Consumer Choice 1

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GR5211 - Microeconomic Analysis 1

- We are now going to think a lot more about a particular type of choice we introduced last lecture
- **Choice from Budget Sets**
  - Objects of choice are **commodity bundles**

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

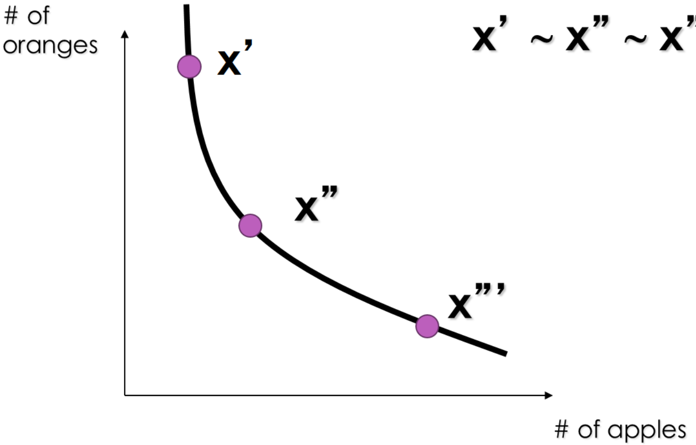
  - Consumers are **price takers**
    - Treat prices and incomes as fixed
  - They can choose any bundle which satisfies their budget constraint

$$\left\{ x \in \mathbb{R}^n_+ | \sum_{i=1}^{n} p_i x_i \leq w \right\}$$

- Why are such choices so interesting?
    - Many economic interactions can be characterized this way
    - Will form the basis of the study of equilibrium in the second half of the class

Indifference Curves

- When dealing with preferences over commodity bundles it will be useful to think about **Indifference Curves**
- These are curves that link bundles that are considered indifferent by the consumer
- Useful for presenting 3 dimensional information on a two dimensional graph



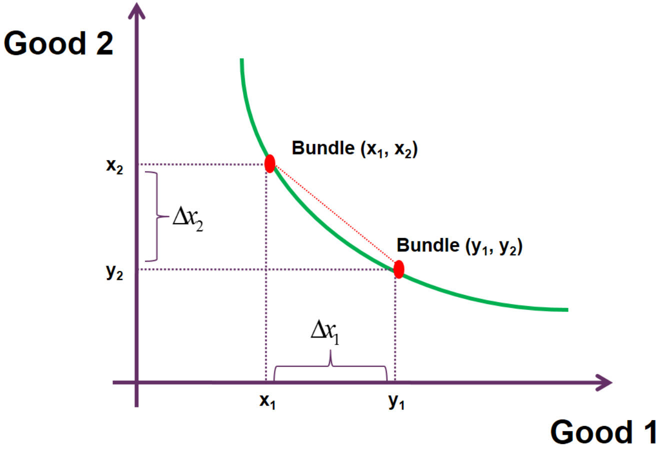
$x' \sim x'' \sim x'''$

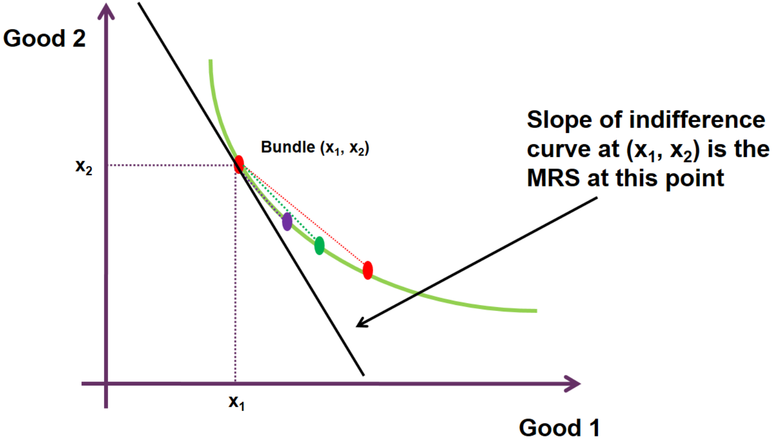
- A couple of properties of indifference curves

1. Two different indifference curves cannot cross (why?)
2. The 'slope' of the indifference curve represents the (negative of the) **marginal rate of substitution**
   - The rate at which two goods can be traded off while keeping the subject indifferent

$$MRS(x_2, x_1) = -\lim_{\Delta(x_1)\to 0} \frac{\Delta(x_2)}{\Delta(x_1)}$$

$$\text{such that } (x_1.x_2) \sim (x_1 + \Delta(x_1), x_2 + \Delta(x_2))$$


Good 2
Bundle (x1, x2)
x2
Δx2
Bundle (y1, y2)
y2
Δx1
x1
y1
Good 1



- Question: Is MRS always well defined?

