results are general

The Linear Quadratic Gaussian Case

- One case in which this problem becomes more tractable is if the input and output signal are both normal
- The entropy of a normal variable $X \sim N(\mu, \sigma_x^2)$ is given by

$$H(Y) = \ln(2\pi e \sigma_x^2)$$

- If Y and X are both normal, then

$$H(Y|X) = \int_x f(x) \int_y f(y|x) \ln(y|x) d(y) d(x)$$

- As $y|x$ is distributed normally with variance $(1-\rho^2)\sigma_y^2$, this becomes

$$\begin{aligned} H(Y|X) &= \int_x f(x) \ln(2\pi e \sigma_{y|x}^2) d(x) \\ &= \frac{1}{2} \ln(2\pi e(1-\rho^2)\sigma_y^2) \end{aligned}$$

The Linear Quadratic Gaussian Case

- As mutual information is given by

$$
\begin{aligned}
& H(Y) - H(Y|X) \\
= \; & \ln(2\pi e \sigma_y^2) - \frac{1}{2}\ln(2\pi e(1-\rho^2)\sigma_y^2)
\end{aligned}
$$

- In this case, the mutual information is given by

$$\frac{1}{2}\ln(1-\rho^2)$$

- So information costs depend only on the covariance of the two signals!
- It turns out that joint normality is optimal if the utility function is quadratic in the relationship between the objective and subjective state
  - Choice of variance on some normally distributed error term
- However, note that some papers *assume* normality (this is bad)

Discrete Choice of Actions

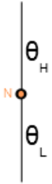
- Outside the linear quadratic case, often the optimal solution has discrete number of chosen actions
- Even if
  - State space is continuous
  - Action space is continuous
- See Sims [2006], Matejka [2008]
- Despite the fact that the state of the world is continuous, prices may jump between a discrete number of values
- Foundation for sticky prices?

- Sequential pricing game
  - One buyer, one seller, one product of uncertain quality
  - Seller gets free info on quality, sets price
  - Buyer gets free info on price and can obtain costly info on quality, decides to buy or not

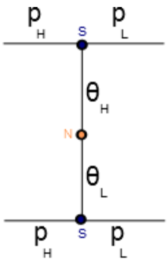
Market Setting

- Once off sales encounter
    - One buyer, one seller, one product

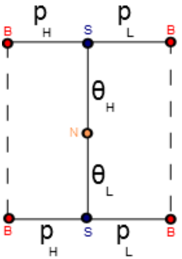
- Nature determines quality $\theta \in \{\theta_L, \theta_H\}$, in $\mathbb{R}_+$
    - Prior $\lambda = \Pr(\theta_H)$



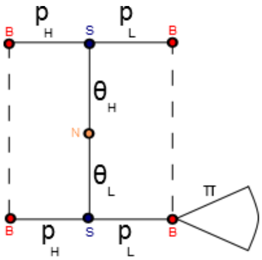
- Seller learns quality, sets price $p \in \{p_L, p_H\}$, in $\mathbb{R}_+$
    - Generalizes to many, internalized first and fully



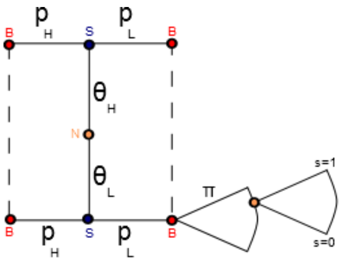
- Buyer learns p, forms interim belief β_p of high quality
  - Based on prior λ (brand) and seller strategies



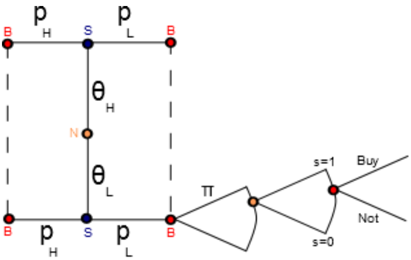
- Choose information technology $\pi \in \Pi^{\beta_p}$
  - $\pi : \{\theta_L, \theta_H\} \to \Delta(S)$, finite support, $S = [0, 1]$ posterior beliefs



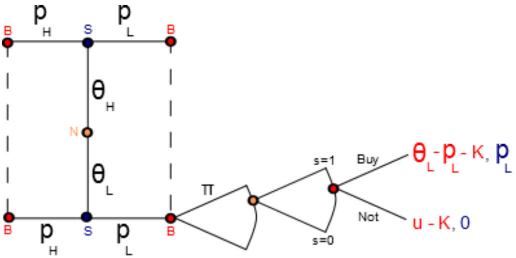
- Nature determines a posterior belief $s \in [0, 1]$
  - Posterior belief about product being high quality