- If preferences can be represented by a utility function, then the equation of an indifference curve is given by

$$u(x) = \bar{u}$$

- Thus, if the utility function is differentiable we have

$$\sum_{i=1}^{N} \frac{\partial u(x)}{\partial x_i} dx_i = 0$$

- And so, in the case of two goods, the slope of the indifference curve is

$$\frac{dx_2}{dx_1} = -\frac{\frac{\partial u(x)}{\partial x_1}}{\frac{\partial u(x)}{\partial x_2}} = -MRS$$

which is another way of characterizing the MRS

Preferences over Commodity Bundles

- When thinking about preferences over commodity bundles it might be natural to assume that preferences have properties other than just
  - Completeness
  - Transitivity
  - Reflexivity
- Some of these we have come across before
  - (Strict) Monotonicity

$$\begin{aligned} x_n &\geq y_n \text{ for all } n \text{ and } x_n > y_n \text{ for some } n \\ \text{implies that } x &\succ y \end{aligned}$$

  - Monotonicity

$$\begin{aligned} x_n &\geq y_n \text{ for all } n \text{ implies } x \succeq y \\ x_n &> y_n \text{ for all } n \text{ implies } x \succ y \end{aligned}$$

  - Local Non-Satiation
- Examples?

- Another property often assumed is **convexity**
  - The preference relation $\succeq$ is convex if the upper contour set $U_{\succeq}(x) = \{y \in X | y \succeq x\}$ is convex
  - i.e. for any x, z, y such that $y \succeq x$ and $z \succeq x$ and $\alpha \in (0, 1)$

$$(\alpha y + (1 - \alpha)z) \succeq x$$

- A preference relation is **strictly convex** if x, z, y such that $y \succeq x$ and $z \succeq x$ and $\alpha \in (0, 1)$

$$(\alpha y + (1 - \alpha)z) \succ x$$

- What is the economic intuition of convexity?
- What do convex indifference curves look like?

Fact

A complete preference relation with a utility representation is convex if and only if it can be represented by a quasi concave utility function - i.e., for every x the set

$$\{y \in X | u(y) \geq u(x)\}$$

is convex

- A another property that preferences **can** have is **homotheticity**
  - The preference relation $\succeq$ is homothetic if $x \succeq y$ implies $\alpha x \succeq \alpha y$ for any $\alpha \geq 0$

Fact

A complete, increasing, continuous homothetic preference relation with a utility representation can be represented with a utility function which is homogenous of degree 1, i.e.

$$u(\alpha x_1, ...\alpha x_n) = \alpha u(x_1, ...x_n)$$

- What do homothetic indifference curves look like?
- What is their economic intuition?

- Finally, we might be interested in preferences that are **quasi linear**
  - The preference relation $\succeq$ is quasi linear in commodity 1 if $x \succeq y$ implies
$$(x + \varepsilon e_1) \succeq (y + \varepsilon e_1)$$
for $\varepsilon > 0$ and
$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Fact

A complete, increasing, strictly monotonic, quasi linear preference relation with a utility representation can be represented with a utility function of the form

$$u(x) = v(x_2, ...x_k) + x_1$$

The Consumer's Problem

- We are now in the position to think about what the solution to the consumer's problem looks.like
- We will think of the consumer's problem as defined by
  - A set of preferences $\succeq$
  - A set of prices $p \in \mathbb{R}^N_{++}$
  - A wealth level w
- With the problem being

$$\begin{aligned} \text{choose } x &\in \mathbb{R}^N_+ \\ \text{in order to maximize } &\succeq \\ \text{subject to } \sum_{i=1}^{N} p_i x_i &\leq w \end{aligned}$$

Existence

- **Question:** is the consumer's problem guaranteed to have a solution?
- Not without some further assumptions
- Here is a simple example
    - Let $N = 1$, $w = 1$ and $p_1 = 1$
    - Let preferences be such that higher numbers are preferred so long as they are less that 1, so

$$\begin{aligned} \text{If } x &< 1 \text{ then } x \succeq y \text{ iff } x \geq y \\ \text{If } x &\geq 1 \text{ the } x \succeq y \text{ iff } y \geq x \end{aligned}$$

- We need to add something else
- Any guesses what?

Theorem

If preferences $\succeq$ are continuous then the consumer's problem has a solution

- Proof follows fairly directly from Weierstrass Theorem!

Theorem

Any continuous function evaluated on a compact set has a maximum and a minimum

- Means that in order to guarantee existence we need three properties
  - Continuity of the function (comes from continuity of preferences)
  - Closedness of the budget set (comes from the fact that it is defined using weak inequalities)
  - Boundedness of the budget set (comes from the fact that we insist prices are strictly positive)

The Walrasian Demand Correspondence

- We are now in a position to define the Walrasian demand correspondance
- This is the amount of each good that the consumer will demand as a function of prices and income
- $x(p, w) \subset \mathbb{R}_+^N$ is the (set of) solution to the consumer's maximization problem when prices are p and wealth is w
    - i.e. the set of all bundles that maximize preferences (or equivalently utility) when prices are p and wealth is w
- Here are some straightforward properties of x when we maintain the assumptions of
    - Continuity
    - Local non-satiation

Fact

x is homogeneous of degree zero (i.e. $x(\alpha p, \alpha w) = x(p, w)$ for $\alpha > 0$)

- This follows from the fact that

$$\begin{aligned} & \left\{ x \in \mathbb{R}^n_+ | \sum_{i=1}^{n} p_i x_i \leq w \right\} \\ = & \left\{ x \in \mathbb{R}^n_+ | \sum_{i=1}^{n} \alpha p_i x_i \leq \alpha w \right\} \end{aligned}$$

Fact

Walras Law:

$$\sum_{i=1}^{n} p_i x_i = w$$

for any $x \in x(p, w)$

- This follows directly from local non-satiation

Properties of the Demand Correspondance

- Our final two properties are going to involve uniqueness and continuity of x
- Further down the road it will be very convenient for
  - x to be a function (not a correspondance)
  - x to be continuous
- What can we assume to guarantee this?

- First: do we have uniqueness?
- No! (see diagram)
- Here, convexity will come to our rescue

Fact

If $\succeq$ is convex then $x(p, w)$ is a convex set. If $\succeq$ is strictly convex then $x(p, w)$ is a function

- Proof comes pretty much directly from the definition and the fact that the budget set is convex

Properties of the Demand Correspondance

- In fact, if x is a function then we also get continuity

Fact

If x is single values and $\succeq$ is continuous then x is continuous

- Proof comes directly from the theorem of the maximum

Theorem (The Theorem of the Maximum)

Let

- X *and* Y *be metric spaces (*Y *will be the set of things that are chosen,* X *the set of parameters)*
- $\Gamma : X \rightrightarrows Y$ *be compact valued and continuous (this is the budget set)*
- $f : X \times Y \to \mathbb{R}$ *be continuous, (this is the utility function)*
  Now define $y^* : X \rightrightarrows Y$ *as the set of maximizers of* f *given parameters* x

  $$y^*(x) = \arg \max_{y \in \Gamma(x)} f(x, y)$$

  and define $f^* : X \rightrightarrows Y$ *as the maximized value of* f *for* f *given parameters* x

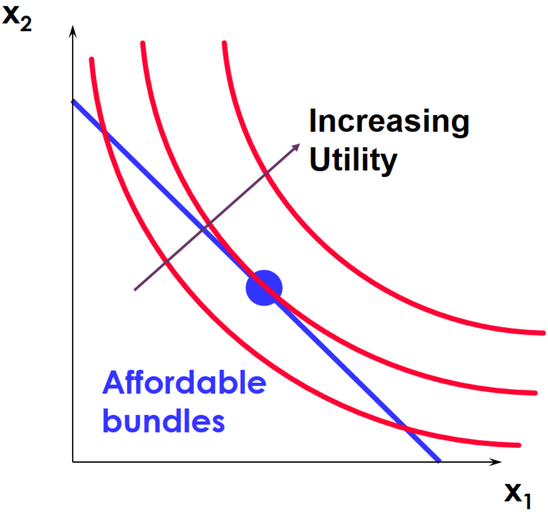
  $$f^*(x) = \max_{y \in \Gamma(x)} f(x, y)$$

Theorem (The Theorem of the Maximum)

Then

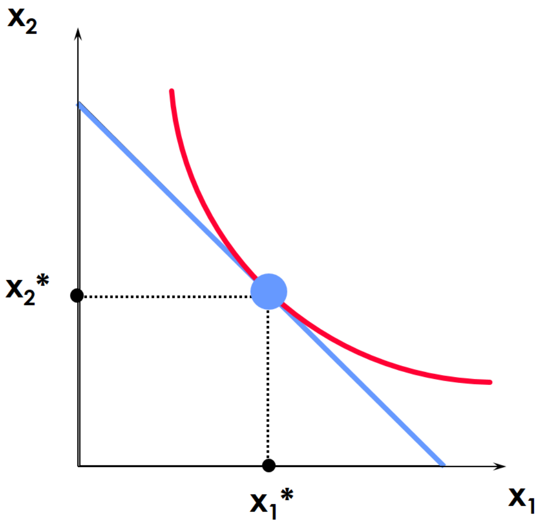
1. y^* *is upper hemi-continuous and compact valued*
2. f^* *is continuous*