- Decides whether to buy or not
  - Just a unit of the good



- Standard utility and profit functions (risk neutral EU)
  - $u \in \mathbb{R}_+$ is outside option, $K \in \mathbb{R}_+$ is Shannon cost



- Shannon cost for information technology π, cost κ, and interim beliefs β_p

$$K\left(\pi, \kappa, \beta_p\right) =$$

$$\kappa \sum_{s \in S(\pi)} \pi(s) \left(s \ln(s) + (1-s)\ln(1-s)\right)$$

$$-\kappa\left(\beta_p \ln\left(\beta_p\right) + \left(1-\beta_p\right)\ln\left(1-\beta_p\right)\right)$$

Equilibrium

- Only two mixed strategy PBE w/ rational inattention:
  - Always exists "Pooling low"
    - High quality sellers charge a *low price* with probability 1
    - Low quality sellers charge a *low price* with probability 1
    - Strategic ignorance: Buyers never attend, strong beliefs
  - Always exists "Mimic high"
    - High quality sellers charge a *high price* with probability 1
    - Low quality sellers charge a *high price* with probability $\eta \in [0, 1]$ (mimicking)
    - Buyers typically attend at high prices

Theorem

For every cost κ, there exists an equilibrium ("mimic high") where high quality sellers price high with probability 1 and low quality sellers price high with a unique probability $\eta \in [0, 1]$.

- Why unique mimicking η?
- When $\eta \in (0, 1)$, need low quality seller indifference:

$$d_{p_H}^{\theta_L} \times p_H = p_L \Rightarrow d_{p_H}^{\theta_L} = \frac{p_L}{p_H}$$

  where $d_{p_H}^{\theta_L}$ is conditional demand
- As η increases, $d_{p_H}^{\theta_L}$ strictly decreases, so single crossing with $\frac{p_L}{p_H}$ *if any*
- Why is $d_{p_H}^{\theta_L}$ strictly decreasing in η?

- Threshold posterior for each action: $s^0_{p_H}$ (not buy at p_H) and $s^1_{p_H}$ (buy at p_H)

$$\ln\left(\frac{s^1_{p_H}}{s^0_{p_H}}\right) = \frac{(\theta_H - p_H) - u}{\kappa}$$

$$\ln\left(\frac{1 - s^1_{p_H}}{1 - s^0_{p_H}}\right) = \frac{(\theta_L - p_H) - u}{\kappa}$$

- Key: Thresholds do not depend on beliefs
  - Property of rational inattention

- Let β_{p_H} be the prior probability that the good is of high quality given that it is of high price
- By Bayes Rule

$$
\begin{aligned}
s^1_{p_H} &= \frac{(1-\beta_{p_H})d^{\theta_L}_{p_H}}{(1-\beta_{p_H})d^{\theta_L}_{p_H} + \beta_{p_H} d^{\theta_H}_{p_H}} \\
s^0_{p_H} &= \frac{(1-\beta_{p_H})(1-d^{\theta_L}_{p_H})}{(1-\beta_{p_H})(1-d^{\theta_L}_{p_H}) + \beta_{p_H}(1-d^{\theta_H}_{p_H})}
\end{aligned}
$$

$$
d^{\theta_L}_{p_H} = \frac{\left(\frac{1-s^1_{p_H}}{s^1_{p_H}-s^0_{p_H}}\right)\left(\beta_{p_H} - s^0_{p_H}\right)}{\left(1-\beta_{p_H}\right)}
$$

- Because thresholds do not depend on beliefs, conditional demand is
    - Strictly increasing in interim beliefs β_{p_H}
    - So strictly decreasing in mimicking η

- What is the unique value of η when $\eta \in (0,1)$?

$$\eta = \frac{\lambda}{1-\lambda} \frac{\left(1 - s^0_{p_H}\right)\left(1 - s^1_{p_H}\right)}{s^0_{p_H}\left(1 - s^1_{p_H}\right) + \frac{p_L}{p_H}\left(s^1_{p_H} - s^0_{p_H}\right)}$$

- As $\kappa \to 0$, $\eta \to 0$
- As $\kappa \to \infty$, $\eta \to ?$