- Graphically, what does the solutions to the consumer's problem look like?
- Here it is useful to think in two dimensions



x_2
Increasing
Utility
Affordable
bundles
x_1

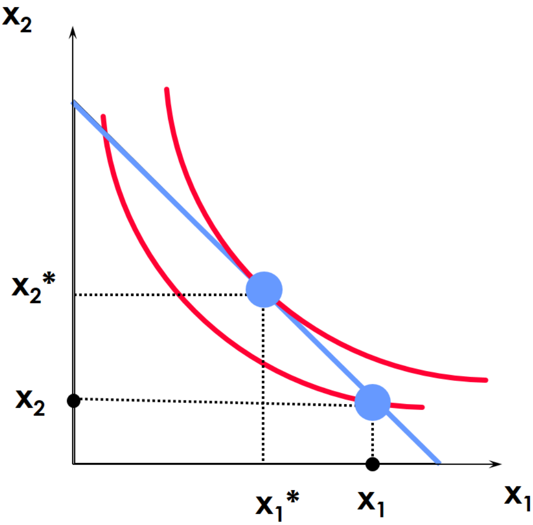
Tangency Conditions



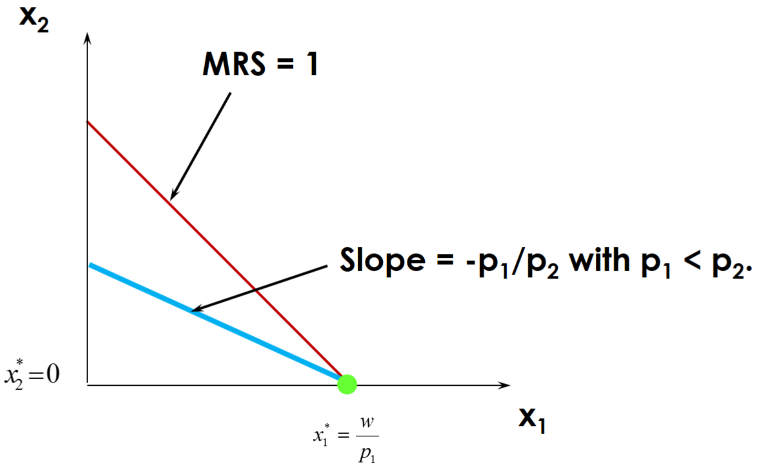
- If the solution to the consumer's problem is interior, then
    - The indifference curve
    - The budget line

  are tangent to each other
- Implies that the marginal rate of substituion is the same as the price ratio
- This makes intuitive sense
    - The rate at which goods can be traded off against each other in the market
    - is equal to the rate at which they can be traded off leaving the consumer indifferent
    - If not, then utility could be increased by switching to the 'cheaper' good

Tangency Conditions



- What about corner solutions?
- For example, none of good 2 is purchased
- Here, the indifference curve and the price line need not be equal
- But the price line must be **shallower** than the slope of indifference curve


x_2
MRS = 1
Slope = $-p_1/p_2$ with $p_1 < p_2$.
$x_2^* = 0$
x_1
$x_1^* = \frac{w}{p_1}$

- In the case in which utility is continuously differentiable, we can use the **Kuhn Tucker** (necessary) conditions to capture this intuition
- For the problem

$$
\begin{aligned}
& \max u(x) \\
\text{subject to } \sum_{i=1}^{n} p_i x_i - w &= 0 \\
-x_i &\leq 0 \ \forall i
\end{aligned}
$$

- We can set up the Lagrangian for the problem

$$
u(x) - \lambda \left(\sum_{i=1}^{n} p_i x_i - w \right) - \sum_{i=1}^{n} \mu_i \left(-x_i \right)
$$

- A necessary condition of a solution to the optimization problem x^* is the existence of λ, and $\mu_i \geq 0$ such that

$$\begin{aligned} \frac{\partial u(x^*)}{\partial x_i} - \lambda p_i + \mu_i &= 0 \\ \text{and } x_i^* . \mu_i &= 0 \text{ for all } i \end{aligned}$$

- So, if $x_i^* > 0$ then $\mu_i = 0$ and

$$\frac{\partial u(x^*)}{\partial x_i} = \lambda p_i$$

- If $x_i^* = 0$ then $\mu_i \leq 0$ and so

$$\frac{\partial u(x^*)}{\partial x_i} \leq \lambda p_i$$

- This can be summarized compactly by saying, that for a solution x^*

$$\begin{aligned} \nabla u(x^*) &\leq \lambda p \\ x^* \left[\nabla u(x^*) - \lambda p \right] &= 0 \end{aligned}$$

- Note that this implies that

$$\frac{\frac{\partial u(x^*)}{\partial x_i}}{\frac{\partial u(x^*)}{\partial x_j}} = \frac{p_i}{p_j}$$

- if x_i and x_j are both strictly positive